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Entropic Lattice Boltzmann Method: an implicit Large-Eddy Simulation?

G. Tauzin, L. Biferale, M. Sbragaglia,
A. Gupta, F. Toschi, A. Bartel, M. Ehrhardt

Università degli studi di Roma Tor Vergata
Bergische Universität Wuppertal

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Motivations

Context

Implicit SGS arising when stabilizing for $\nu \to 0$ a Lattice Boltzmann Method by equipping it with a H-theorem

Aim

Study the physical properties of this implicit sub-grid scale (SGS) model

Tool

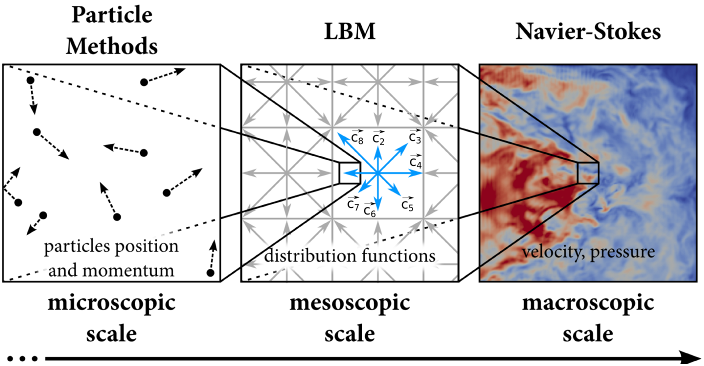
Development a tool based on balance equations to check hydrodynamic recovery of simulated flows accross scales

Introduction to Lattice Boltzmann Method (LBM)

LBM Equation with a relaxation time $\tau \equiv \tau_0$ fixed

$$f_i(\vec{x} + \vec{c}_i \Delta t, t + \Delta t) - f_i(\vec{x}, t) = -\frac{1}{\tau_0}\left[f_i(\vec{x}, t) - f_i^{eq}(\vec{x}, t)\right]$$

Macroscopic quantities: Density: $\rho = \sum_i f_i$ Momentum: $\rho\vec{u} = \sum_i f_i \vec{c}_i$



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$\Downarrow$ Chapman-Enskog expansion: $\mathcal{O}(K_n^2)$, $\mathcal{O}(M_a^3)$ $\Downarrow$

$$\nu = c_s^2(\tau - 0.5)\Delta t$$

Weakly compressible Navier-Stokes with viscosity $\nu \equiv \nu_0$ fixed

$$\rho \partial_t u_i + \rho u_j \partial_j u_i = -\partial_i p + \partial_j \rho \nu \left(\partial_j u_i + \partial_i u_j\right) + \mathcal{O}(M_a^3) + \mathcal{O}(K_n^2)$$

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Instabilities arising as $\tau_0 \to 0.5 \iff \nu_0 \to 0$ at a fixed resolution

Can we get rid of those instabilities?

Entropic LBM: Unconditionnal stability and implicit SGS

- Entropic LBM (ELBM) equips a H-theorem by locally adaptating τ

$\tau = \tau^{eff}(\vec{x},\, t) = \dfrac{2\,\tau_0}{\alpha(f_i(\vec{x},\, t)\,)}$ where α has a non-linear dependency on $f_i(\vec{x},\, t)$

[Karlin *et al.*, 1999]

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In practice unconditionnally stable

$\nu_{eff}(\vec{x}, t) = c_s^2(\tau_{eff}(\vec{x}, t) - 0.5)\Delta t$ expression in terms of macroscopic quantities?

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$$\nu_{eff}(\vec{x},\, t) = \nu_0 + \nu_t^M(\vec{x},\, t) \text{ with } \nu_t^M(\vec{x},\, t) \propto -\frac{S_{\theta\kappa}S_{\kappa\gamma}S_{\gamma\theta}}{S_{\lambda\mu}S_{\lambda\mu}}$$

where $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$

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Similar to a Smagorinsty model: $\nu_t(\vec{x},\, t) = C\sqrt{S_{\theta\kappa}S_{\theta\kappa}}$

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Objective: Numerically check the existence of this implied SGS

