Burgers equation and Fourier Fractal Decimation

Observatoire de Nice
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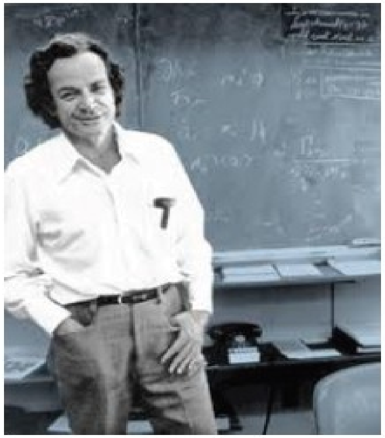
Outline:

1) Why are **Navier-Stokes** equations interesting for Theoretical Physics?

- Strongly non perturbative field Theory (Classical)
- Anomalous Scaling (Non-Gaussian Statistics)



"With turbulence, it's not just a case of physical theory being able to handle only simple cases—we can't do any. We have no good fundamental theory at all." (Feynman, 1979, Omni Magazine, Vol. 1, No.8).

2) Why do we need a model for Navier-Stokes?

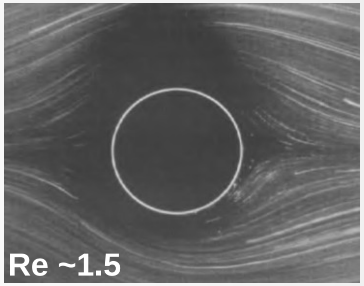
3) Burgers' equation and Fourier Fractal Decimation

Probabilistic description for fully developed Turbulence

Navier-Stokes, (N-S), equations:

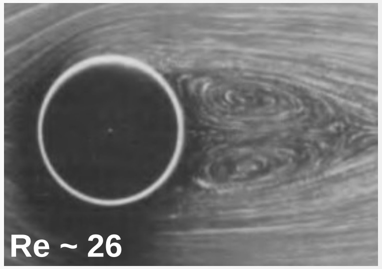
$$\begin{cases} \frac{\partial \mathbf{v}(\mathbf{x},t)}{\partial t} + \mathbf{v}(\mathbf{x},t)\cdot\nabla_x \mathbf{v}(\mathbf{x},t) = -\nabla_x p(\mathbf{x},t) + \nu\Delta_x \mathbf{v}(\mathbf{x},\mathbf{t}) + \mathbf{f}(\mathbf{x},t) \\ \nabla_x \cdot \mathbf{v}(\mathbf{x},t) = 0 \end{cases}$$

$$\begin{cases} \hat{t} = t/t_0 \\ \hat{x} = x/l_0 \\ \hat{v} = v/v_0 \end{cases} \qquad \partial_t \hat{v} + \hat{v}\cdot\partial\hat{v} = -\partial\hat{P} + \frac{1}{Re}\partial^2\hat{v} \qquad Re = \frac{l_0 v_0}{\nu} \qquad Re \sim \frac{\hat{v}\partial\hat{v}}{\nu\partial^2\hat{v}}$$

