Optimal Control tools to minimize dispersion in turbulent flows

APS DFD 2022
75th Annual Meeting
November 20 – 22, 2022
Indianapolis
APS

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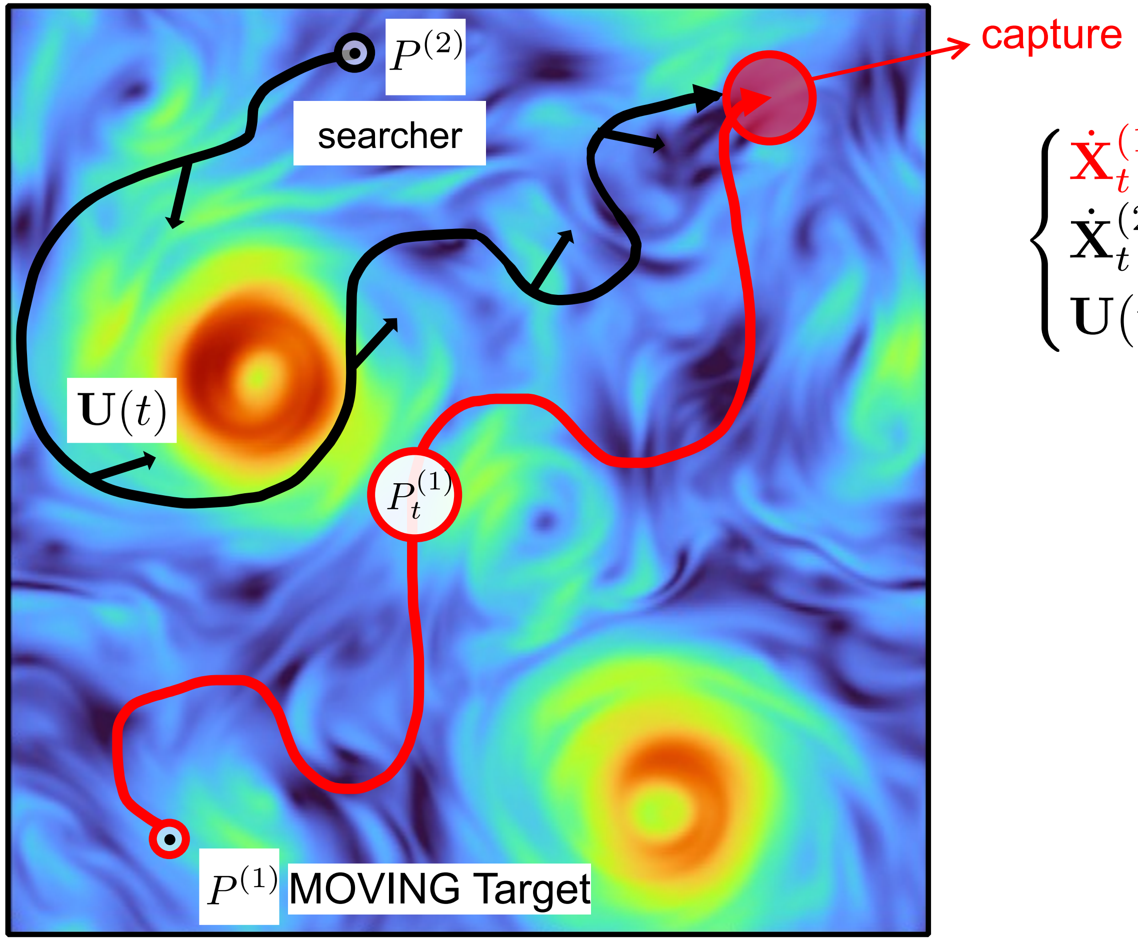
C. Calascibetta, L. Biferale, F. Borra, A. Celani, M. Cencini. **Optimal Control tools to minimize dispersion in turbulent flows**, (*in preparation* 2022).

2 AGENTS



Goal: **minimize** the **separation/capture** in a finite time horizon

Problem setup

$$\begin{cases} \dot{\mathbf{X}}_t^{(1)} = \boldsymbol{v}(\mathbf{X}_t^{(1)}, t) \\ \dot{\mathbf{X}}_t^{(2)} = \boldsymbol{v}(\mathbf{X}_t^{(2)}, t) + \mathbf{U}(t) \\ \mathbf{U}(t) = V_s \hat{\boldsymbol{n}}(t) \end{cases} \longrightarrow \mathbf{U}(t) = ?$$

$$\hat{\boldsymbol{n}}(t) = (\cos[\theta_t], \sin[\theta_t])$$

Tools:
(1) Heuristic policies
(2) Optimal Control (OC) theory
(3) Reinforcement Learning (RL)

$$\begin{cases} \dot{\mathbf{X}}_t^{(1)} = \boldsymbol{v}(\mathbf{X}_t^{(1)}, t) \\ \dot{\mathbf{X}}_t^{(2)} = \boldsymbol{v}(\mathbf{X}_t^{(2)}, t) + \mathbf{U}(t) \\ \mathbf{U}(t) = V_s \hat{\boldsymbol{n}}(t) \end{cases}$$

$$\mathbf{U}(t) = ?$$

$$\begin{cases} \boldsymbol{R}_t = \mathbf{X}_t^{(2)} - \mathbf{X}_t^{(1)} \\ L = \text{ characteristic scale of the flow} \end{cases}$$

(1) (semi) Heuristic policies

Trivial Policy: constantly chooses the direction that points towards the moving target, $\hat{\boldsymbol{n}}(t) = -\hat{\boldsymbol{R}}_t$

Surfing Policy*:
- constant gradients for a time τ_s (free parameter);
- maximization of the searcher displacement along the $\boldsymbol{R}_t$ direction;
- good for slowly varying $\boldsymbol{R}_t$ (i.e. at large scales)

$$\hat{\boldsymbol{n}}(t) = -\frac{\left[e^{(\tau_s - t)\nabla \boldsymbol{v}_{t_0}}\right]^{\mathrm{T}} \cdot \hat{\boldsymbol{R}}_{t_0}}{\left\| \left[e^{(\tau_s - t)\nabla \boldsymbol{v}_{t_0}}\right]^{\mathrm{T}} \cdot \hat{\boldsymbol{R}}_{t_0} \right\|}$$

Perturbative Policy:
- 0^{th} order OC with constant gradients for a time τ_p (free parameter) ;
- valid at small scales

$$\hat{\boldsymbol{n}}(t) = -\frac{\left[e^{(\tau_p - t)\nabla \boldsymbol{v}_{t_0}}\right]^{\mathrm{T}} \cdot e^{(\nabla \boldsymbol{v})_{t_0} \tau_p} \cdot \hat{\boldsymbol{R}}_{t_0}}{\left\| \left[e^{(\tau_p - t)\nabla \boldsymbol{v}_{t_0}}\right]^{\mathrm{T}} \cdot e^{(\nabla \boldsymbol{v})_{t_0} \tau_p} \cdot \hat{\boldsymbol{R}}_{t_0} \right\|}$$

*Monthiller, Rémi, et al. **Surfing on Turbulence: A Strategy for Planktonic Navigation**. *Phys. Rev. Lett.* ***129**, 064502* (2022)

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$$\begin{cases} \boldsymbol{R}_t = \mathbf{X}_t^{(2)} - \mathbf{X}_t^{(1)} \\ L = \text{ characteristic scale of the flow} \end{cases}$$

(1) (semi) Heuristic policies

Surfing policy* - derivation

- Approximate linearly the underlying flow, $\boldsymbol{v}(\mathbf{X}_t^{(2)}, t)$ for $t_0 < t < \tau_s$; *(Assuming constant gradients for a time τ_s)*

$$\dot{\mathbf{X}}_t^{(2)} = \boldsymbol{v}_{t_0} + (\nabla \boldsymbol{v})_{t_0} \cdot \left(\mathbf{X}_t^{(2)} - \mathbf{X}_{t_0}^{(2)}\right) + \left(\frac{\partial \boldsymbol{v}}{\partial t}\right)_{t_0} (t - t_0) + \mathbf{U}(t) \,,$$

- Find $\mathbf{U}(t)$ such that $-(\mathbf{X}_{\tau_s}^{(2)} - \mathbf{X}_{t_0}^{(2)}) \cdot \hat{\boldsymbol{R}}_{t_0}$ is maximum;

$$\hat{\boldsymbol{n}}(t) = -\frac{\left[e^{(\tau_s - t)\nabla \boldsymbol{v}_{t_0}}\right]^{\mathrm{T}} \cdot \hat{\boldsymbol{R}}_{t_0}}{\left\| \left[e^{(\tau_s - t)\nabla \boldsymbol{v}_{t_0}}\right]^{\mathrm{T}} \cdot \hat{\boldsymbol{R}}_{t_0} \right\|}$$

(Assuming constant the direction $\hat{\boldsymbol{R}}_t$ for a time τ_s)

↓

$\|\boldsymbol{R}_t\| \gg L$

- Numerically optimize the free parameter τ_s.

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$$\mathbf{U}(t) = ?$$

$$\begin{cases} \boldsymbol{R}_t = \mathbf{X}_t^{(2)} - \mathbf{X}_t^{(1)} \\ L = \text{ characteristic scale of the flow} \end{cases}$$

(1) (semi) Heuristic policies

Perturbative policy - derivation

- Consider linearity between the two agents, i.e., $\boldsymbol{v}(\mathbf{X}_t^{(2)}, t) \simeq \boldsymbol{v}(\mathbf{X}_t^{(1)}, t) + \nabla \boldsymbol{v}_t \boldsymbol{R}_t \,, \longrightarrow \|\boldsymbol{R}_t\| \ll L$

$$\dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t) \rightarrow \quad \boldsymbol{R}_{\tau_p} = \underbrace{e^{[(\nabla \boldsymbol{v})_{t_0} \tau_p]} \boldsymbol{R}_{t_0}} + V_s \int_{t_0}^{\tau_p} dt \, e^{[(\nabla \boldsymbol{v})_{t_0} (\tau_p - t)]} \hat{\boldsymbol{n}}(t) \, ;$$

(Assuming constant gradients for a time τ_p)

- Find $\mathbf{U}(t)$ such that $\boldsymbol{R}_{\tau_p} \cdot \boldsymbol{R}_{\tau_p}^{free}$ is minimum;

$$\hat{\boldsymbol{n}}(t) = - \frac{\left[e^{(\tau_p - t) \nabla \boldsymbol{v}_{t_0}}\right]^{\mathrm{T}} \cdot e^{(\nabla \boldsymbol{v})_{t_0} \tau_p} \cdot \hat{\boldsymbol{R}}_{t_0}}{\left\| \left[e^{(\tau_p - t) \nabla \boldsymbol{v}_{t_0}}\right]^{\mathrm{T}} \cdot e^{(\nabla \boldsymbol{v})_{t_0} \tau_p} \cdot \hat{\boldsymbol{R}}_{t_0} \right\|}$$

- Numerically optimize the free parameter τ_p.

(2) **Optimal Control theory** – Pontryagin minimum principle

state variables *control variables*

Minimize $J = C_F(\mathbf{X}(t_f)) + \int_{t_0}^{t_f} dt\,[L(\mathbf{X}(t), \mathbf{U}(t), t)]$

performance index *Lagrangian function*

Imposing $\dot{\mathbf{X}}_t = \boldsymbol{f}(\mathbf{X}(t), \mathbf{U}(t), t)$

and other possible constraints,

e.g.: $\begin{cases} \mathbf{X}(t_f) = \mathbf{X}_\star\,, \quad \mathbf{X}(t_0) \le \mathbf{X}_\star\,, \\ \|\mathbf{U}(t)\|^2 = 1\,, \quad \|\mathbf{U}(t)\|^2 \le 1\,, \text{ exc.} \end{cases}$

- **Model based** and analytical tool
- Perfect knowledge required

$\|\boldsymbol{R}_{t_0}\| \sim \frac{V_s}{\lambda_{lyapunov}}$ border of controllability

In our case:

$\|\boldsymbol{R}^\star\| = \|\boldsymbol{R}_{t_0}\|/100$

capture's distance

Minimize $J = \|\boldsymbol{R}_{t_f}\|^2 + c\int_{t_0}^{t_f} dt\,\theta(\|\boldsymbol{R}_t\|^2 - \|\boldsymbol{R}^\star\|^2)$

Imposing $(*)$ and the control constraint $\|\hat{\boldsymbol{n}}(t)\|^2 = 1$

$$(*)\begin{cases} \dot{\mathbf{X}}_t^{(1)} = \boldsymbol{v}(\mathbf{X}_t^{(1)}, t) \\ \dot{\mathbf{X}}_t^{(2)} = \boldsymbol{v}(\mathbf{X}_t^{(2)}, t) + \mathbf{U}(t) \\ \mathbf{U}(t) = V_s \hat{\boldsymbol{n}}(t) \end{cases}$$

Minimize trajectories' separation

Minimize time of arrival at the desired distance

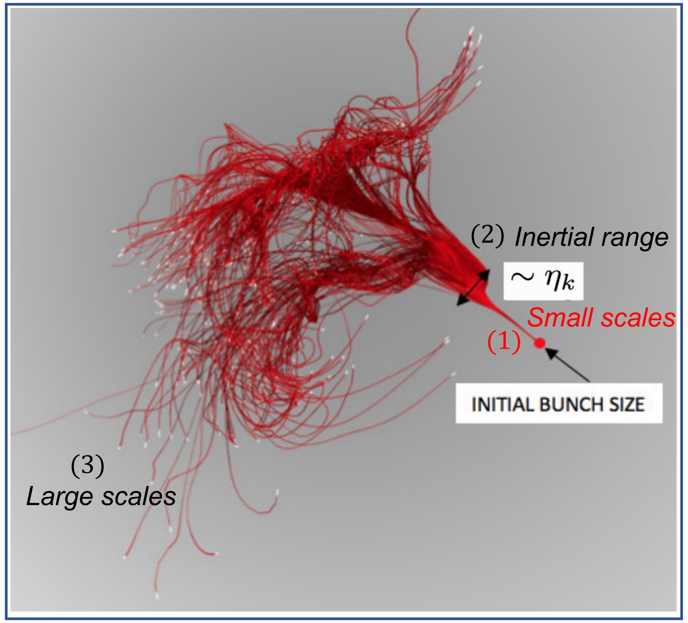
Optimal Control vs heuristic policies at **small scales**

Velocity field*

3D Direct Numerical Simulations $N = 1024^3$

NSEs: $$\begin{cases} \partial_t \boldsymbol{v} = -\nabla \mathrm{p} - (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \nu\nabla^2\boldsymbol{v} + \mathbf{F}\,, \\ \nabla\cdot\boldsymbol{v} = 0 \end{cases}$$