Kinetic energy balancing averaged over a sub-volume V

Averaged kinetic energy balance equation for $\nu = \nu_0$ fixed

$$LHS_V^E = \partial_t \left\langle \frac{\rho u_i u_i}{2} \right\rangle_V$$

$$= -\left\langle u_i \partial_i p \right\rangle_V - \nu_0 \left\langle \rho \left(\partial_j u_i + \partial_i u_j\right) \partial_j u_i \right\rangle_V + \nu_0 \left\langle \partial_j \rho u_i \left(\partial_j u_i + \partial_i u_j\right) \right\rangle_V$$

$$- \left\langle \partial_j \frac{\rho u_i u_i}{2} u_j \right\rangle_V + \left\langle u_i F_i \right\rangle_V$$

$$= RHS_V^{E,1} + RHS_V^{E,2} + RHS_V^{E,3} + RHS_V^{E,4} + RHS_V^{E,5} = RHS_V^E$$

where $\langle \dots \rangle_V$ denotes the average over a sub-volume V

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where $\langle \dots \rangle_V$ denotes the average over a sub-volume V

Simulations of 2D homogeneous isotropic turbulence

Periodic 256×256 grid using a D2Q9 lattice

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Forcing on a shell of wavenumber

$$F_{\Psi}^{T} = F_{0}^{T} \sum_{\|\vec{k}\|=5}^{7} \cos\left(\frac{2\,\pi}{L}\vec{k}.\vec{x} + \phi\right)$$

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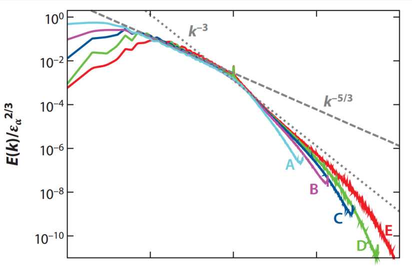

When forcing at k_f, we have:

- a backward energy cascade to large scales
- a forward enstrophy cascade to small scales

[Boffetta & Ecke, 2012]

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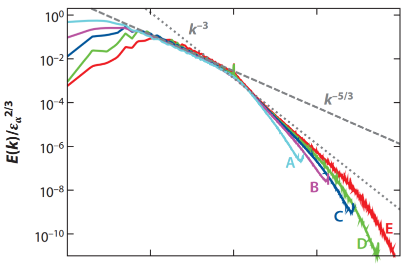
Energy removal at large scale

$$\vec{F^E}(\vec{x}, t) = -F_0^E \sum_{\|\vec{k}\|=1}^{2} \hat{\vec{u}}(\vec{k}, t) e^{\frac{2\pi}{L}\vec{k}\cdot\vec{x}}$$



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Statistical analysis of the balancing errors δ_V^E

Kinetic energy balancing error

$$\delta_V^E(t) = \frac{\left|RHS_V^E(t) - LHS_V^E(t)\right|}{\max_i \left|RHS_V^{E,\,i}(t)\right|}$$

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For a scale L, we gather statistics of δ_L^E:
Balancing error over sub-volumes of shape $V = L \times L$ in space and time

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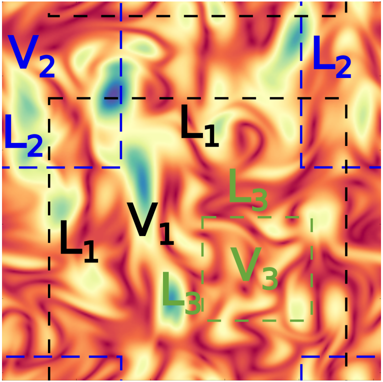
Example of sub-volumes shown on a vorticity field



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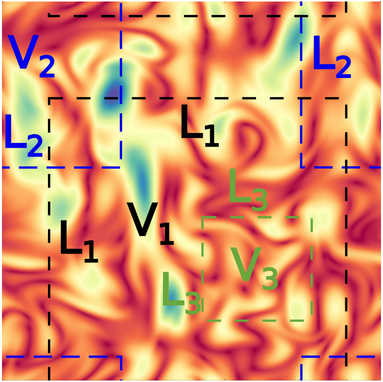

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We process outputs from simulations that have reached statistical stationnarity

