Eulerian statistics in homogeneous, anisotropic flows

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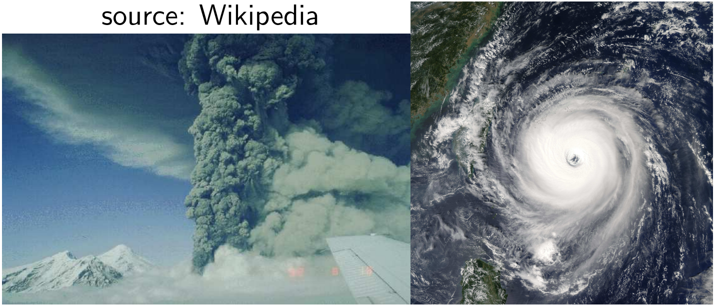
Supercomputing resources at CINECA

source: Wikipedia



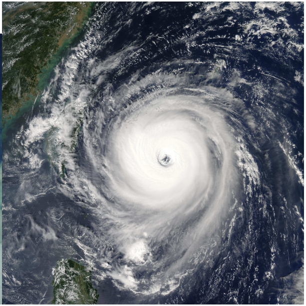
- Incompressible NSE invariant under Rotation + Translation

At large scales:
forcing, B.C
break rotation invariance

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v}\cdot\partial\mathbf{v} = -\partial p + \nu\partial^2\mathbf{v} + \mathbf{f} \\ \partial\cdot\mathbf{v} = 0 \\ + \ \text{boundary conditions} \end{cases}$$

- Is breaking of rotational symmetry passed down-scale ?

source: Wikipedia



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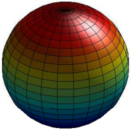
Universal signatures in small-scale fluctuations?

Longitudinal structure function: $S^{(n)}(\mathbf{r}) \equiv \left\langle \left[\left(\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x}) \right) \cdot \hat{\mathbf{r}} \right]^n \right\rangle$

$$\boxed{S^{(n)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{m=j} S_{jm}^{(n)}(r) Y_{jm}(\hat{\mathbf{r}})}$$

Arad et. al. PRL'98

$Y_{00} = 1/\sqrt{4\pi}$

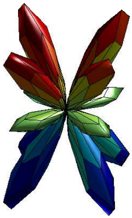


$Y_{20} = \sqrt{\frac{5}{16\pi}}(3cos^2\theta - 1)$

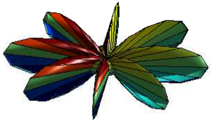
$Y_{22} = \sqrt{\frac{15}{16\pi}}(\sin^2\theta \cos 2\phi)$

Y_{40}

Y_{42}



Y_{44}



rotational invariant operators

$$\boxed{\partial_t S^{(2)}(\mathbf{r}) + \Gamma^{(3)} S^{(3)}(\mathbf{r}) - 2\nu \nabla^2 S^{(2)}(\mathbf{r}) = f^{(2)}(\mathbf{r})}$$

$r \ll L$

Universality at small scales

$$\partial_t S^{(2)}(\mathbf{r}) + \Gamma^{(3)} S^{(3)}(\mathbf{r}) - 2\nu \nabla^2 S^{(2)}(r) \sim 0$$

+ SO(3)

Weak Anisotropy

$$S_j^{(2)}(\mathbf{r}) = \sum_{m=-j}^{m=+j} Y_{jm}(\hat{\mathbf{r}}) \int S_{jm}^{(2)}(r) Y_{jm}(\hat{\mathbf{r}}) d\hat{\mathbf{r}}$$

$$\boxed{\partial_t S_j^{(2)}(\mathbf{r}) + \Gamma_j^{(3)} S_j^{(3)}(\mathbf{r}) - 2\nu \nabla^2 S_j^{(2)}(\mathbf{r}) \sim 0} \quad j = 0, 1, 2, \ldots$$

FOLIATION ???

$$S^{(n)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{m=j} S_{jm}^{(n)}(r) Y_{jm}(\hat{\mathbf{r}})$$

$$S^{(n)}(\mathbf{r}) \sim \Lambda_0 \left(\frac{r}{L}\right)^{\xi_0^n} + \Lambda_1 \left(\frac{r}{L}\right)^{\xi_1^n} + \Lambda_2 \left(\frac{r}{L}\right)^{\xi_2^n} + + \dots$$

$$S^{(n)}(\mathbf{L}) \sim \Lambda_0 + \Lambda_1 + \Lambda_2 + \dots$$

Prefactors cannot be universal!

$$S^{(n)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{m=j} S^{(n)}_{jm}(r) Y_{jm}(\hat{\mathbf{r}})$$

$$S^{(n)}(\mathbf{r}) \sim \Lambda_0 \big(\frac{r}{L}\big)^{\xi_0^n} + \Lambda_1 \big(\frac{r}{L}\big)^{\xi_1^n} + \Lambda_2 \big(\frac{r}{L}\big)^{\xi_2^n} + + \ldots$$

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Prefactors cannot be universal!