$\downarrow$ homogeneous and isotropic forcing



DNS parameters

$$\eta_k = 0.0043$$

$$\tau_\eta = 0.023$$

$$Re \simeq 17000$$

(1) LINEAR REGIME $\|\boldsymbol{R}_{t_0}\| < \eta_k$

$$\begin{cases} \dot{\mathbf{X}}_t^{(1)} = \boldsymbol{v}(\mathbf{X}_t^{(1)}, t) \\ \dot{\mathbf{X}}_t^{(2)} = \boldsymbol{v}(\mathbf{X}_t^{(2)}, t) + \mathbf{U}(t) \\ \mathbf{U}(t) = V_s\hat{\boldsymbol{n}}(t) \end{cases}$$

$$\boldsymbol{v}(\mathbf{X}_t^{(2)}, t) \simeq \boldsymbol{v}(\mathbf{X}_t^{(1)}, t) + \nabla\boldsymbol{v}_t\boldsymbol{R}_t$$

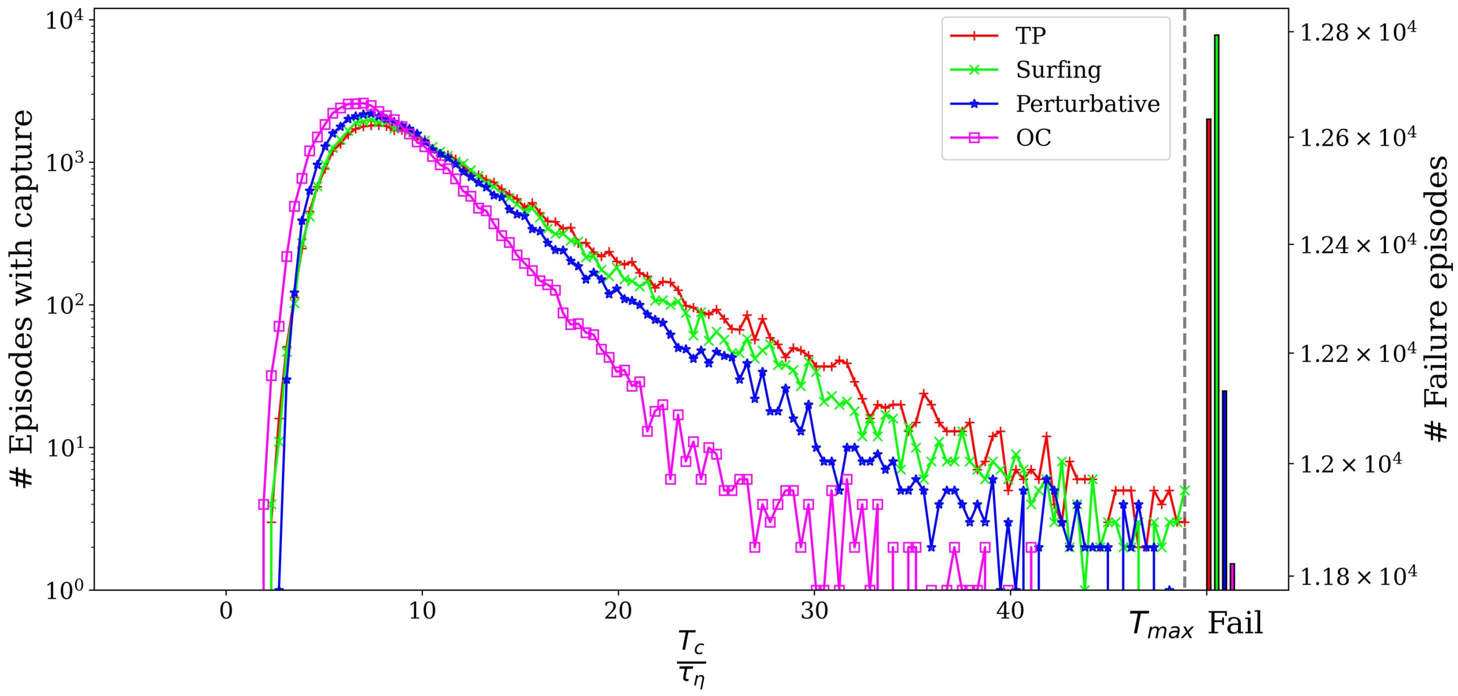
$$\dot{\boldsymbol{R}}_t = \nabla\boldsymbol{v}_t\boldsymbol{R}_t + \mathbf{U}(t)$$

*Buzzicotti et al. **Lagrangian statistics for Navier–Stokes turbulence under Fourier-mode reduction: fractal and homogeneous decimations.** *New J. Phys., 18 (11)* (2016), *p. 113047*

Optimal Control vs heuristic policies in linear regime

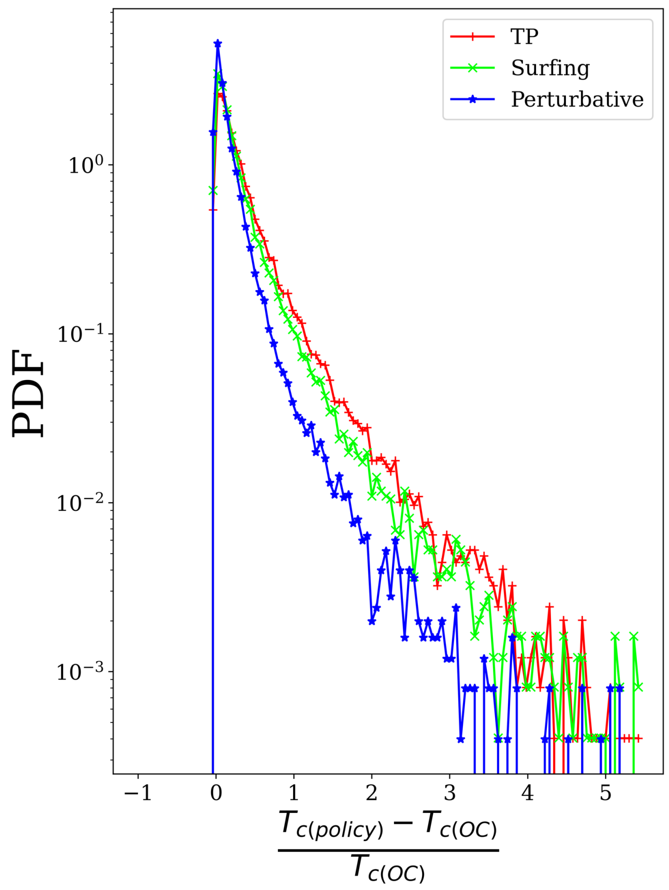
$\dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t)$

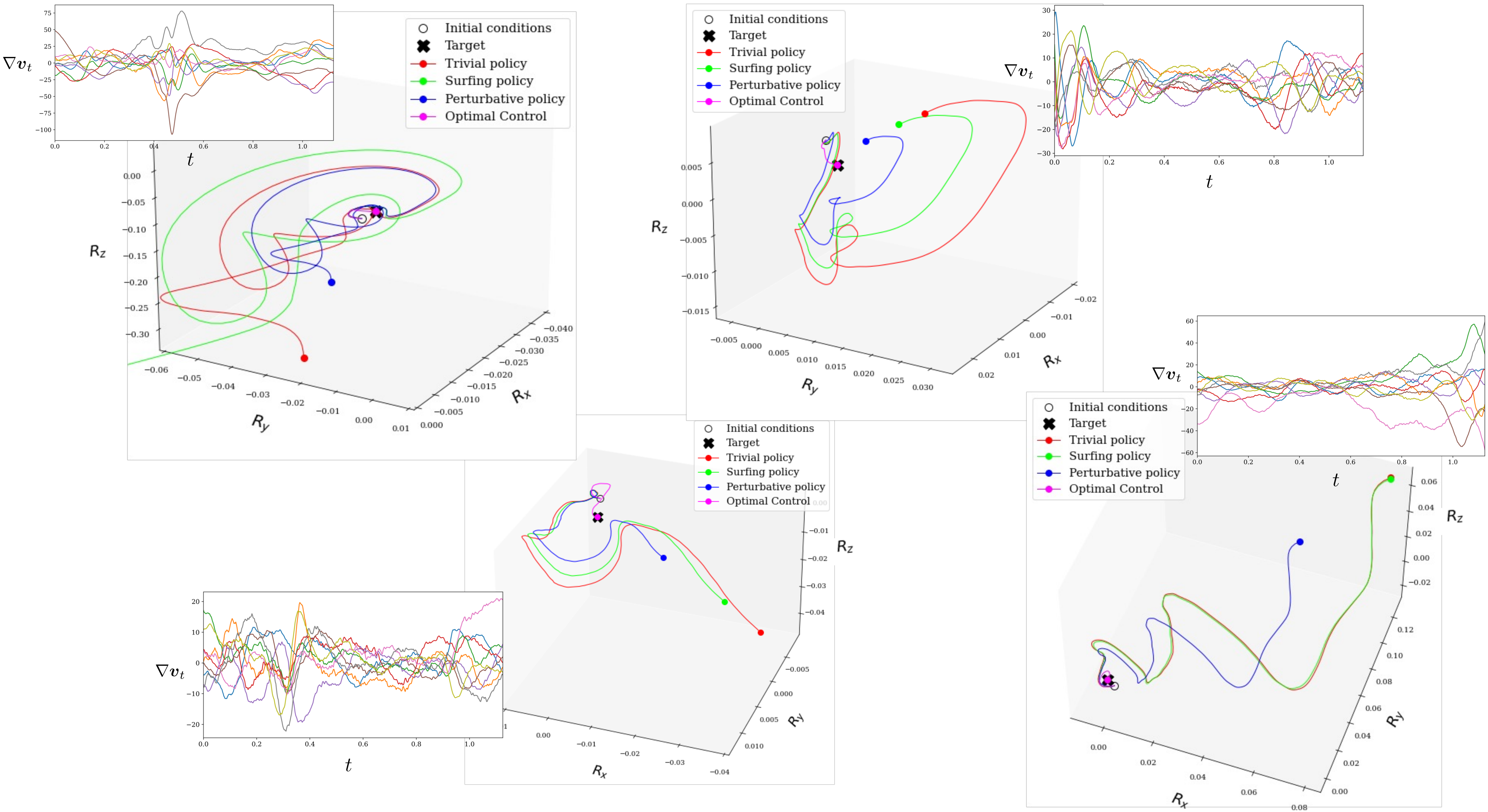
$T_c =$ **Capture** time: (time of arrival at the desired distance)



PRELIMINARY UNPUBLISHED

PDF of normalized capture time





Initial conditions
Target
Trivial policy
Surfing policy
Perturbative policy
Optimal Control
∇v_t
t
R_x
R_y
R_z
Initial conditions
Target
Trivial policy
Surfing policy
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t
R_x
R_y
R_z

pros & cons

Optimal Control

+ It is optimized
- It is model based and needs perfect information from the environment
- It is sensitive to variation of the initial condition
- It is difficult to consider a decision time in the control variable

Heuristic policies

- They are not optimized
+ They need only partial information
+ They are stable wrt variation of the initial condition
+ They work also with a discrete decision time

Next step: Reinforcement Learning

+ It is optimized
+ It is model free
+ It needs partial information
- It is data-hungry

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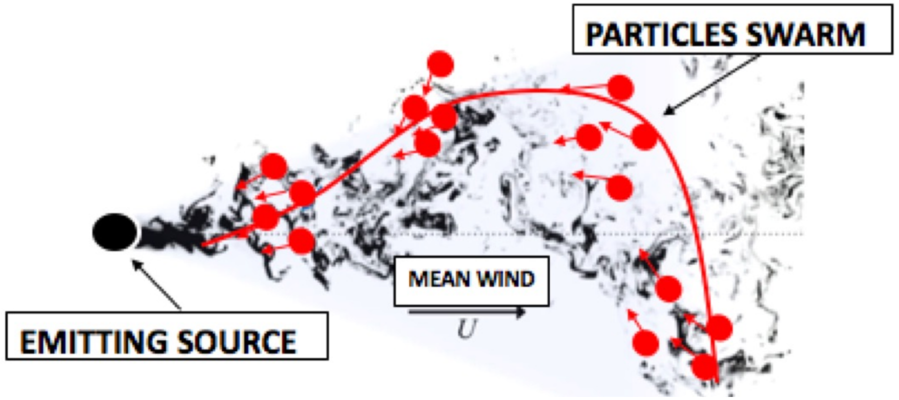
Conclusions



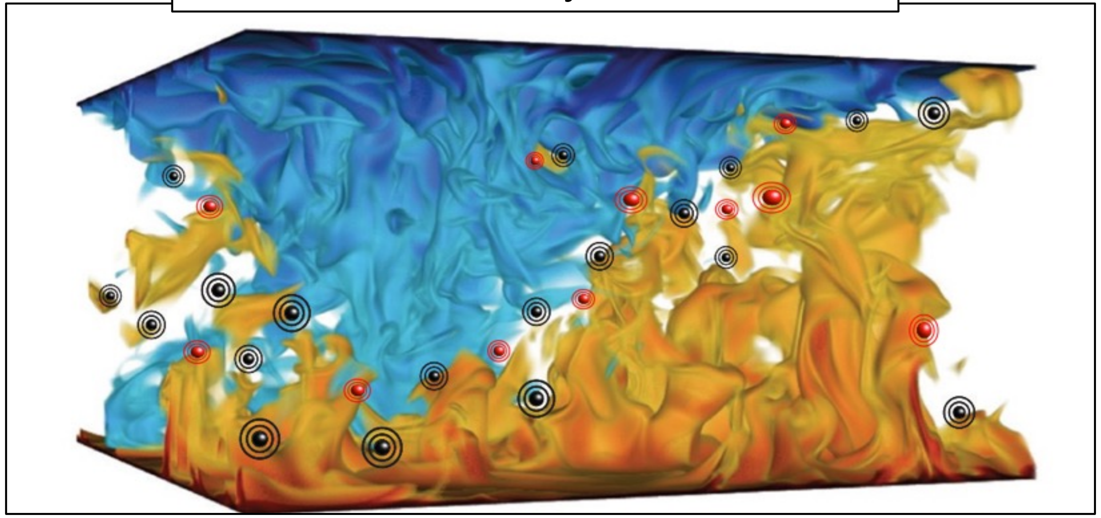
Open questions:

1. How to control a multi-agent system to minimize turbulent dispersion in realistic geophysical flows (beyond the linear regime) ?
2. Can we identify the key degrees-of-freedom to control the agents' trajectories (key flow structures)?
3. Are the agents able to collaborate with each-other during the navigation?