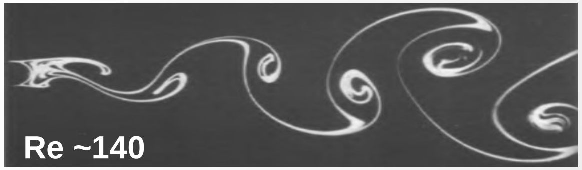


Re

Left-right invariance is broken — ~ 10 — **Recirculating standing eddies**

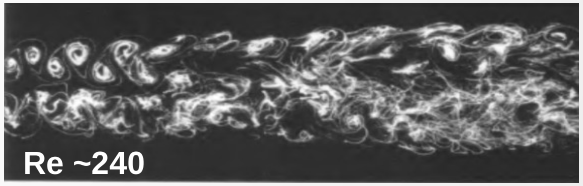


Z-invariance is broken, **discrete time invariance** — **Kàrmàn** street

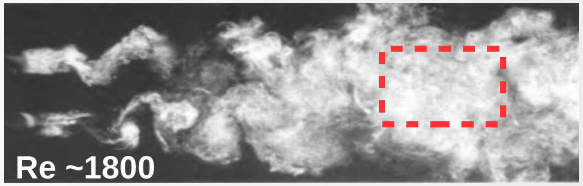


$\sim 10^2$

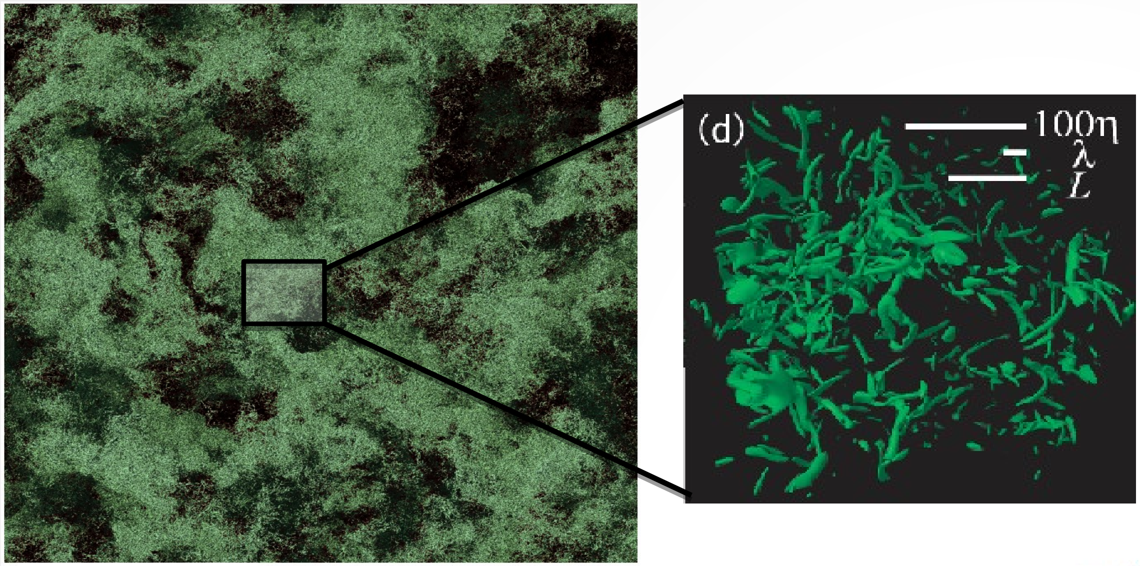
At high Re symmetries are **spontaneously broken** — Flow becomes **chaotic** in its time-dependence



Restored symmetries; (in a statistical sense) — $\sim 10^3$ — Homogeneous-isotropic **fully developed turbulence**