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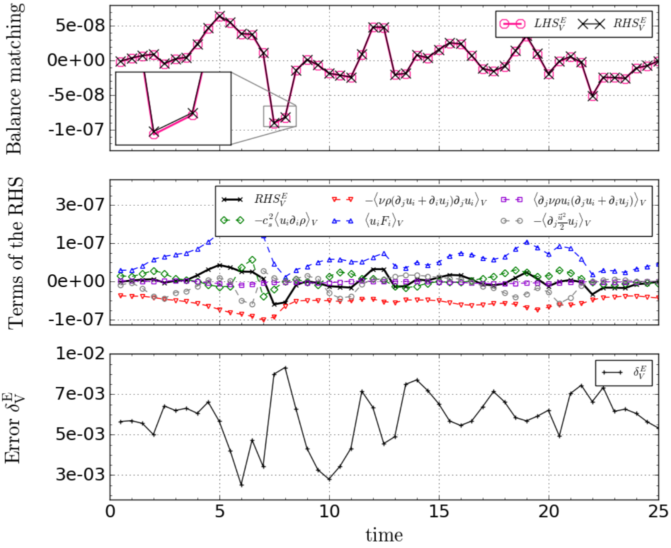
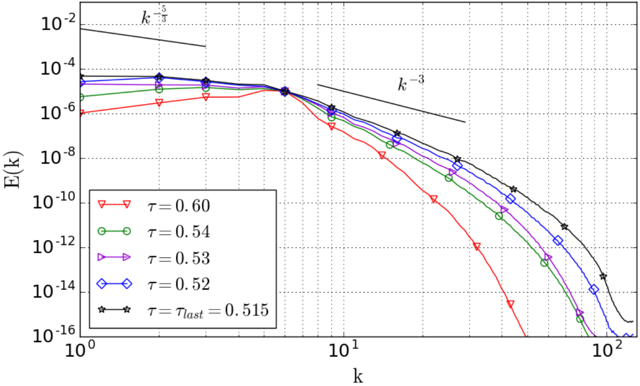
Evolution of the kinetic energy balancing on a single sub-volume $V = 181 \times 181$ for a LBM simulation with $\tau \equiv \tau_0 = 0.55$ fixed



LBM for different fixed $\tau \equiv \tau_0 \to 0.5$: Superposed spectrum

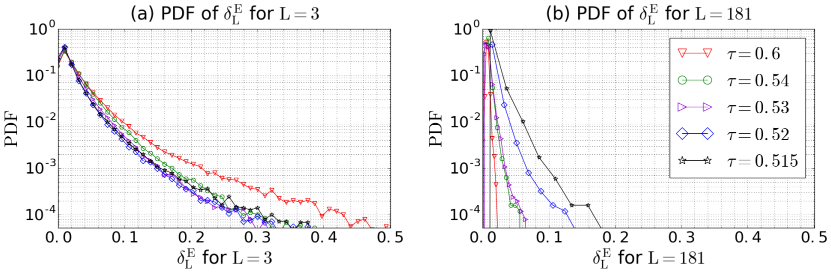
$\tau \equiv \tau_0$ fixed & Constant forcing amplitudes

Time-averaged superposed spectrum



where $\tau_{last} = 0.515$ is the lowest τ ensuring a stable LBM simulation at this resolution and forcing amplitude

LBM for different fixed $\tau \equiv \tau_0 \to 0.5$: δ_L^E and $Ma_L = \frac{<u_{RMS}>_L}{c_s}$



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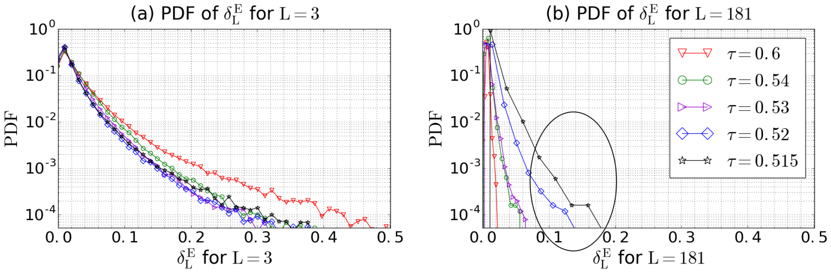