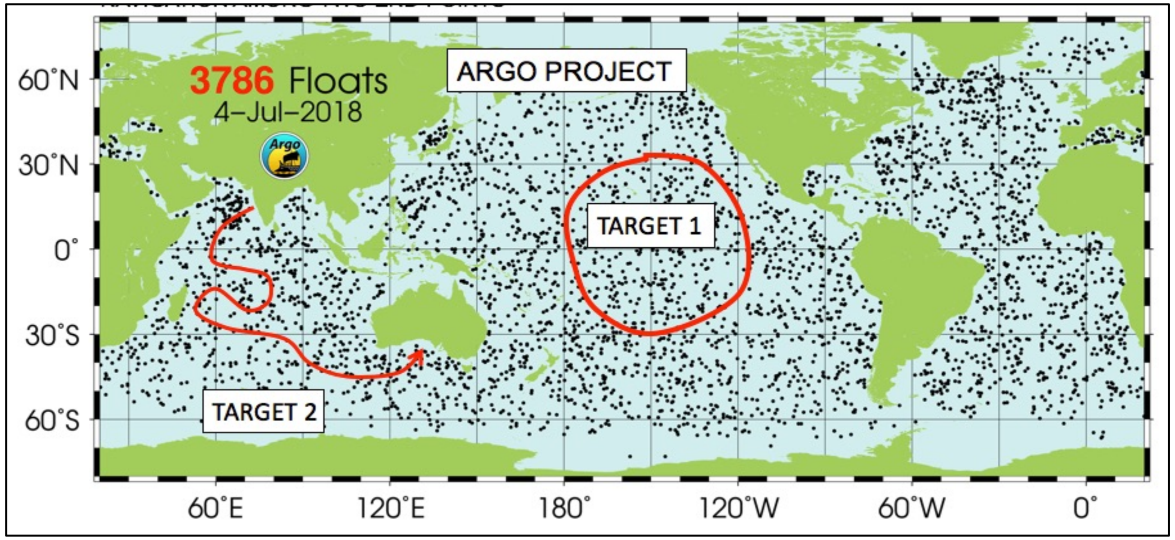
Tools:

- We can use RL to control autonomous swimmers in a realistic way (i.e., with a limited knowledge of the underlying flow - only local or instantaneous features);
- We can use OC as a benchmark to test the RL solutions.

Smart two-way feedback



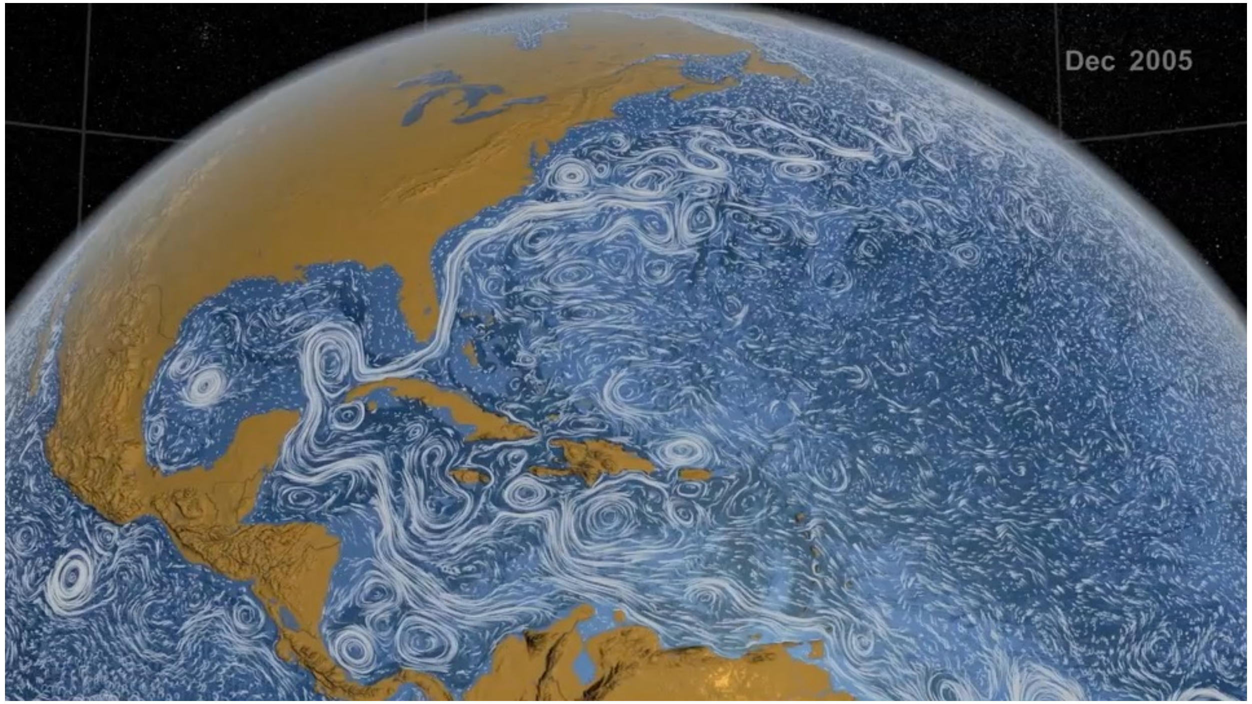
http://stilton.tnw.utwente.nl/people/stevensr/afid.html



https://argo.ucsd.edu/

Backup slides

Turbulent flows



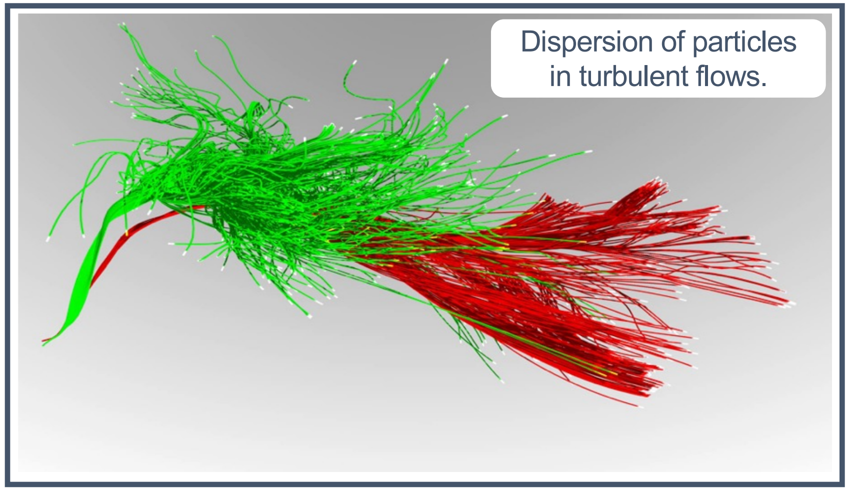
https://svs.gsfc.nasa.gov/3827

*How to exploit **coherent structures**?*

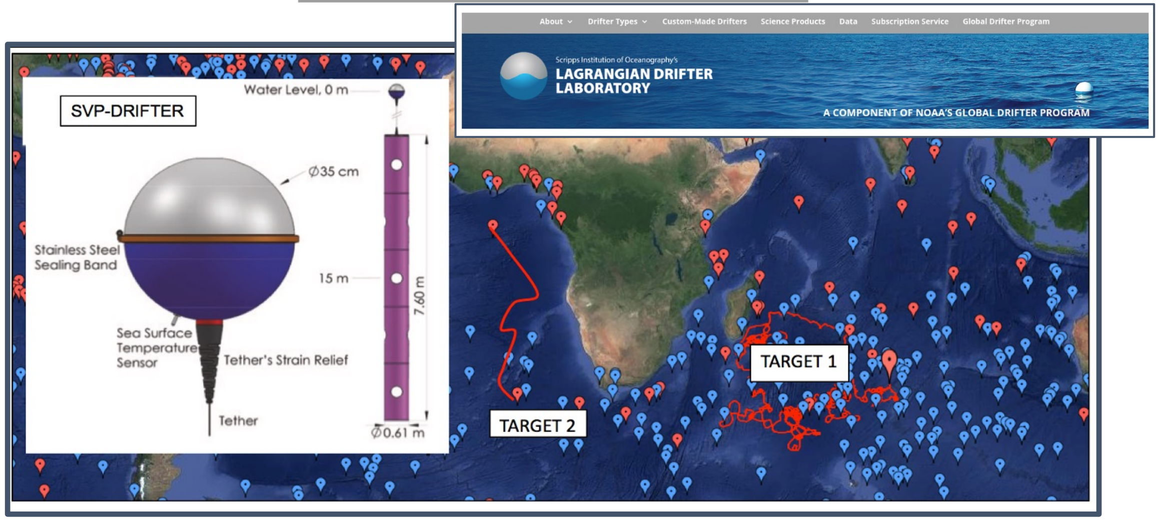
*How to avoid (or exploit) **intense fluctuations** when navigating inside the flow?*

*Which is the best **limited-control** to navigate in such complex flows?*

Theoretical interests:



Engineering applications:

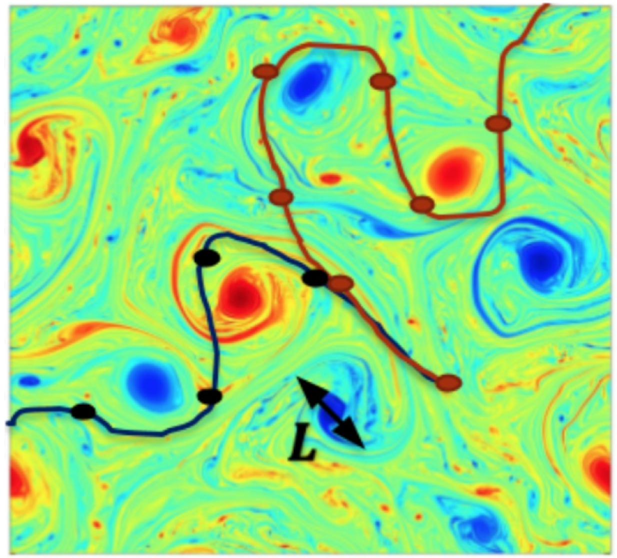


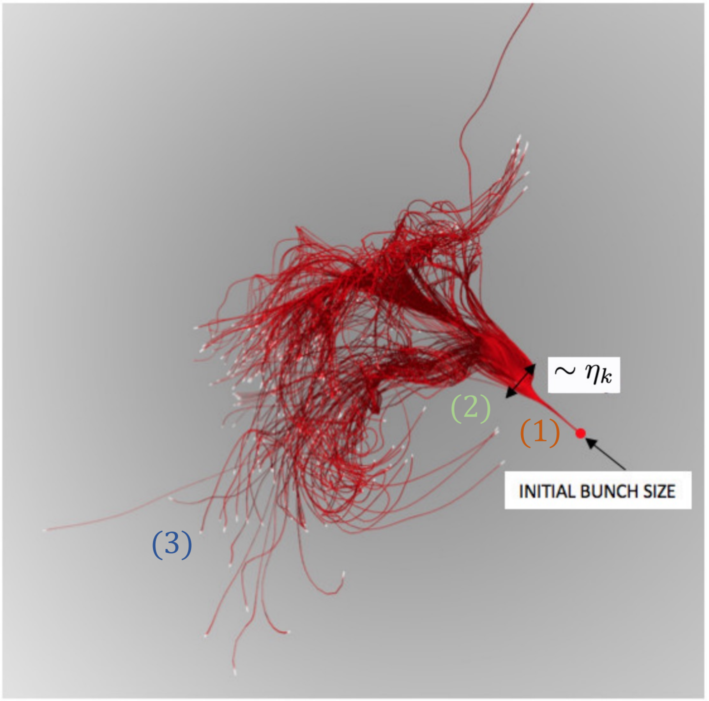
Particles dispersion in complex flows

Lagrangian approach

$$\dot{\mathbf{X}} = \boldsymbol{v}(\mathbf{X}_t, t)$$

Eq. of motion of a tracer





Trajectories separation:

$$\delta R_t = \|\mathbf{X}_t^2 - \mathbf{X}_t^1\|$$

(1) Dispersion at small scales

$$\delta R_t \sim \delta R_0 \, e^{\lambda t}$$

Lagrangian Chaos

$$\lambda = \lim_{t\to\infty} \lim_{\delta R_0 \to 0} \frac{1}{t} ln \frac{\delta R_t}{\delta R_0}$$

Lyapunov exponent

(2) Dispersion at intermediate scales

(Inertial range)