Anomalous Exponents, Small-Scales Intermittency

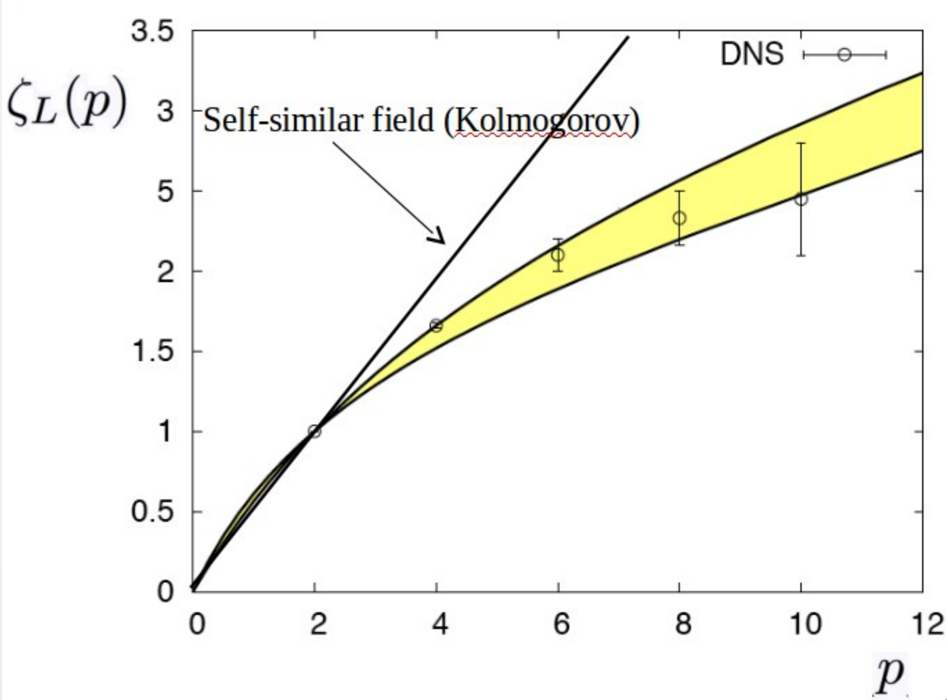


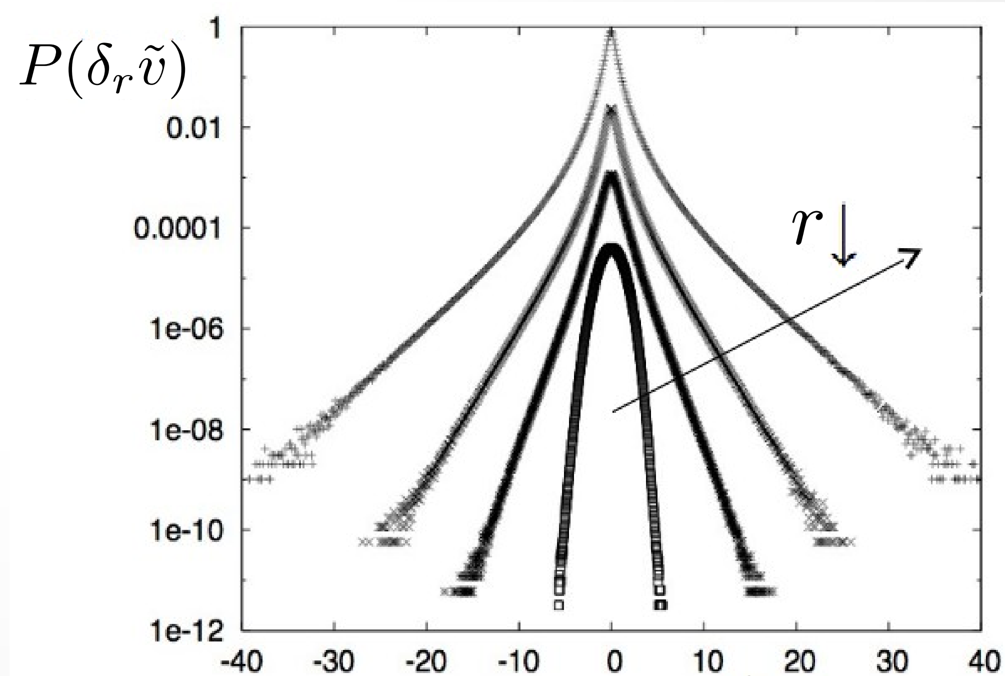
H1) **Restored symmetries** (in a statistical sense).

H2) **Self-similarity** at small scales.

$$S_p(r) =< (\delta_r v)^p > \sim r^{\zeta(p)}$$

$$\delta_r v = v(x+r) - v(x)$$





$$\delta_r \tilde{v} = \frac{\delta_r v}{< (\delta_r v)^2 >^{1/2}}$$

Benzi, Biferale, Fisher, Lamb and Toschi, JFM (2010).

..a model for Turbulence

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

Burgers' equation

u(t,x): velocity field, depending on a variable of time (***t***), and on a variable of space (***x***) | ***ν***: kinematic viscosity

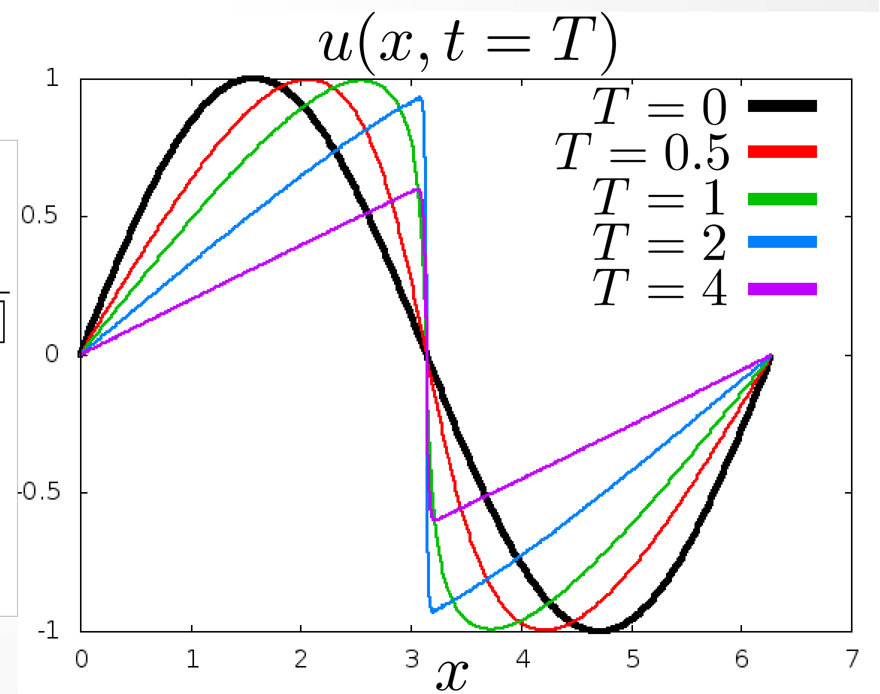
Burgers **produces a singularity**, (**shock**).

Lagrangian observation

$$\begin{cases} u(t, X(t,a)) = u_0(a) \\ X(t,a) = a + t u_0(a) \end{cases} ; J(t,a) = \frac{\partial X}{\partial a} = 1 + t u_0'(a) \; ; t^* = \frac{1}{-inf_a[u_0'(a)]}$$