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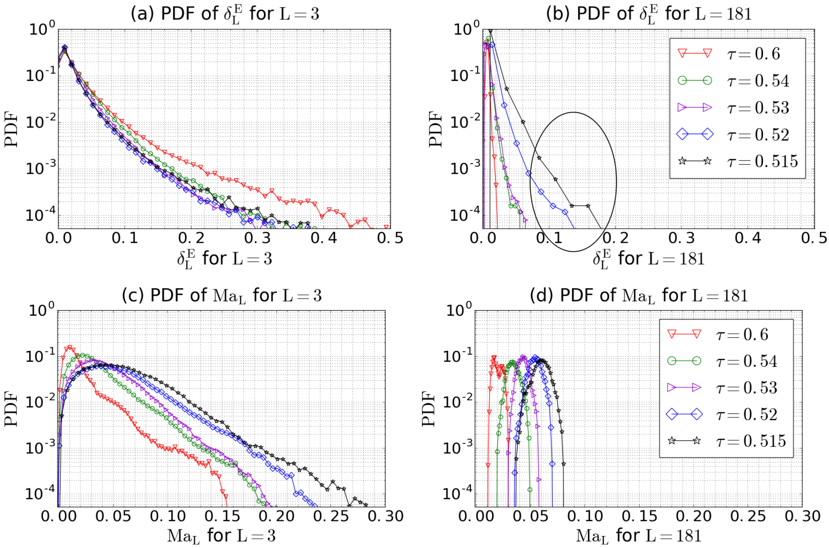
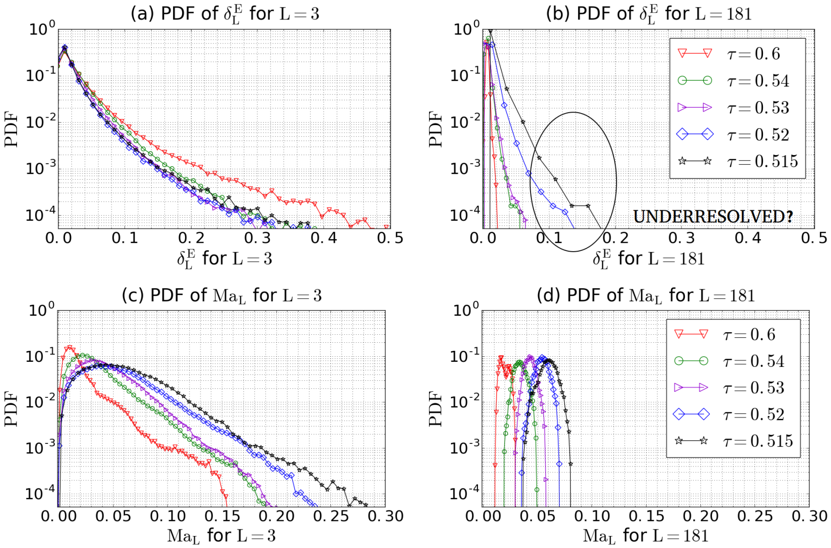
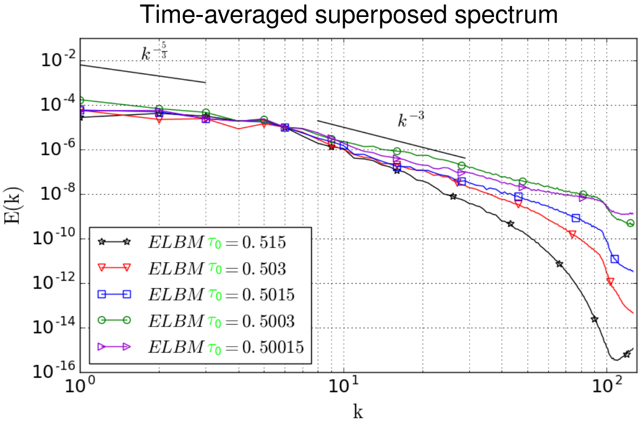


LBM for different fixed $\tau \equiv \tau_0 \rightarrow 0.5$: δ_L^E and $Ma_L = \frac{<u_{RMS}>_L}{c_s}$



Entropic LBM for different $\tau_0 \to 0.5$: Superposed spectrum

$$\tau = \tau^{eff}(\vec{x}, t) = \frac{2\,\tau_0}{\alpha(f_i(\vec{x}, t))}$$ & Constant forcing amplitudes



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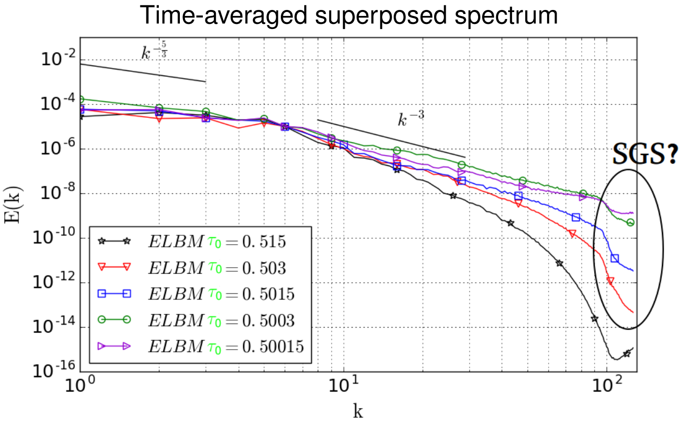
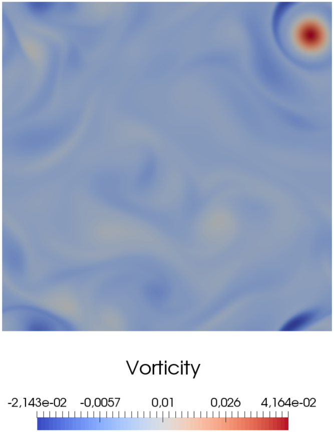