$$\langle(\delta R_t)^2\rangle \sim t^3$$

non-differentiable velocity field

If $Re \to \infty$ Fully Developed Turbulence
Richardson's Dispersion

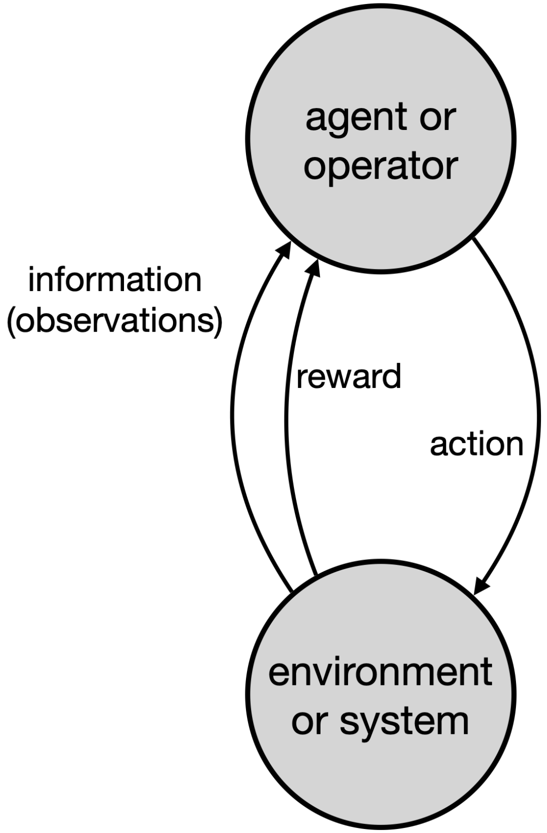
(3) Dispersion at large scales

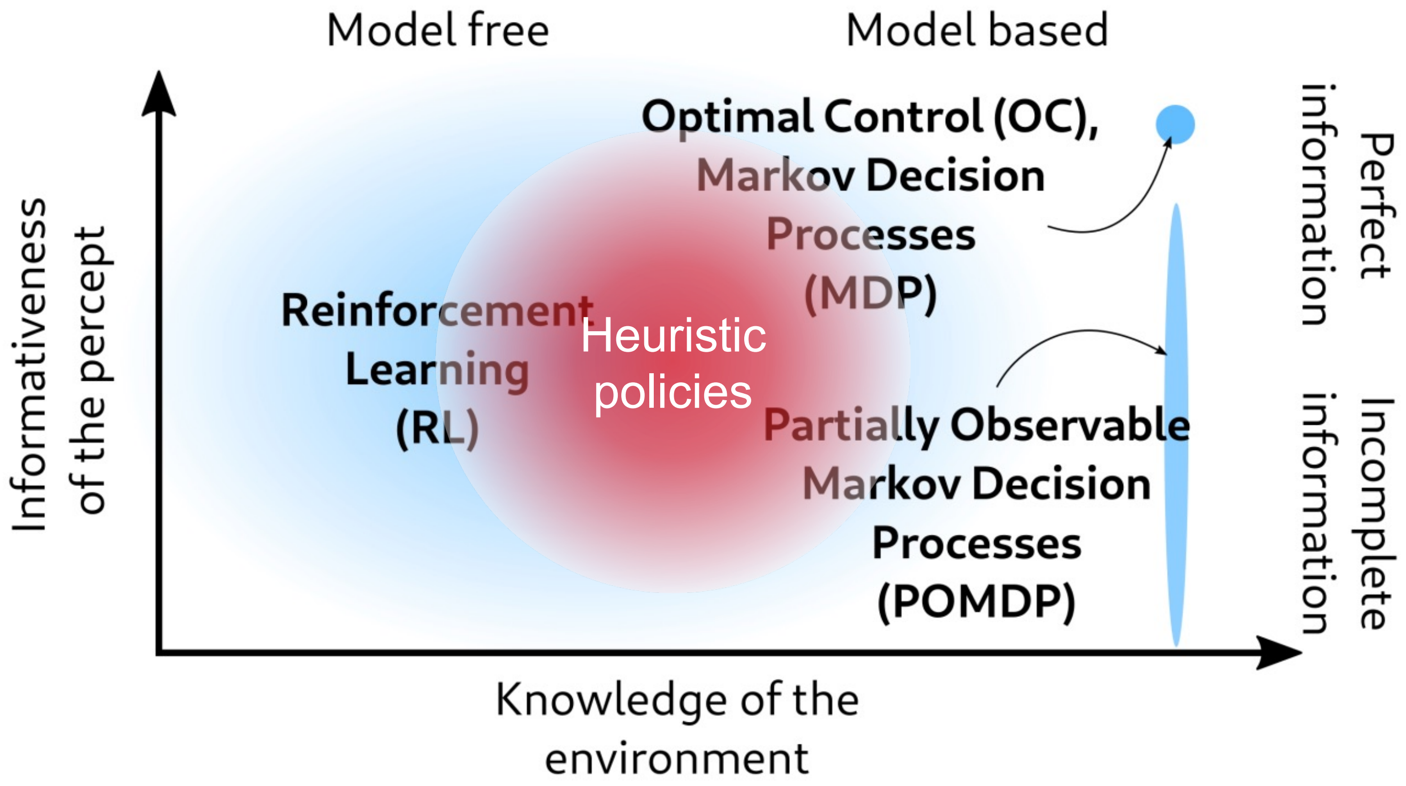
Advection
+
molecular diffusion

$$\langle(\delta R_t)^2\rangle \sim D^E t$$

effective diffusion

Control Theory





Tools:
Optimal Control (**OC**) theory
Reinforcement Learning (**RL**)
Heuristic policies

Optimal Control vs heuristic policies at **small scales**

$$\dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t)$$

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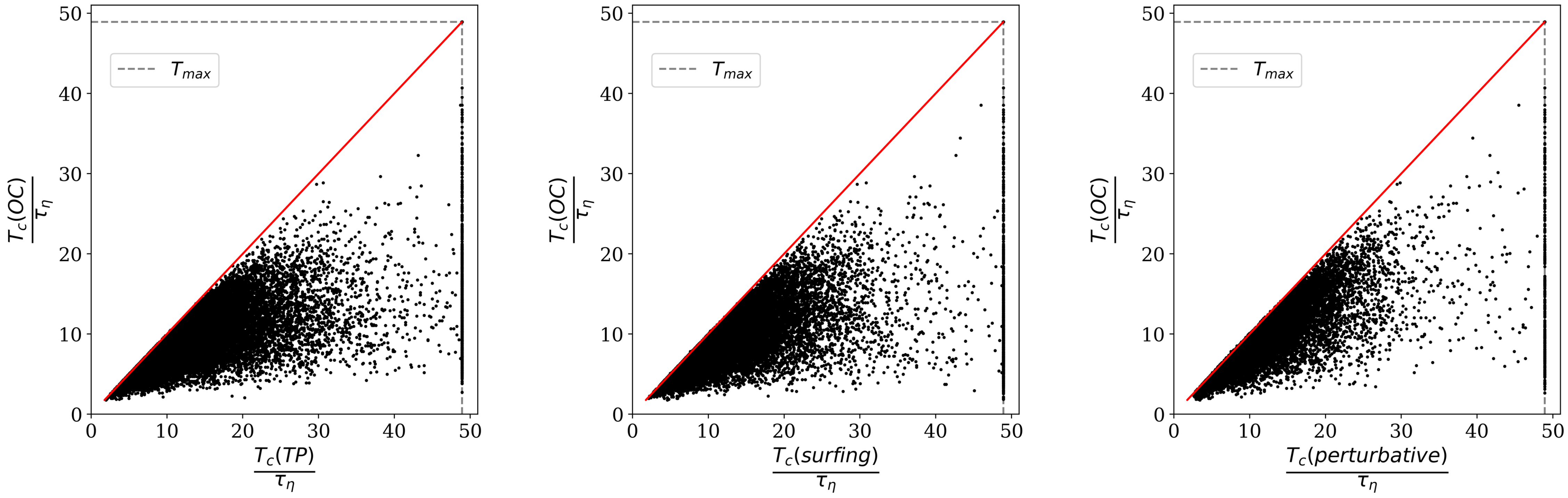
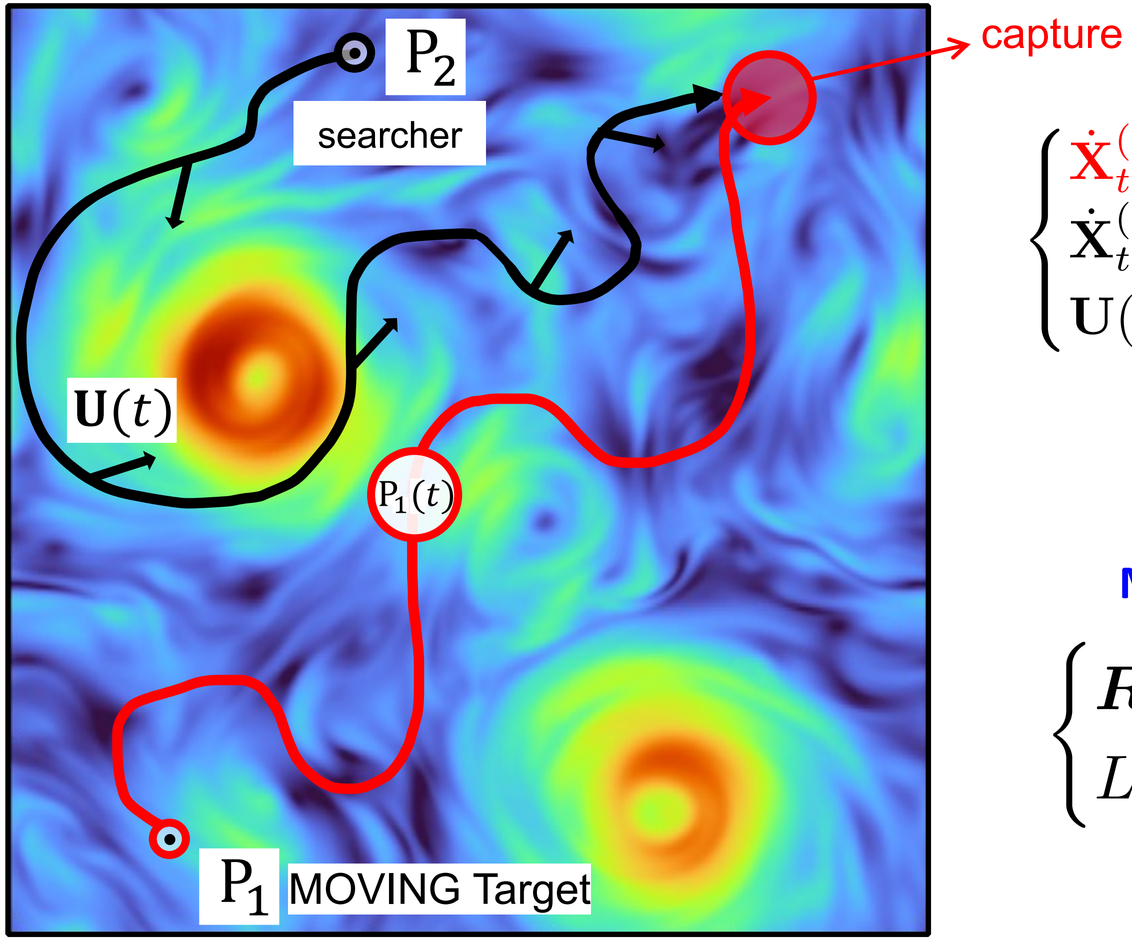


C. Calascibetta, L. Biferale, F. Borra, A. Celani, M. Cencini, **Optimal Control tools to minimize dispersion in chaotic flows**, (*in preparation*).

2 AGENTS



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Problem setup

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$$\hat{\boldsymbol{n}}(t) = (\cos[\theta_t], \sin[\theta_t])$$

MAIN POINT: Different approaches for different range of scales:

$$\begin{cases} \boldsymbol{R}_t = \mathbf{X}_t^{(2)} - \mathbf{X}_t^{(1)} \\ L = \text{Characteristic scale of the flow} \end{cases}$$

$\|\boldsymbol{R}_t\| \ll L$ Small scales

$\|\boldsymbol{R}_t\| \gg L$ Large scales

Tools:
(1) **Heuristic policies**
(2) Optimal Control (**OC**) theory
(3) Reinforcement Learning (**RL**)

$$\begin{cases} \dot{\mathbf{X}}_t^{(1)} = \boldsymbol{v}(\mathbf{X}_t^{(1)}, t) \\ \dot{\mathbf{X}}_t^{(2)} = \boldsymbol{v}(\mathbf{X}_t^{(2)}, t) + \mathbf{U}(t) \\ \mathbf{U}(t) = V_s \hat{\boldsymbol{n}}(t) \end{cases}$$

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Perturbative Policy: - 0^{th} order OC with constant gradients for a time τ_p (free parameter) ;
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Surfing policy* - derivation

$$\dot{\mathbf{X}}_t^{(2)} = \boldsymbol{v}_{t_0} + (\nabla\boldsymbol{v})_{t_0}\cdot\left(\mathbf{X}_t^{(2)} - \mathbf{X}_{t_0}^{(2)}\right) + \left(\frac{\partial\boldsymbol{v}}{\partial t}\right)_{t_0}(t-t_0) + V_s\hat{\boldsymbol{n}}(t)$$

$$\mathbf{X}_{\tau_s}^{(2)} = \mathbf{X}_{t_0}^{(2)} + \left[e^{\tau_s(\nabla\boldsymbol{v})_{t_0}} - \mathbb{I}\right]\cdot(\nabla\boldsymbol{v}_{t_0})^{-1}\cdot\left[\boldsymbol{v}_{t_0} + (\nabla\boldsymbol{v}_{t_0})^{-1}\cdot\left(\frac{\partial\boldsymbol{v}}{\partial t}\right)_{t_0}\right] - \\ -\tau_s(\nabla\boldsymbol{v}_{t_0})^{-1}\cdot\left(\frac{\partial\boldsymbol{v}}{\partial t}\right)_{t_0} + V_s\int_{t_0}^{\tau_s} dt\, e^{(\tau_s-t)\nabla\boldsymbol{v}_{t_0}}\cdot\hat{\boldsymbol{n}}(t)$$

find $\hat{\boldsymbol{n}}(t)$ such as $-(\mathbf{X}_{\tau_s}^{(2)} - \mathbf{X}_{t_0}^{(2)})\cdot\hat{\boldsymbol{R}}_{t_0}$ is maximum,

means find $\hat{\boldsymbol{n}}(t)$ such that $-\int_{t_0}^{\tau_s} dt\left[e^{(\tau_s-t)\nabla\boldsymbol{v}_{t_0}}\cdot\hat{\boldsymbol{n}}(t)\cdot\hat{\boldsymbol{R}}_{t_0}\right]$ is maximum (i.e. by maximizing the integrand).

$$\underbrace{\left[e^{(\tau_s-t)\nabla\boldsymbol{v}_{t_0}}\cdot\hat{\boldsymbol{n}}(t)\cdot\hat{\boldsymbol{R}}_{t_0}\right]}_{-[e^{(\tau_s-t)\nabla\boldsymbol{v}_{t_0}}]^{\mathrm{T}}\cdot\hat{\boldsymbol{R}}_{t_0}\cdot\hat{\boldsymbol{n}}(t)}$$

$\hat{\boldsymbol{n}}(t)$ must be collinear to $-[e^{(\tau_s-t)\nabla\boldsymbol{v}_{t_0}}]^{\mathrm{T}}\cdot\hat{\boldsymbol{R}}_{t_0}$ →

$$\hat{\boldsymbol{n}}(t) = -\frac{\left[e^{(\tau_s-t)\nabla\boldsymbol{v}_{t_0}}\right]^{\mathrm{T}}\cdot\hat{\boldsymbol{R}}_{t_0}}{\left\|\left[e^{(\tau_s-t)\nabla\boldsymbol{v}_{t_0}}\right]^{\mathrm{T}}\cdot\hat{\boldsymbol{R}}_{t_0}\right\|}$$

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(2) **Optimal Control theory** – Pontryagin minimum principle

state variables *control variables*

Minimize $J = C_F(\mathbf{X}(t_f)) + \int_{t_0}^{t_f} dt\,[L(\mathbf{X}(t), \mathbf{U}(t), t)]$

performance index *Lagrangian function*

Imposing $\dot{\mathbf{X}}_t = \boldsymbol{f}(\mathbf{X}(t), \mathbf{U}(t), t)$

and other possible constraints,

e.g.: $\begin{cases} \mathbf{X}(t_f) = \mathbf{X}_\star\,, \quad \mathbf{X}(t_0) \leq \mathbf{X}_\star\,, \\ \|\mathbf{U}(t)\|^2 = 1\,, \quad \|\mathbf{U}(t)\|^2 \leq 1\,, \; exc. \end{cases}$

Observe that this is a **constrained** minimization

Lagrangian multipliers

$$\tilde{J} = C_F(\mathbf{X}(t_f)) + \int_{t_0}^{t_f} dt\,[L(\mathbf{X}(t), \mathbf{U}(t), t) + \boldsymbol{\lambda}^{\mathrm{T}} \cdot (\boldsymbol{f} - \dot{\mathbf{X}}) + \ldots]$$

$$+ \mu(t)(1 - \|\mathbf{U}(t)\|^2)$$

We impose minimum in $\mathbf{X}(\cdot)$, $\mathbf{U}(\cdot)$, $\boldsymbol{\lambda}(\cdot)$, i.e., $d\tilde{J} \leq 0$:

$$\begin{cases} \dfrac{\delta \tilde{J}}{\delta \mathbf{X}(t)} = 0 \implies \dot{\boldsymbol{\lambda}} = -\boldsymbol{\partial}_{\mathbf{X}} L - (\boldsymbol{\partial}_{\mathbf{X}} f)^{\mathrm{T}} \boldsymbol{\lambda}(t)\,, \\ \dfrac{\delta \tilde{J}}{\delta \mathbf{X}(t_f)} = 0 \implies \boldsymbol{\lambda}(t_f) = \boldsymbol{\partial}_{\mathbf{X}} C_F(\mathbf{X}(t_f))\,, \\ \dfrac{\delta \tilde{J}}{\delta \mathbf{U}(\mathbf{X},t)} = 0 \implies \mathbf{U}(\mathbf{X}, t) = \dfrac{\boldsymbol{\partial}_{\mathbf{U}} L + (\boldsymbol{\partial}_{\mathbf{U}} \boldsymbol{f})^{\mathrm{T}} \boldsymbol{\lambda}(t)}{2\mu(t)}\,. \end{cases}$$

*computationally heavy

It requires iterative searching with backward and forward integration such as to identify the optimal control

*Lenhart, S., & Workman, J.T. (2007). Optimal Control Applied to Biological Models (1st ed.). Chapman and Hall/CRC.