Gradient in the Eulerian coordinates

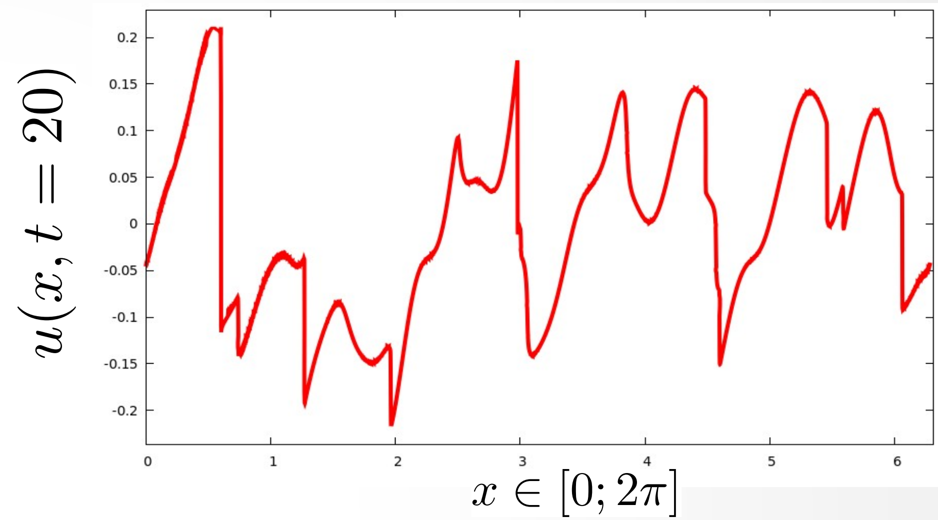
$$\left.\frac{\partial u}{\partial x}\right|_{x^*=a^*} = \left.\frac{\partial u}{\partial a}\right|_{a^*} \left.\frac{\partial a}{\partial x}\right|_{x^*} = u_0'(a)\frac{1}{1+tu_0'(a)} \rightarrow \lim_{t\rightarrow t^*} \frac{u_0'(a)}{1+tu_0'(a)} = \infty$$

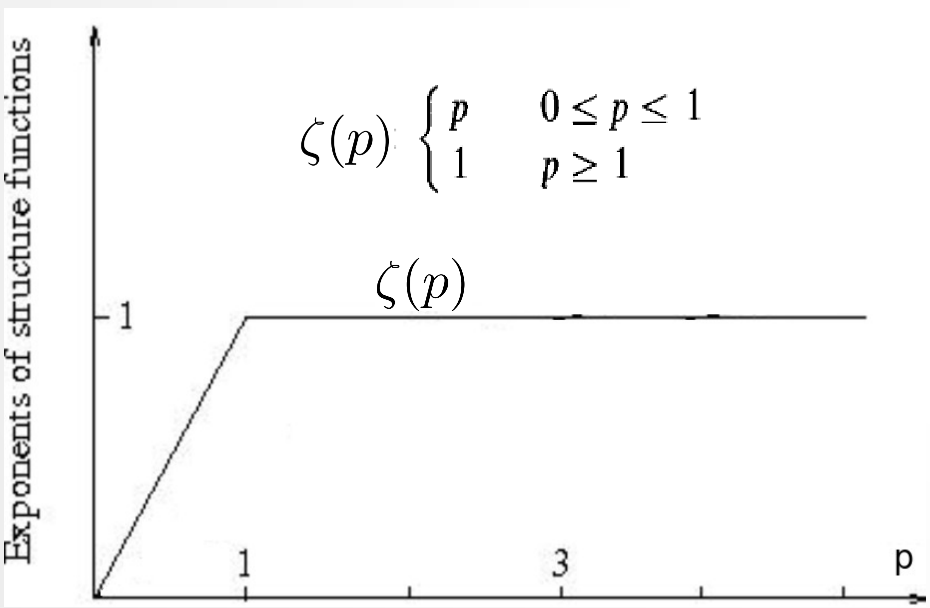


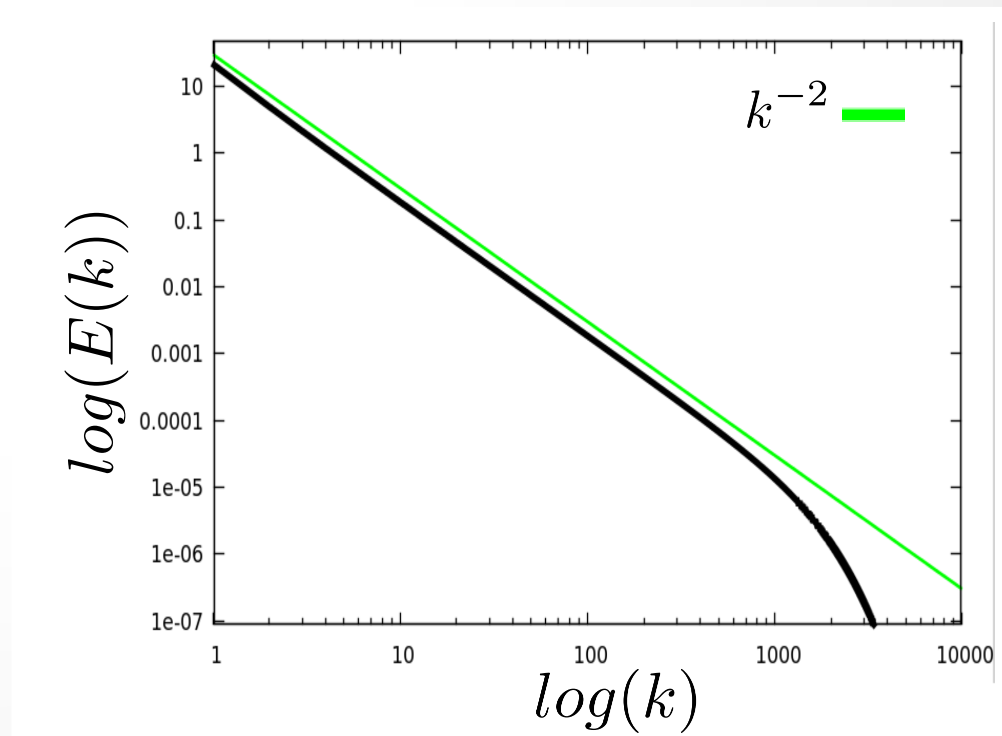
Intermittency on Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2} + f(x,t)$$