ELBM has the dissipative properties expected from a LES

Entropic LBM for $\tau_0 = 0.515$: Correlation between ω and τ^{eff}

Snapshot of $\tau^{eff}(\vec{x}, t) = \frac{2\,\tau_0}{\alpha(f_i(\vec{x}, t))}$

Snapshot of $\omega = \partial_x u_y - \partial_y u_x$



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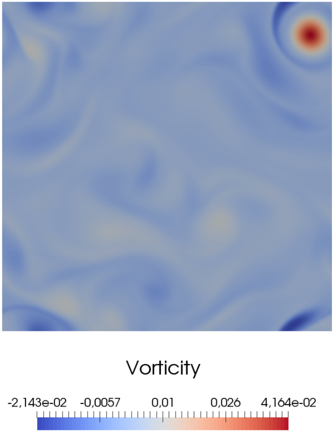
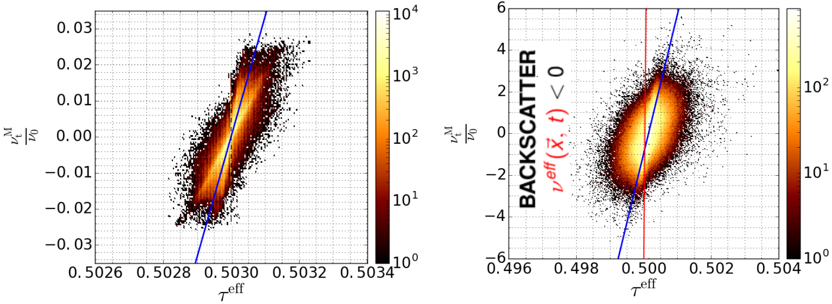


τ^{eff} adapts itself to vorticity peaks

Entropic LBM: Validation of Malaspinas ν_t^M

Is Malaspinas' $\nu_{eff}^M(\vec{x},\,t) = \nu_0 + \nu_t^M(\vec{x},\,t)$ with $\nu_t^M(\vec{x},\,t) \propto -\frac{S_{\theta\kappa}S_{\kappa\gamma}S_{\gamma\theta}}{S_{\lambda\mu}S_{\lambda\mu}}$ an accurate approximation of $\nu_{eff}(\vec{x},\,t) = c_s^2\,(\tau_{eff}(\vec{x},\,t) - 0.5)\,\Delta t$?

Joint PDF between $\frac{\nu_t^M(\vec{x},t)}{\nu_0}$ and $\tau^{eff}(\vec{x},\,t)$ with $\frac{\nu_t(\vec{x},t)}{\nu_0} = \frac{\nu^{eff}(\vec{x},t)-\nu_0}{\nu_0}$ in blue



$\tau_0 = 0.503$ $\qquad$ $\tau_0 = 0.50015$

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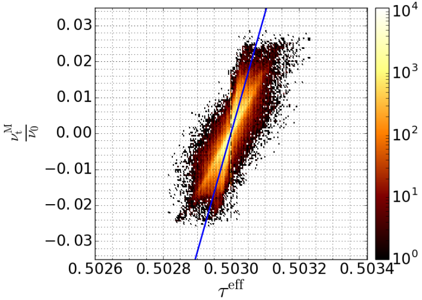
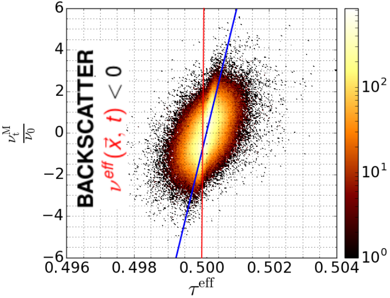


$\tau_0 = 0.503$



$\tau_0 = 0.50015$

ν_t^M is indeed a 1st order expansion of ν_t

Conclusions

- Developped a tool to check numerically the balance of kinetic energy and enstrophy accross scales
- Applied it to standard LBM:
  - Hydrodynamics is well recovered at large scales
  - Recovery at small-scales is less good
  - Enstrophy balance highlights higher order M_a terms
- Preliminary results on Entropic LBM:
  - Dissipative properties as $\tau_0 \to 0.5$ are as expected for a LES
  - Malaspinas' $\nu_t^M(\vec{x},\ t)$ was numerically shown to be a 1st order expansion of $\nu_t(\vec{x},\ t) = \nu^{eff}(\vec{x},\ t) - \nu_0$
  - As $\tau_0 \to 0.5$, τ^{eff} variance increase and ν^{eff} can become locally negative
- Systematic analysis of hydrodynamics recovery for Entropic LBM by adding Malaspinas SGS term to the balance equations is on-going

Thank you for your attention!

Any questions?

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