(2) Optimal Control theory to **minimize Lagrangian particles dispersion** in turbulent flows

state variables *control variables*

Minimize $J = C_F(\mathbf{X}(t_f)) + \int_{t_0}^{t_f} dt\, [L(\mathbf{X}(t), \mathbf{U}(t), t)]$

performance index

Imposing $\dot{\mathbf{X}}_t = \boldsymbol{f}(\mathbf{X}(t), \mathbf{U}(t), t)$

and other possible constraints,

e.g.: $\begin{cases} \mathbf{X}(t_f) = \mathbf{X}_\star\,, \quad \mathbf{X}(t_0) \leq \mathbf{X}_\star\,, \\ \|\mathbf{U}(t)\|^2 = 1\,, \quad \|\mathbf{U}(t)\|^2 \leq 1\,, \text{ exc.} \end{cases}$

In our case:

$$(*) \begin{cases} \dot{\mathbf{X}}_t^{(1)} = \boldsymbol{v}(\mathbf{X}_t^{(1)}, t) \\ \dot{\mathbf{X}}_t^{(2)} = \boldsymbol{v}(\mathbf{X}_t^{(2)}, t) + \mathbf{U}(t) \\ \mathbf{U}(t) = V_s \hat{\boldsymbol{n}}(t) \end{cases}$$

$\|\boldsymbol{R}^\star\| = \|\boldsymbol{R}_{t_0}\|/100$

capture's distance

Minimize $J = \|\boldsymbol{R}_{t_f}\|^2 + c \int_{t_0}^{t_f} dt\, \theta(\|\boldsymbol{R}_t\|^2 - \|\boldsymbol{R}^\star\|^2)$

Imposing $(*)$ and the control constraint $\|\hat{\boldsymbol{n}}(t)\|^2 = 1$

$\|\boldsymbol{R}_{t_0}\| \sim \dfrac{V_s}{\lambda_{lyapunov}}$ border of controllability

Minimize trajectories' separation

Minimize time of arrival at the desired distance

(2) **Optimal Control theory** – Pontryagin minimum principle

state variables *control variables*

Minimize $J = C_F(\mathbf{X}(t_f)) + \int_{t_0}^{t_f} dt\, [L(\mathbf{X}(t), \mathbf{U}(t), t)]$

performance index *Lagrangian function*

Imposing $\dot{\mathbf{X}}_t = \boldsymbol{f}(\mathbf{X}(t), \mathbf{U}(t), t)$

and other possible constraints,

e.g.: $\begin{cases} \mathbf{X}(t_f) = \mathbf{X}_\star\,, \quad \mathbf{X}(t_0) \leq \mathbf{X}_\star\,, \\ \|\mathbf{U}(t)\|^2 = 1\,, \quad \|\mathbf{U}(t)\|^2 \leq 1\,, \text{ exc.} \end{cases}$

- **Model based** and analytical tool
- Perfect knowledge required

$\|\boldsymbol{R}_{t_0}\| \sim \frac{V_s}{\lambda_{lyapunov}}$ border of controllability

In our case:

$\|\boldsymbol{R}^\star\| = \|\boldsymbol{R}_{t_0}\|/100$

capture's distance

Minimize $J = \|\boldsymbol{R}_{t_f}\|^2 + c \int_{t_0}^{t_f} dt\, \theta(\|\boldsymbol{R}_t\|^2 - \|\boldsymbol{R}^\star\|^2)$

Imposing $(*)$ and the control constraint $\|\hat{\boldsymbol{n}}(t)\|^2 = 1$

$$(*) \begin{cases} \dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t)\,, \\ \mathbf{U}(t) = V_S \hat{\boldsymbol{n}}(t)\,. \end{cases}$$

LINEAR REGIME

Optimal Control theory to **minimize Lagrangian particles dispersion** in turbulent flows

Constrained performance index:

$$\tilde{J} = \|\boldsymbol{R}_{t_f}|^2 + \int_{t_0}^{t_f} dt\, \{c[\theta(\|\boldsymbol{R}_t\|^2 - \|\boldsymbol{R}^\star\|^2)] + \underbrace{\boldsymbol{\lambda}^{\mathrm{T}}(t)[\nabla \boldsymbol{v}_t \boldsymbol{R}_t + V_s \hat{\boldsymbol{n}}(t) - \dot{\boldsymbol{R}}] + \mu(t)(1 - \|\hat{\boldsymbol{n}}(t)\|^2)\}}_{\text{Constrained minimization problem}}$$

Constrained minimization problem

Integrating by parts,

$$\tilde{J} = \|\boldsymbol{R}_{t_f}\|^2 - \boldsymbol{\lambda}^{\mathrm{T}}(t_f)\boldsymbol{R}_{t_f} + \boldsymbol{\lambda}^{\mathrm{T}}(t_0)\boldsymbol{R}_{t_0} + \int_{t_0}^{t_f} dt\, [H(\boldsymbol{R}_t, \boldsymbol{\lambda}(t), \hat{\boldsymbol{n}}(t), \mu(t), t) + \dot{\boldsymbol{\lambda}}^{\mathrm{T}} \boldsymbol{R}_t]\,,$$

$$H(\boldsymbol{R}_t, \boldsymbol{\lambda}(t), \hat{\boldsymbol{n}}(t), \mu(t), t) = c[\theta(\|\boldsymbol{R}_t\|^2 - \|\boldsymbol{R}^\star\|^2)] + \boldsymbol{\lambda}^{\mathrm{T}}(t)[\nabla \boldsymbol{v}_t \boldsymbol{R}_t + V_s \hat{\boldsymbol{n}}(t)] + \mu(t)(1 - \|\hat{\boldsymbol{n}}(t)\|^2).$$

Hamiltonian function

Consider variation in $\tilde{J}$:

$$\delta \tilde{J} = \Big[\big(2\boldsymbol{R}_t - \boldsymbol{\lambda}^{\mathrm{T}}(t)\big)\delta \boldsymbol{R}\Big]_{t=t_f} + \Big[\boldsymbol{\lambda}^{\mathrm{T}}(t)\delta\boldsymbol{R}\Big]_{t=t_0} + \int_{t_0}^{t_f} dt\, \Big\{\Big[\frac{\partial H}{\partial \boldsymbol{R}} + \dot{\boldsymbol{\lambda}}^{\mathrm{T}}\Big]\delta\boldsymbol{R} + \frac{\partial H}{\partial \hat{\boldsymbol{n}}}\delta\hat{\boldsymbol{n}}\Big\}$$

Euler-Lagrange equations:

$$\begin{cases} \dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{R}} = -2c\, \boldsymbol{R}_t \delta(\|\boldsymbol{R}_t\|^2 - \|\boldsymbol{R}^\star\|^2) - (\nabla \boldsymbol{v}_t)^{\mathrm{T}} \boldsymbol{\lambda}(t)\,, \\ \boldsymbol{\lambda}(t_f) = 2\boldsymbol{R}_{t_f}\,, \\ \frac{\partial H}{\partial \hat{\boldsymbol{n}}} = 0 \implies \hat{\boldsymbol{n}}(t) = \frac{V_s \boldsymbol{\lambda}(t)}{2\mu(t)} = -\frac{\boldsymbol{\lambda}(t)}{\|\boldsymbol{\lambda}(t)\|}\,. \end{cases}$$

$$\begin{cases} \dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t)\,, \\ \boldsymbol{R}_{t_0} = \text{given}\,, \\ \mathbf{U}(t) = V_S \hat{\boldsymbol{n}}(t)\,. \end{cases}$$

A. E. Bryson and Y. Ho, **Applied optimal control: optimization, estimation and control** (New York: Routledge, 1975).

Optimal Control theory to **minimize Lagrangian particles dispersion** in turbulent flows

Euler-Lagrange equations:

$$\begin{cases} \dot{\boldsymbol{\lambda}} = -\frac{\partial H}{\partial \boldsymbol{R}} = -2c\, \boldsymbol{R}_t \delta(\|\boldsymbol{R}_t\|^2 - \|\boldsymbol{R}^\star\|^2) - (\nabla \boldsymbol{v}_t)^{\mathrm{T}} \boldsymbol{\lambda}(t)\,, \\ \boldsymbol{\lambda}(t_f) = 2\boldsymbol{R}_{t_f}\,, \\ \frac{\partial H}{\partial \hat{\boldsymbol{n}}} = 0 \implies \hat{\boldsymbol{n}}(t) = \frac{V_s \boldsymbol{\lambda}(t)}{2\mu(t)} = -\frac{\boldsymbol{\lambda}(t)}{\|\boldsymbol{\lambda}(t)\|}\,. \end{cases}$$

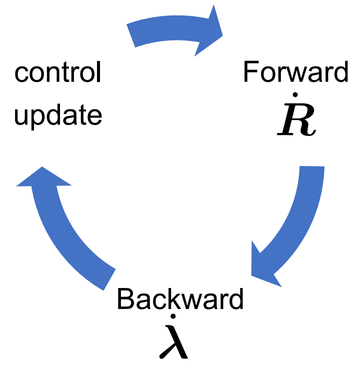
backward integration

$$\begin{cases} \dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t)\,, \\ \boldsymbol{R}_{t_0} = \text{given}\,, \\ \mathbf{U}(t) = V_S \hat{\boldsymbol{n}}(t)\,. \end{cases}$$

Forward integration

computationally heavy

It requires iterative searching with backward and forward integration such as to identify the optimal control

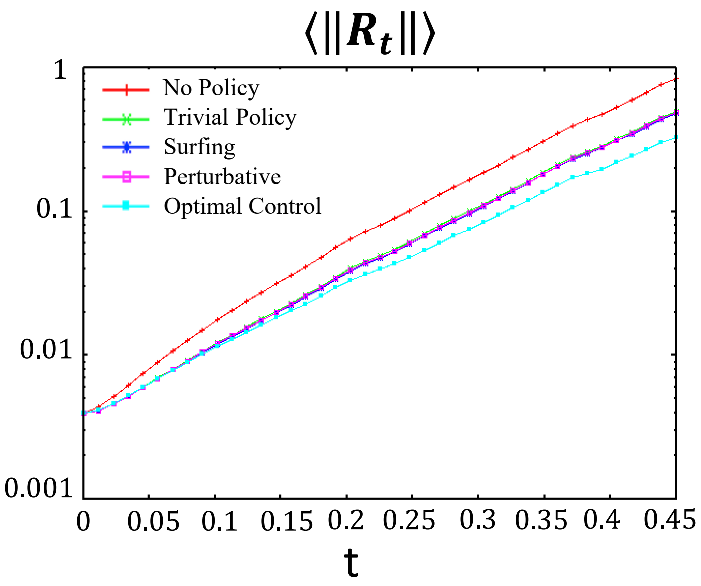


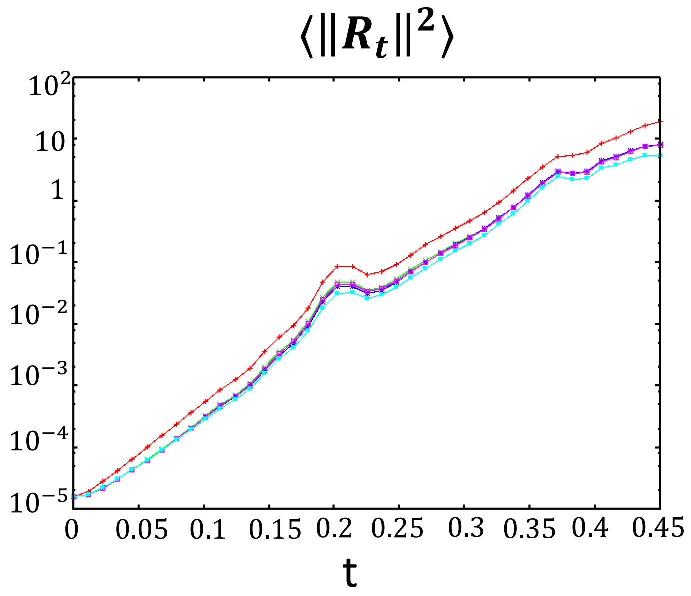
Lenhart, S., & Workman, J.T. (2007). **Optimal Control Applied to Biological Models** (1st ed.). Chapman and Hall/CRC.