$$S_p(r) =< (\delta_r v)^p > \sim r^{\zeta(p)}$$







how many degrees of freedom are related to the singularity?

..Reduce to learn!

Frisch, Pomyalov, Procaccia, and Ray, Turbulence in non-integer dimensions by fractal Fourier decimation. Phys. Rev. Lett. 108, (2012)

FRACTAL FOURIER DECIMATION

$$u(x,t) = \sum_{k \in Z} e^{ikx} u(k,t) \qquad P_D \cdot u(x,t) = \sum_{k \in Z} e^{ikx} \theta_k u(k,t)$$

$$\theta_{\boldsymbol{k}} = \begin{cases} 1 \text{ with probability } h_k \\ 0 \text{ with probability } 1 - h_k \ , \qquad k \equiv |\boldsymbol{k}| \end{cases}$$

$$h_k = (k/k_0)^{D-1}, \quad 0 < D \leq 1$$

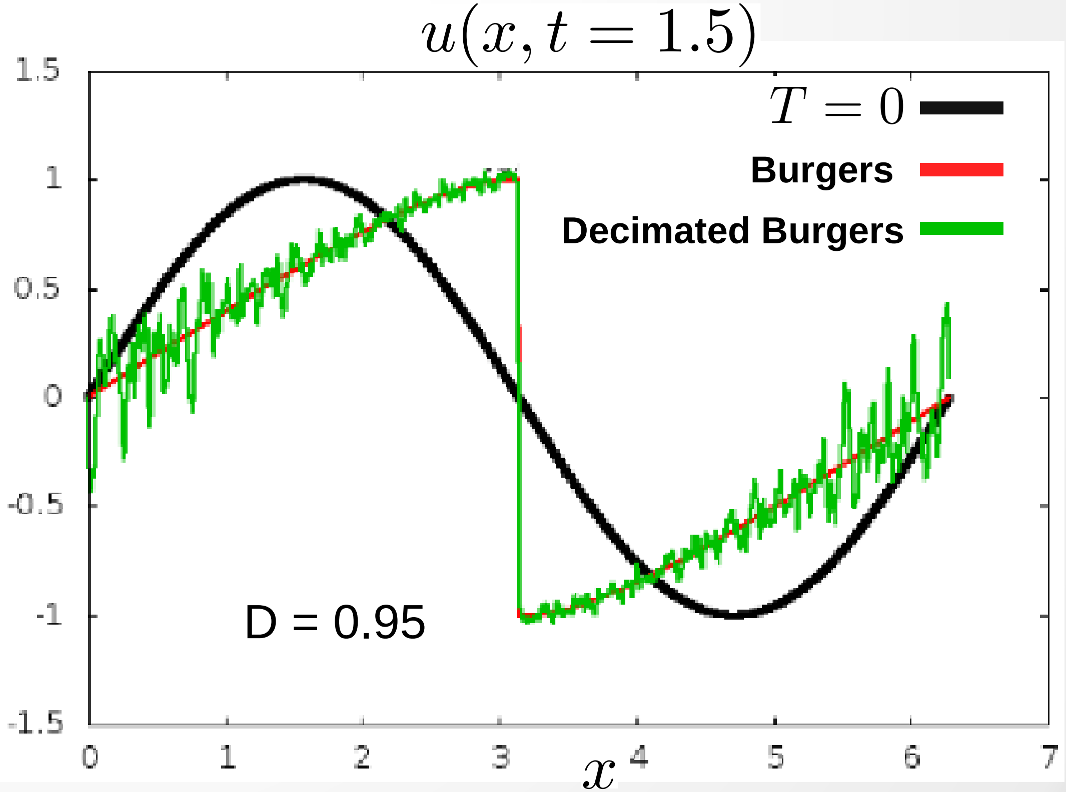
The decimation is Random but Quenched on time,