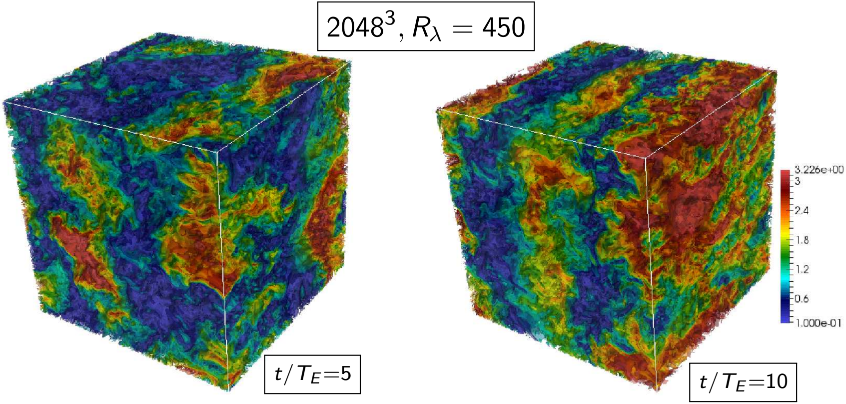
Open Questions

- Are scaling exponents ξ_j^n universal ?
- Return-to-Isotropy ? How fast/slow ... ?

$$\xi_0^n \leq \xi_2^n \leq \xi_4^n \leq \ldots$$

Random Kolmogorov Flow *(Biferale, Toschi PRL'01)*

- RKF: anisotropic, stationary & homogeneous on average



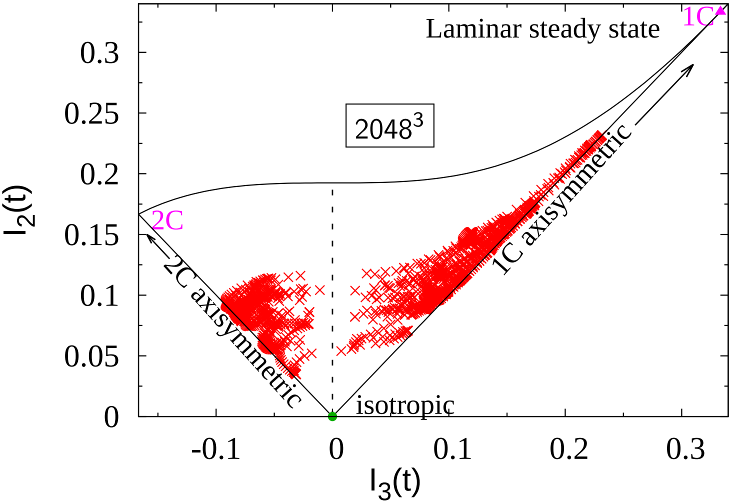
- Forcing: 2^{nd} order Markovian (*Sawford, Phys. Fluids '91*)

$$\boxed{\mathbf{f}_y(\mathbf{k}_{1,2}) = \delta_{i,2} A(t) e^{i\theta_{1,2}(t)}}\,, \mathbf{k}_1 = \pm(1,0,0), \mathbf{k}_2 = \pm(2,0,0)$$

- $A(t)$, $\theta_{1,2}(t)$: RANDOM in time

RKF: test-bed for anisotropic turbulence

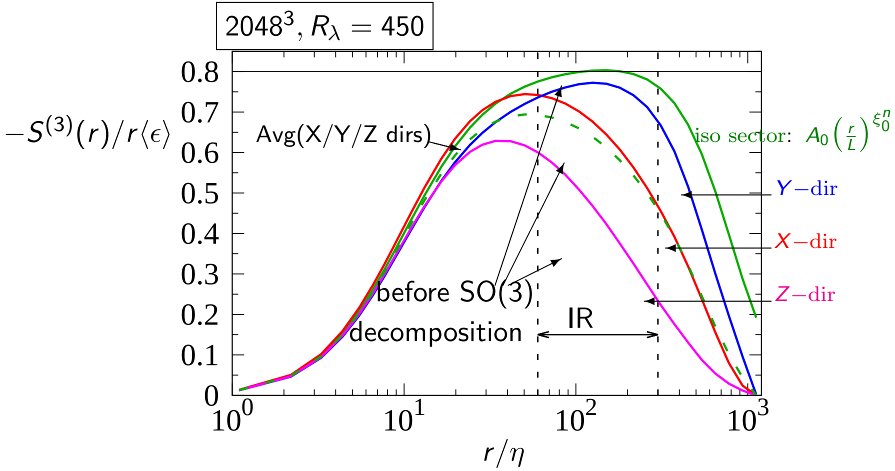
- Reynolds Stress Tensor $b_{ij} \equiv \langle u_i u_j \rangle / \langle u_k u_k \rangle - \delta_{ij}/3$
- RST invariants: $I_2 \equiv (b_{ij} b_{ji}/6)^{(1/2)}$ $I_3 \equiv (b_{ij} b_{jk} b_{ki}/6)^{(1/3)}$