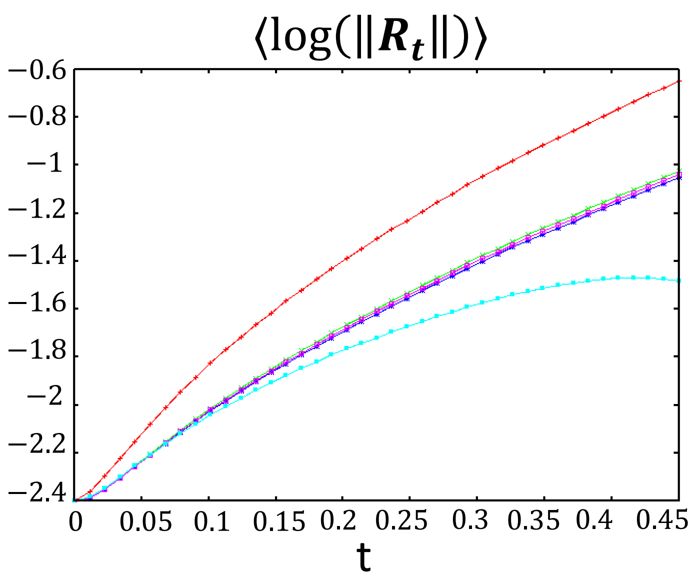
Optimal Control vs heuristic policies at **small scales**

$$\dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t)$$

Failures (no capture):



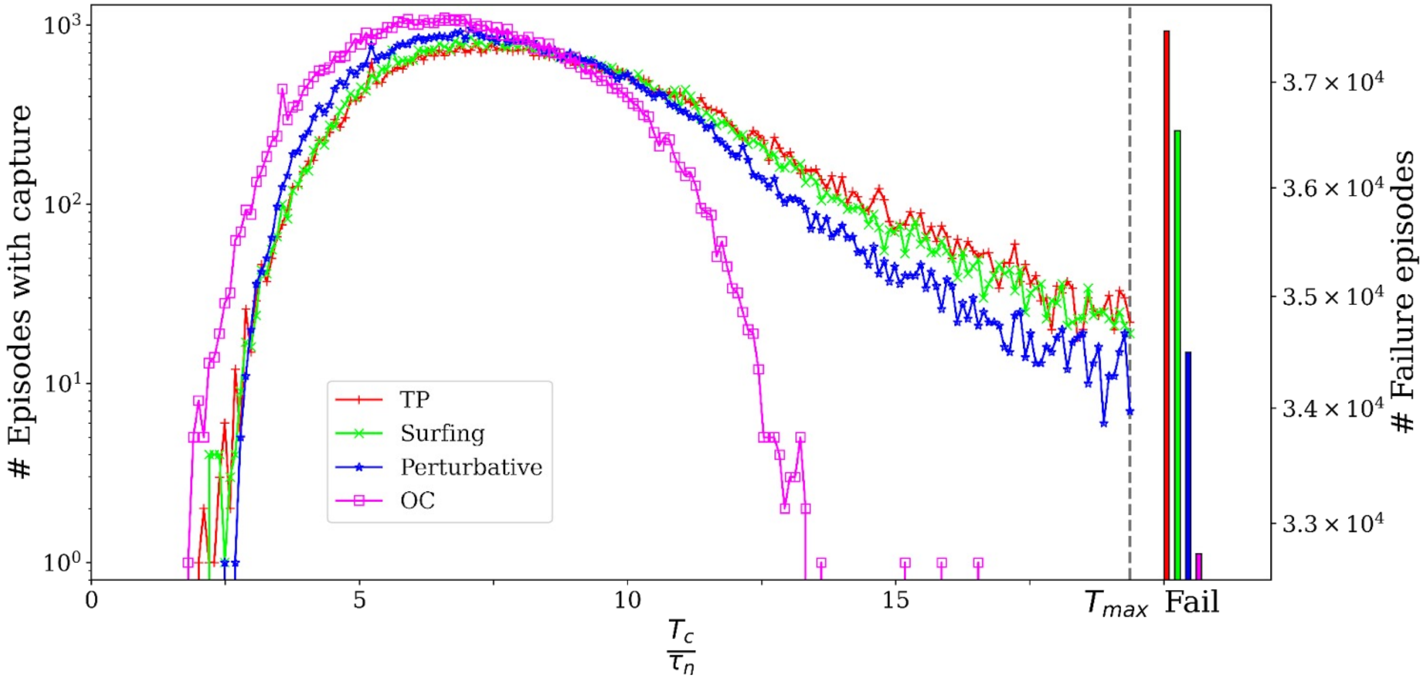




Optimal Control vs heuristic policies in linear regime

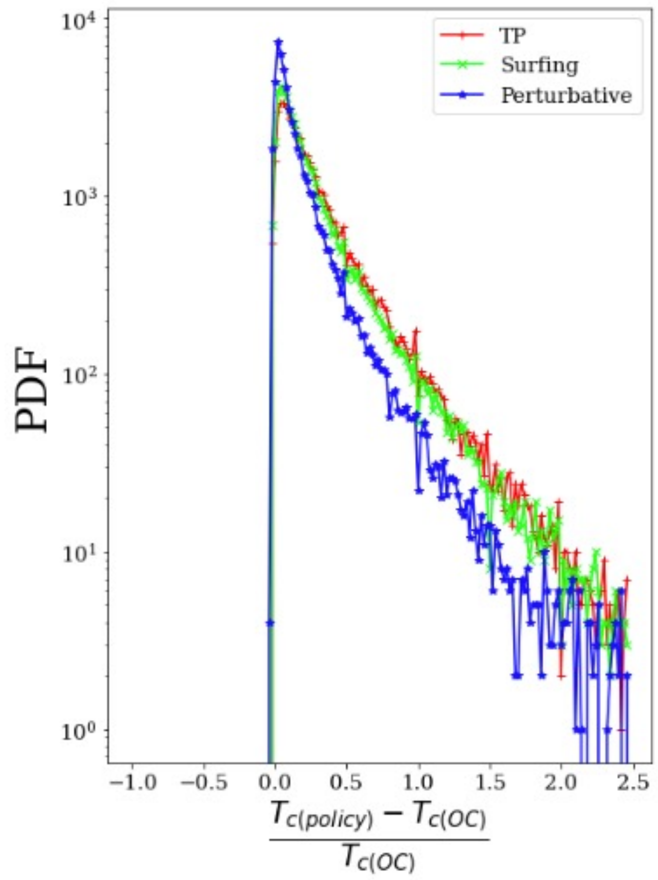
$$\dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t)$$

T_c = **Capture** time: (time of arrival at the desired distance)



PRELIMINARY UNPUBLISHED

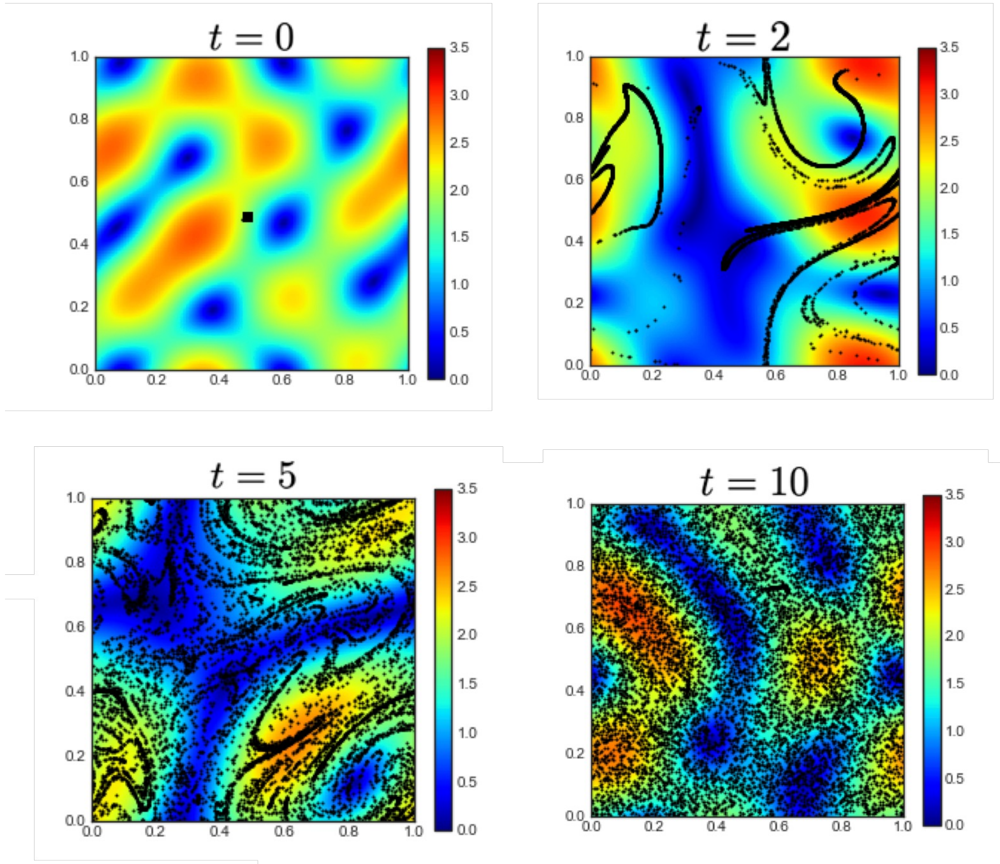
PDF of normalized capture time



Heuristic policies in a 2d stochastic flow (linear and non-linear regime)

Heuristic policies in a 2d stochastic flow

Dispersion of a bunch of particles



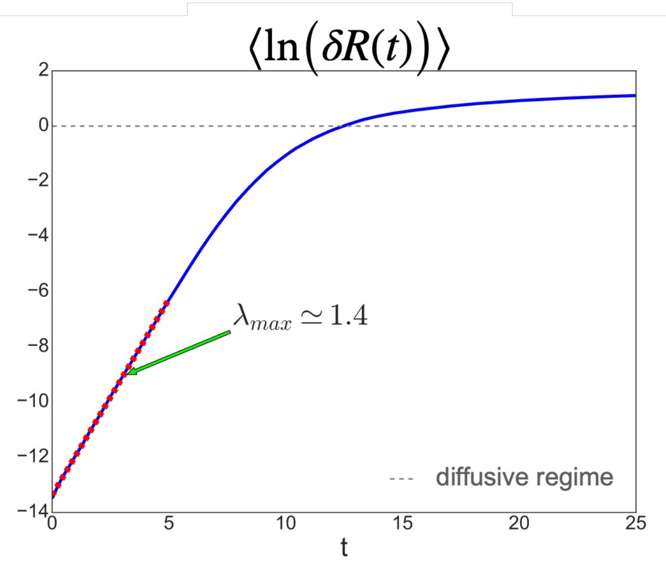
$$\boldsymbol{v}(x, y, t) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right)$$

$$\psi(\boldsymbol{x}, t) = \sum_{\boldsymbol{k} \in \mathscr{K}} \left(A(\boldsymbol{k}, t)e^{i(\boldsymbol{k}\cdot\boldsymbol{x})} + \text{c.c.}\right)$$

Dove $\mathscr{K} = \{(k_s,0), (\pm k_s, k_s), (0,k_s)\}$

Velocity field

- $A(\boldsymbol{k}, t)$ generated by an **Ornstein-Uhlenbeck** process
- $u_{rms} = 1$ (typical velocity)
- $L = \dfrac{2\pi}{k_s} = 1$ (characteristic scale)