leaving on average $N(k) \sim k^D$ **active mode**

Galerkin truncation projection: $k < k_{max}$

⇩

- **Finite number of d.o.f.**
- **Fractal dimension**

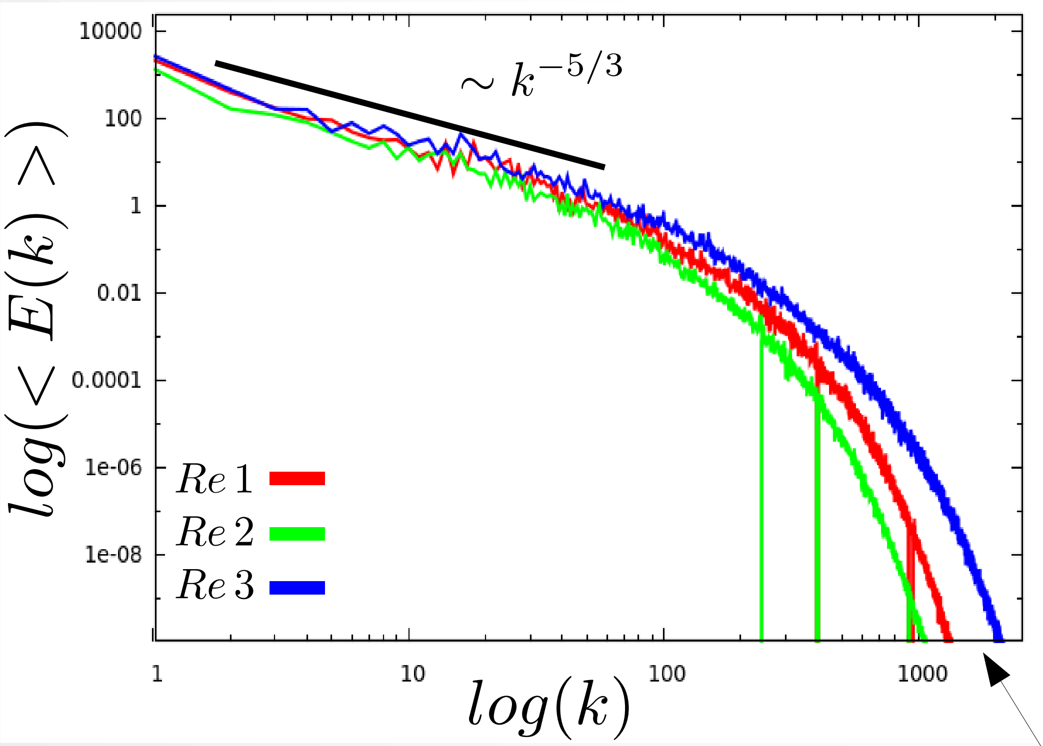


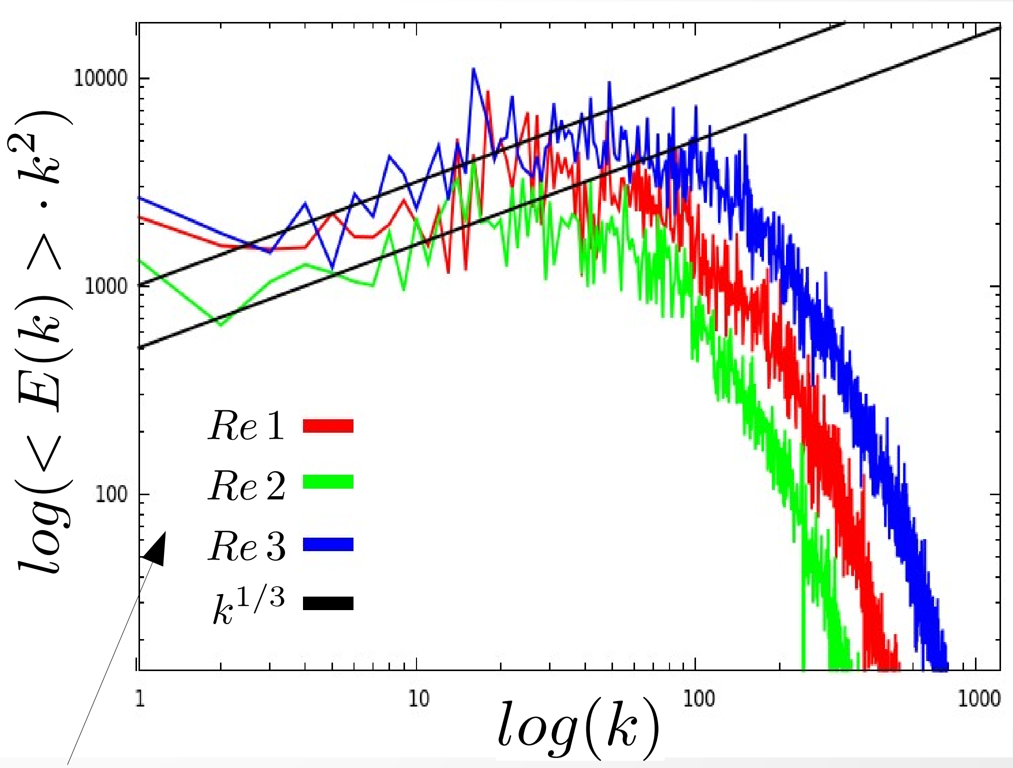
Decimated Energy Spectrum

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

Mean spectra:

1°) Averaged on time in the stationary state
2°) Averaged on different decimation mask (D = 0.95)