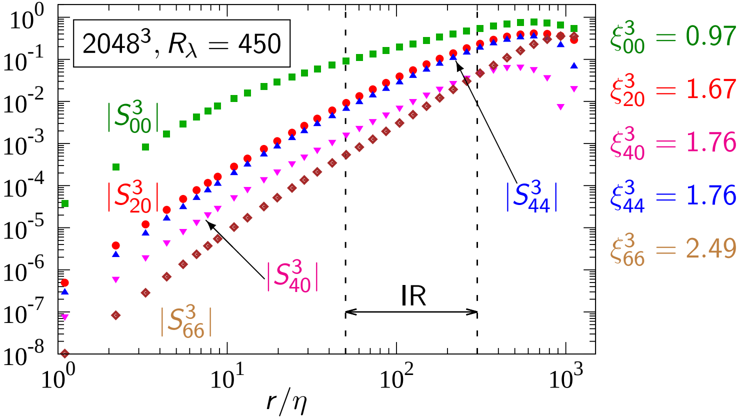
Isotropic sector vs undecomposed structure function

$$S^{(n)}(\mathbf{r}) = A_0\left(\frac{r}{L}\right)^{\xi_0^n} + A_1\left(\frac{r}{L}\right)^{\xi_1^n} + \dots$$

Isotropy in Inertial Range ($\eta \ll r \ll L$) $S^{(3)}(r) = -\frac{4}{5}\langle\epsilon\rangle r$



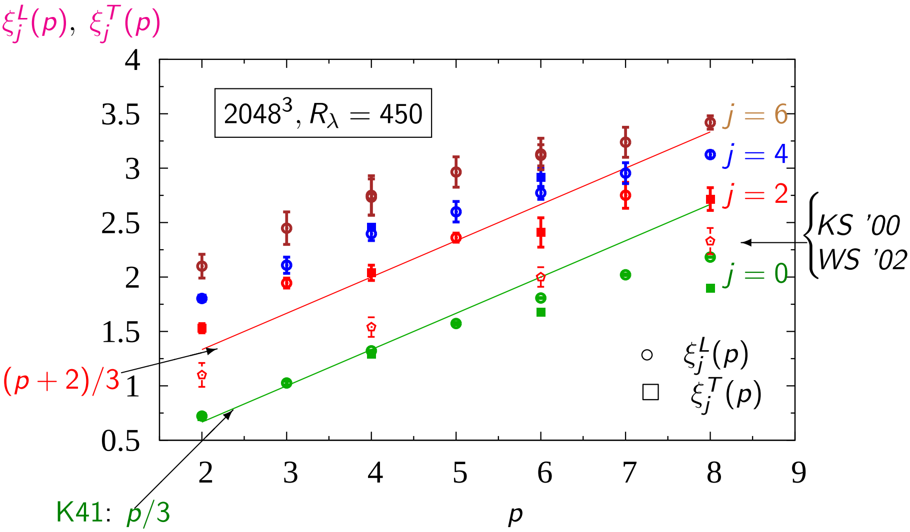
Ansatz for projection coeffs: $\boxed{S^n_{jm}(r) \sim A^n_{jm} r^{\xi^n_j}}$



- $\boxed{\xi^n_0 < \xi^n_{20} < \xi^n_{40} < \xi^n_{66}} \Rightarrow \boxed{(r/L)^{\xi^n_0} > (r/L)^{\xi^n_j}}$ for $r/L \ll 1$
- Isotropic sector dominant. Anisotropic part: sub-leading

$$S^{(p,L)}_{j,m}(r) = \Lambda^{(p,L)}_{j,m}(\tfrac{r}{L})^{\xi^L_j(p)};\ S^{(p,T)}_{j,m}(r) = \Lambda^{(p,T)}_{j,m}(\tfrac{r}{L})^{\xi^T_j(p)}$$

Longitudinal **Transverse**



Conclusions

- Scaling exponents universal in isotropic and anisotropic sectors
- Rate-of-return to isotropy FASTER than previously believed at least at lower orders
- Longitudinal and transverse exponents are (almost) equal in different anisotropic sectors
- Isotropic sector: higher order long/trans exponents differ at $R_\lambda \sim 450$
- Anisotropic effect on Lagrangian statistics: track millions of particles (light/heavy) . . .