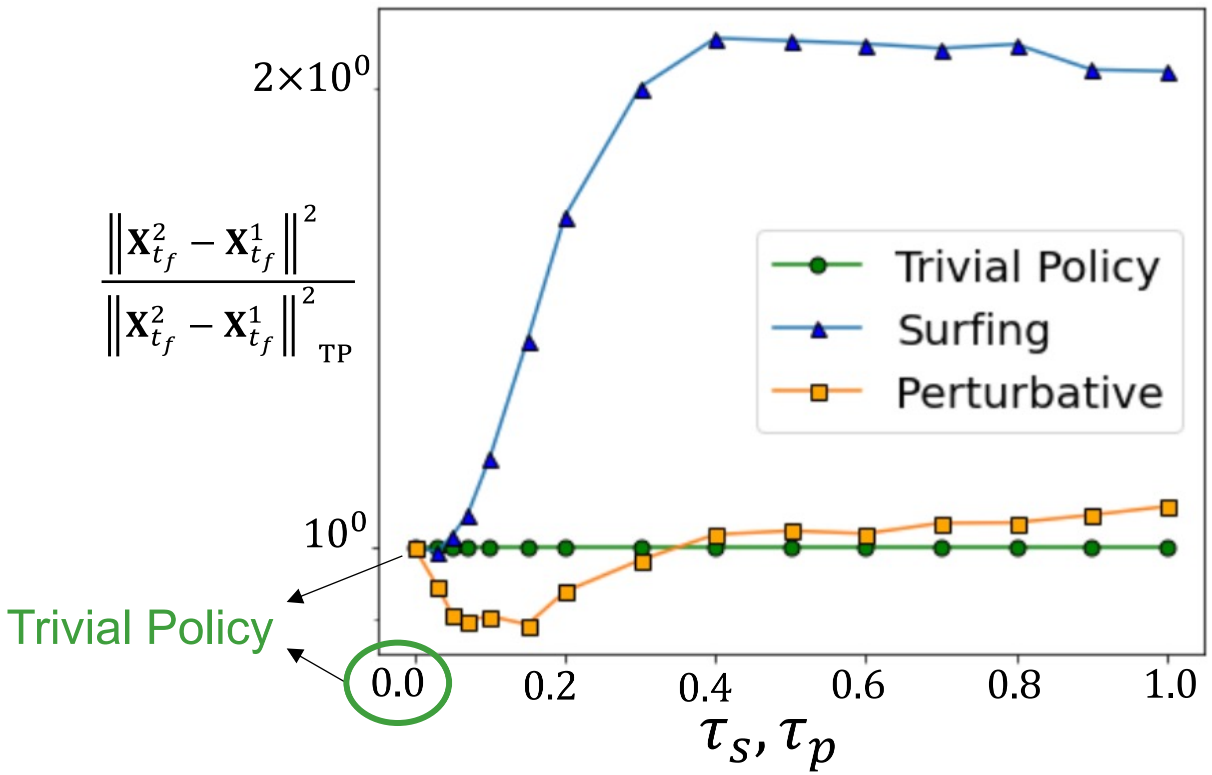


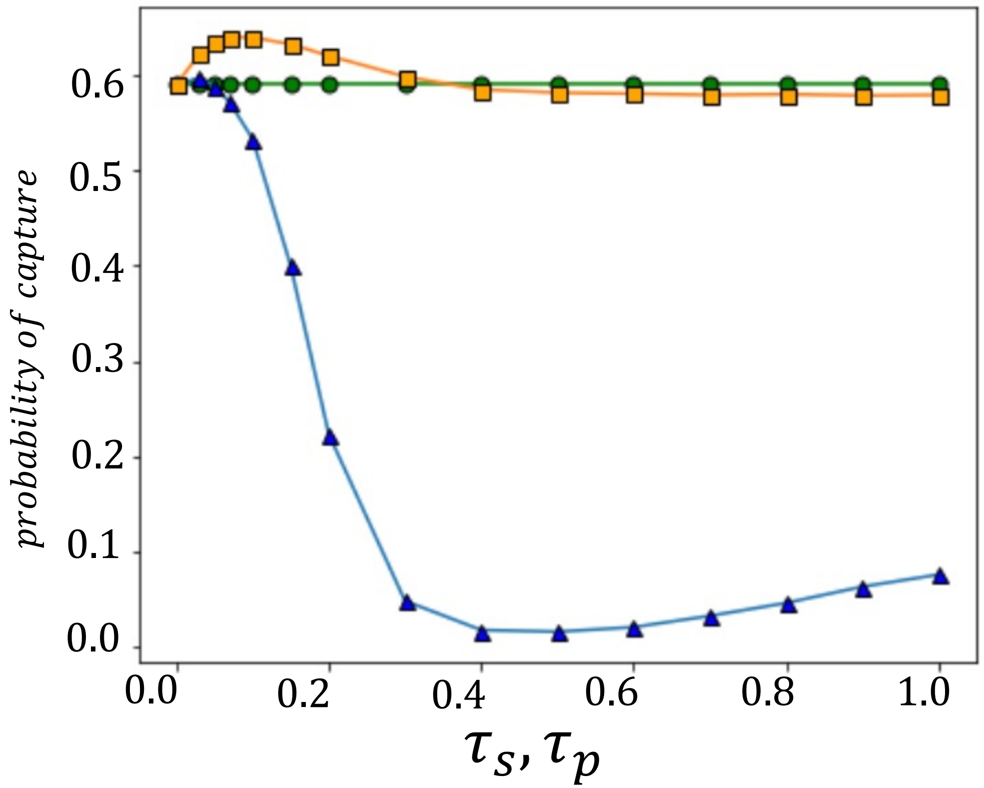
$\lambda > 0$

The Lagrangian dynamics is chaotic

Heuristic policies in a 2d stochastic flow

Performance at small scales $\|R_{t_0}\| \ll L$



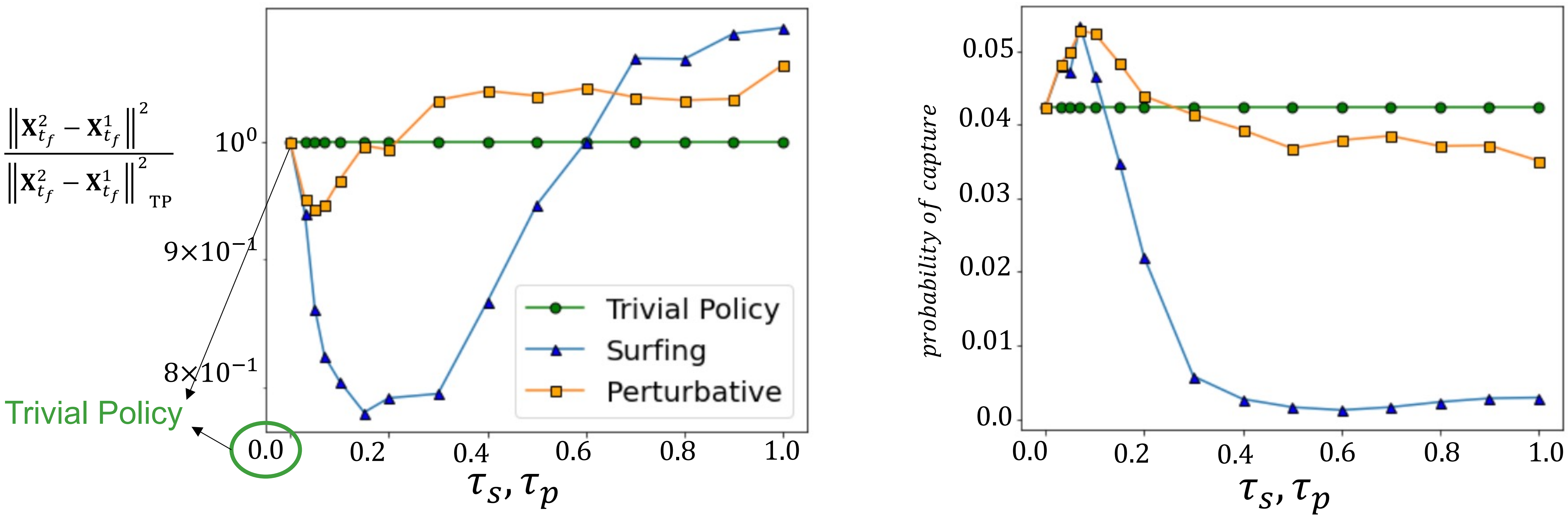


- The surfing policy performs bad at small scales.
- The perturbative policy performs well at small scales, $\exists\ best\ \tau_p \neq 0$.

There is a way to perform better than the Trivial Policy

Heuristic policies in a 2d stochastic flow

Performance at large scales $\left\|R_{t_0}\right\| \gg L$



- The surfing policy performs well at large scales, $\exists\ best\ \tau_s \neq 0$.
- The perturbative policy performs well at large scales, $\exists\ best\ \tau_p \neq 0$.

There is a way to perform better than the Trivial Policy

Heuristic vs OC
Double gyre flow (2d)
(linear regime)

Optimal Control vs heuristic policies at **small scales**

Velocity field (double gyre flow*)

$$f(x,t) = a(t)x^2 + b(t)x$$

$$\phi(x,y,t) = A\sin(\pi f(x,t))\sin(\pi y),$$

$$a(t) = \epsilon \sin(\omega t)$$

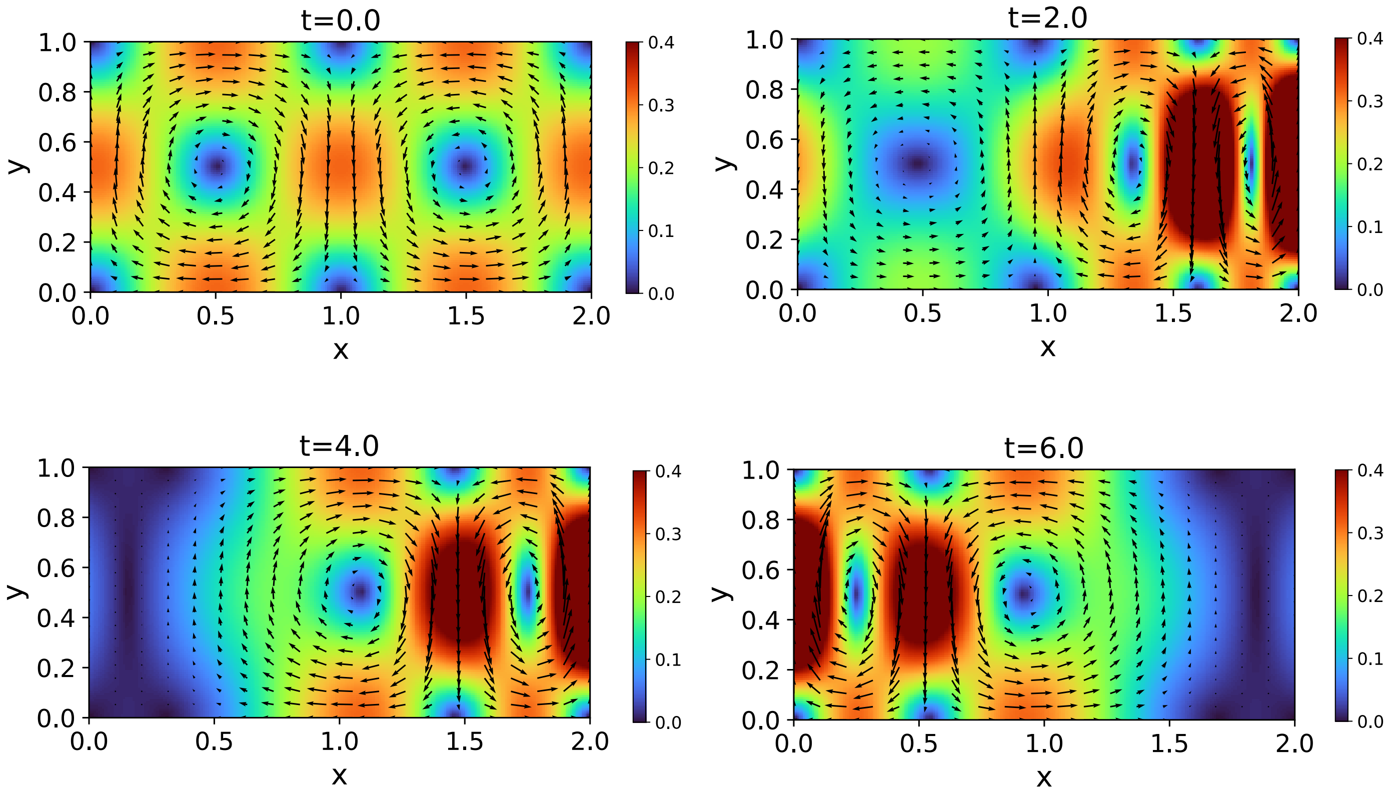
$$b(t) = 1 - 2\epsilon \sin(\omega t).$$

Linear regime

$$\begin{cases} \dot{\mathbf{X}}_t^1 = \boldsymbol{v}(\mathbf{X}_t^1) \\ \dot{\mathbf{X}}_t^2 = \boldsymbol{v}(\mathbf{X}_t^2) + \mathbf{U}(t) \\ \mathbf{U}(t) = V_s\, \hat{\boldsymbol{n}}(t) \end{cases}$$

$$\boldsymbol{v}(\mathbf{X}_t^2) \simeq \boldsymbol{v}(\mathbf{X}_t^1) + \nabla \boldsymbol{v}_t \boldsymbol{R}_t$$

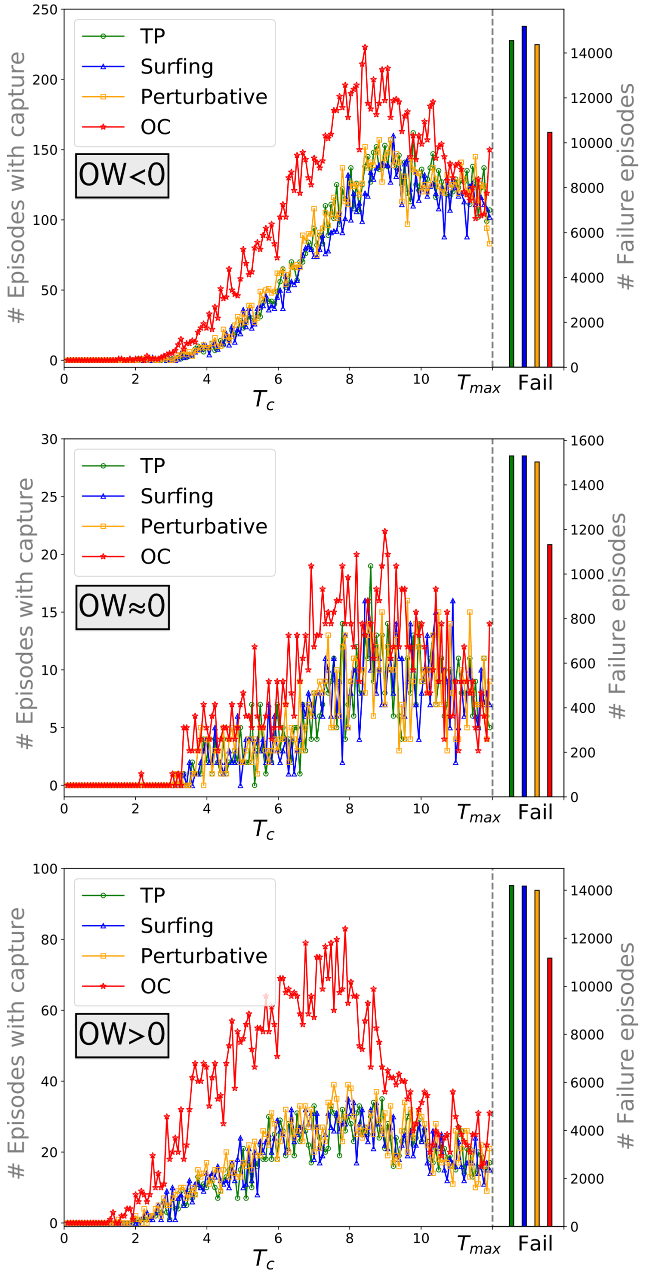
$$\dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t)$$



*Krishna K, Song Z, Brunton SL. 2022 **Finite-horizon, energy-efficient trajectories in unsteady flows**. *Proc. R. Soc. A* **478**: 20210255.

Optimal Control vs heuristic policies in linear regime

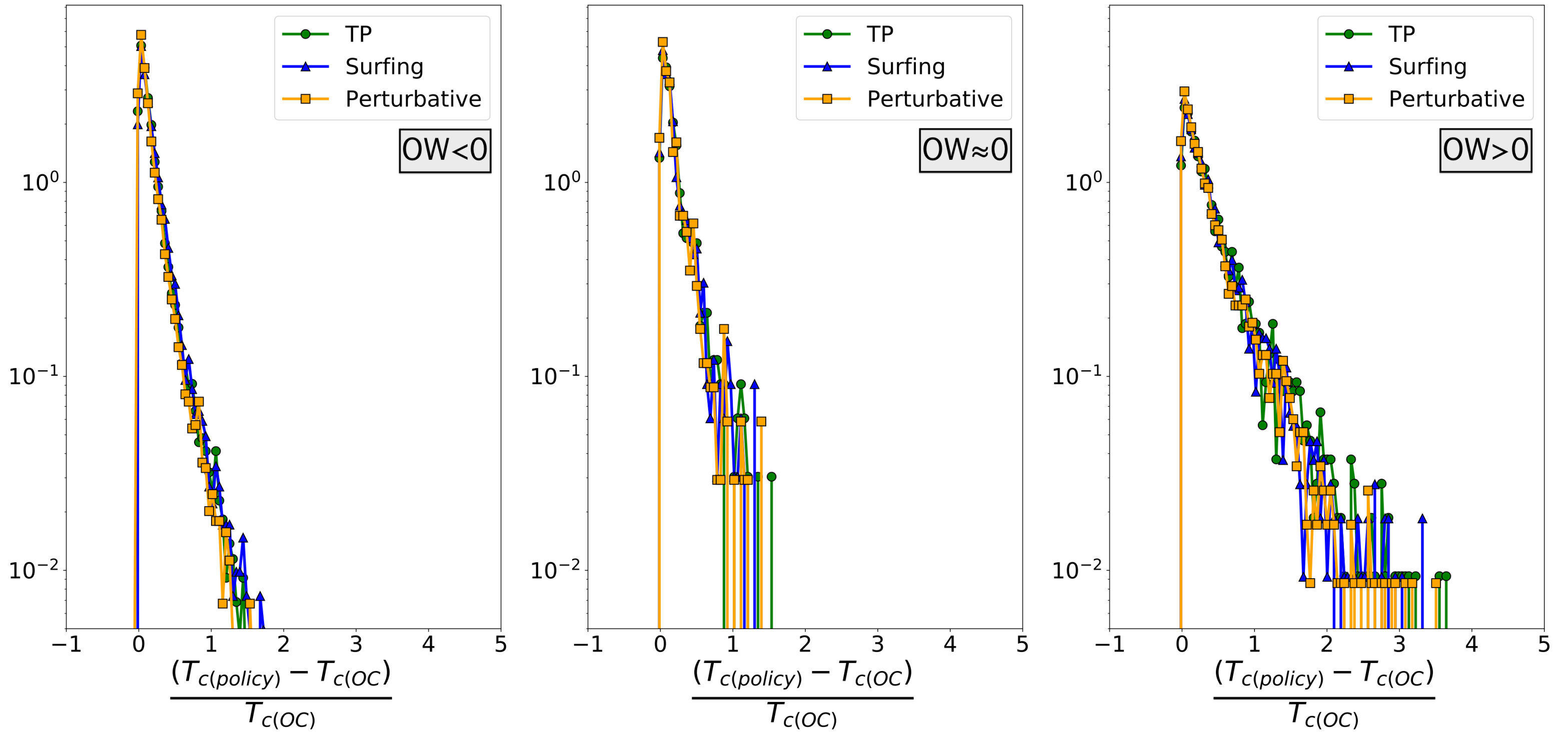
$$\dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t)$$