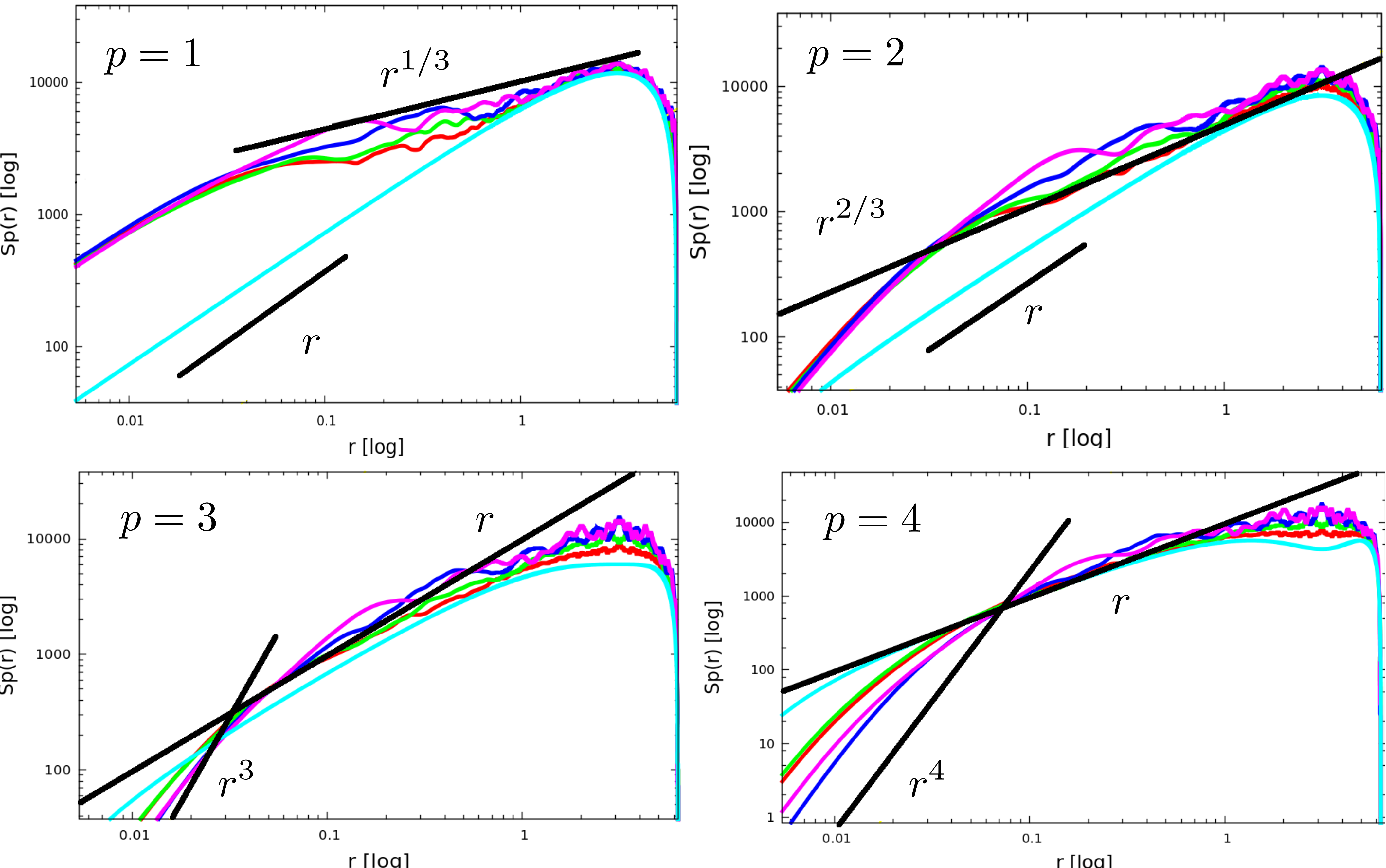




Different Re

Structure Functions

$$S_p(r) =< (\delta_r v)^p > \sim r^{\zeta(p)}$$

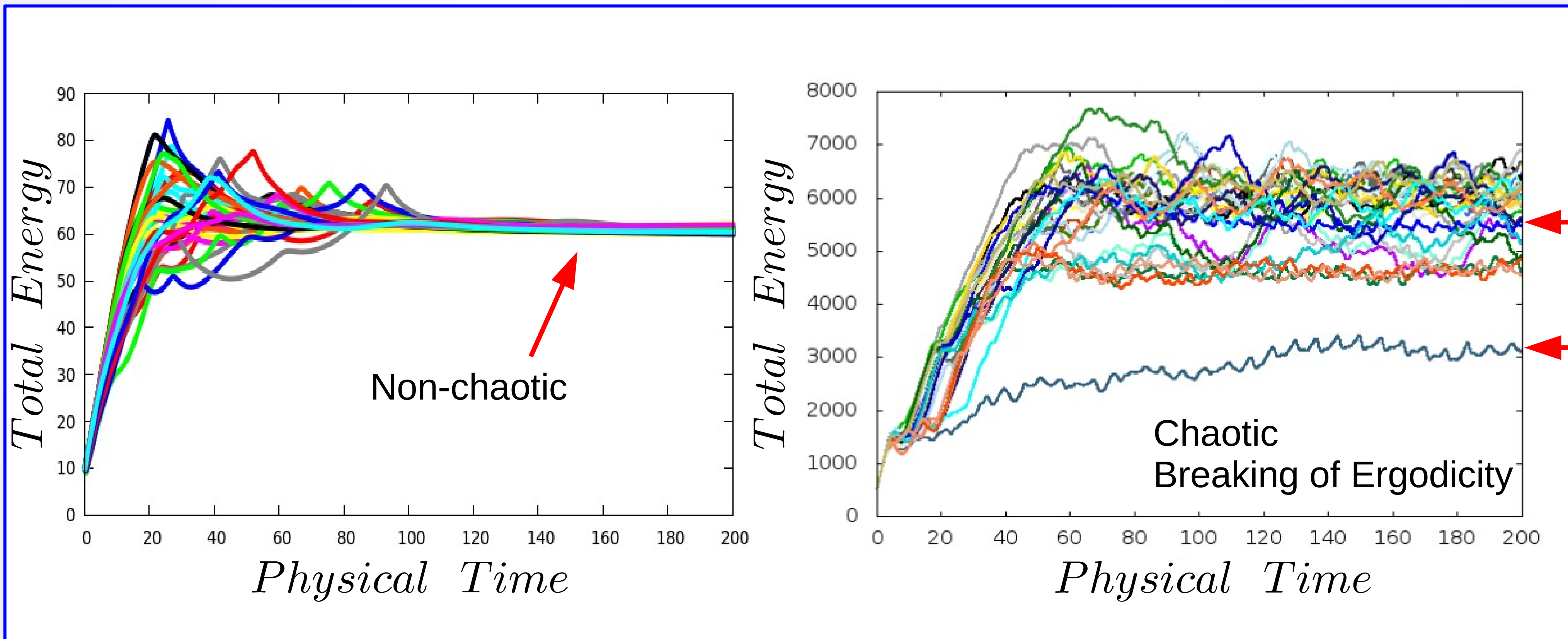