T_c = **Capture** time: (time of arrival at the desired distance)

OW = Average of the Okubo Weiss parameter: < 0 vorticity dominated; > 0 strain dominated

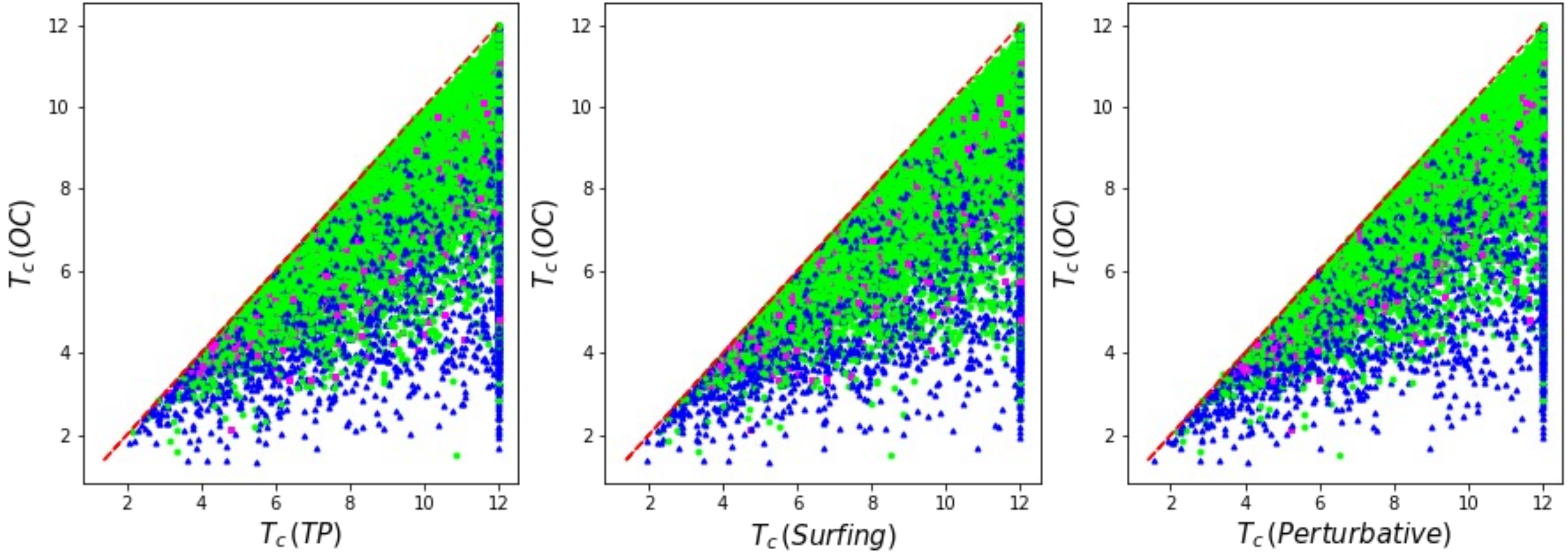
PDF of normalized capture time



$$\dot{\boldsymbol{R}}_t = \boldsymbol{\nabla v}_t \boldsymbol{R}_t + \mathbf{U}(t)$$

Optimal Control vs heuristic policies at **small scales**

$T_c =$ **Capture** time: (time of arriving at the desired distance)

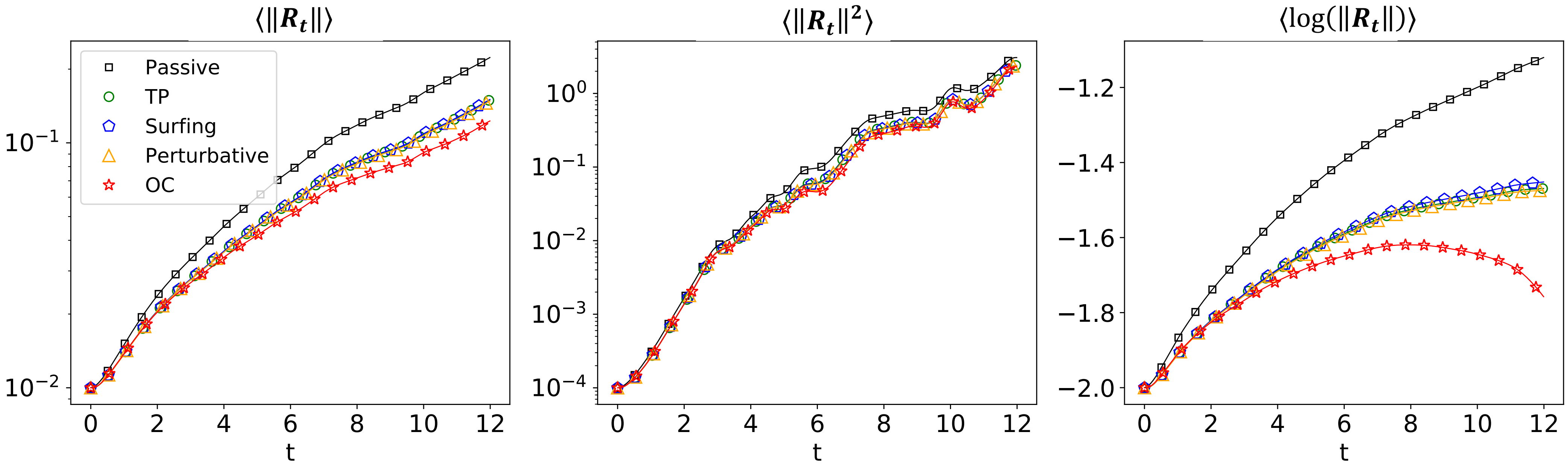


- ▲ $OW > 0$
- ■ $OW \simeq 0$
- ● $OW < 0$

Optimal Control vs heuristic policies at **small scales**

$$\dot{\boldsymbol{R}}_t = \boldsymbol{\nabla v}_t \boldsymbol{R}_t + \mathbf{U}(t)$$

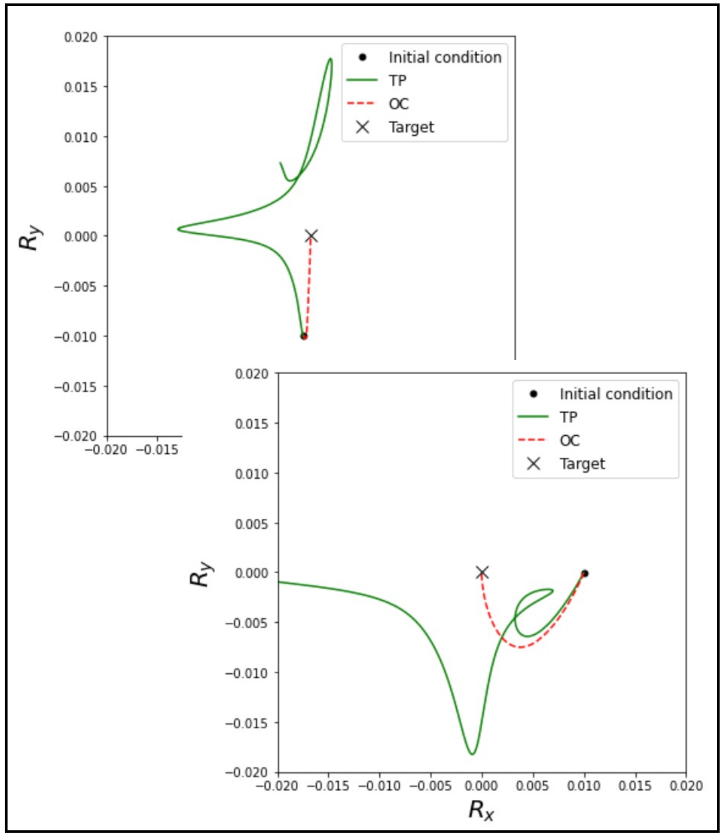
Failures (no capture):



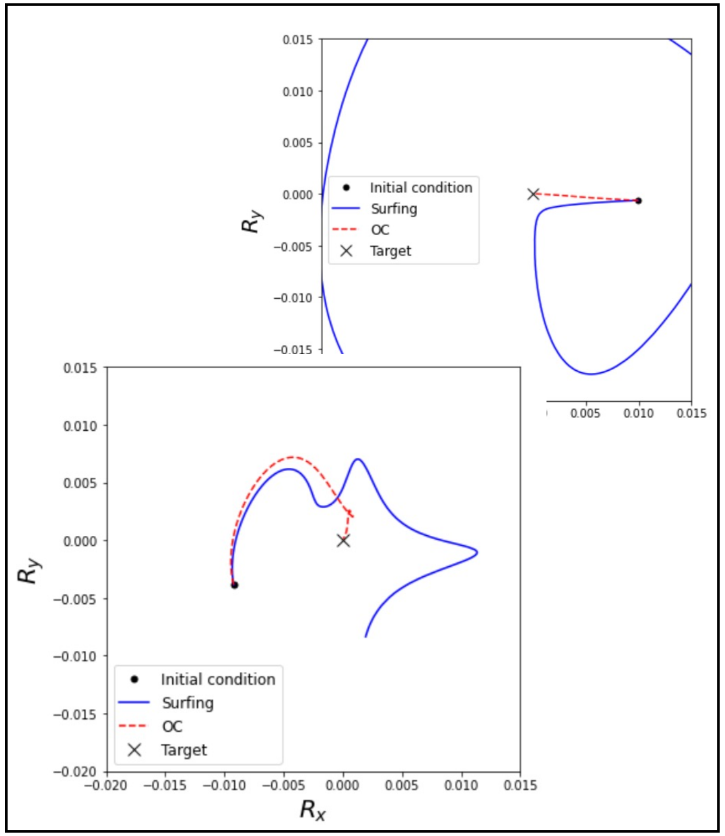
Trajectories examples

$$\dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t)$$

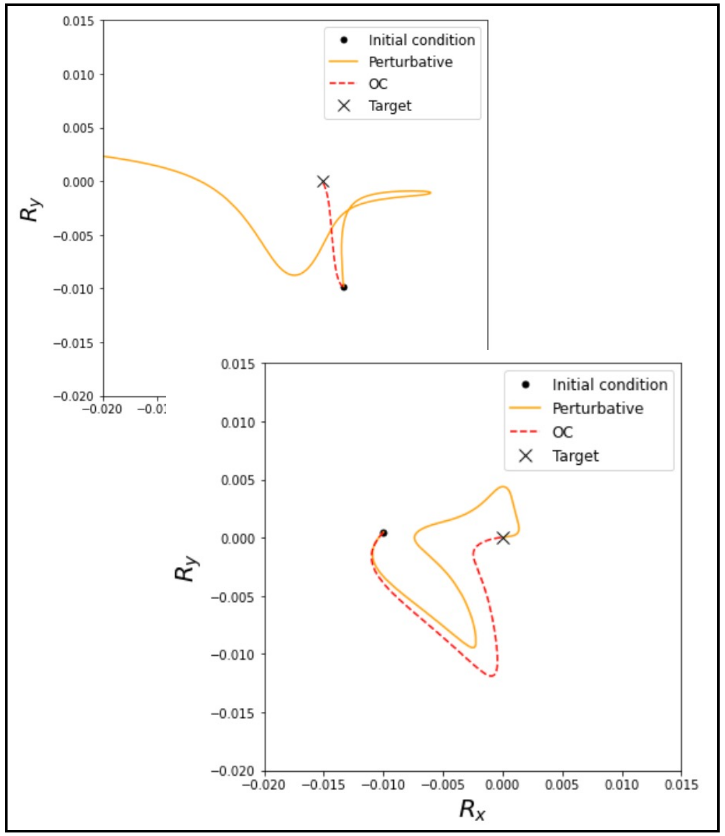
OC vs Trivial Policy



OC vs Surfing



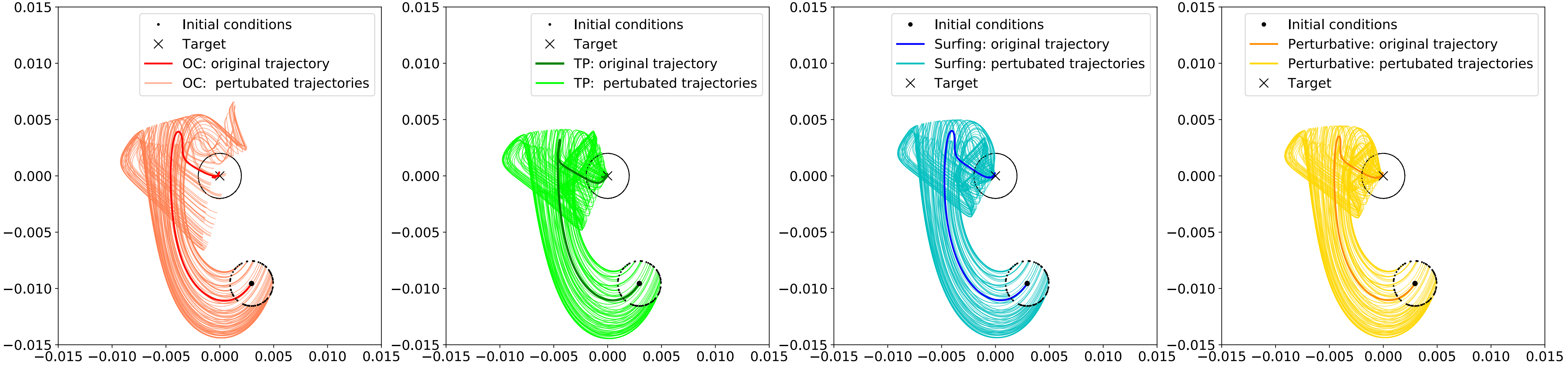
OC vs Perturbative



Policies' stability

$$\dot{\boldsymbol{R}}_t = \nabla \boldsymbol{v}_t \boldsymbol{R}_t + \mathbf{U}(t)$$

Perturbation of the initial condition



PDF (only capture episodes)