Attractor differences (and properties ..?)

Different Initial Conditions:

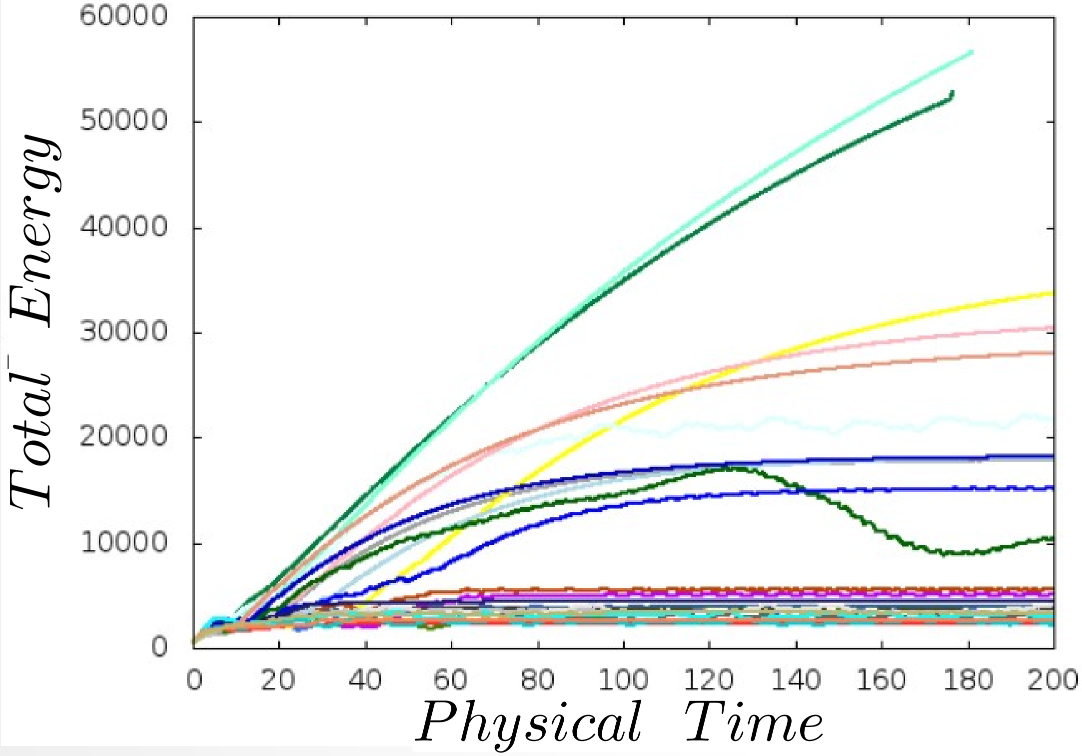
Forced Burgers

Forced Decimated Burgers
(same mask; D = 0.95)

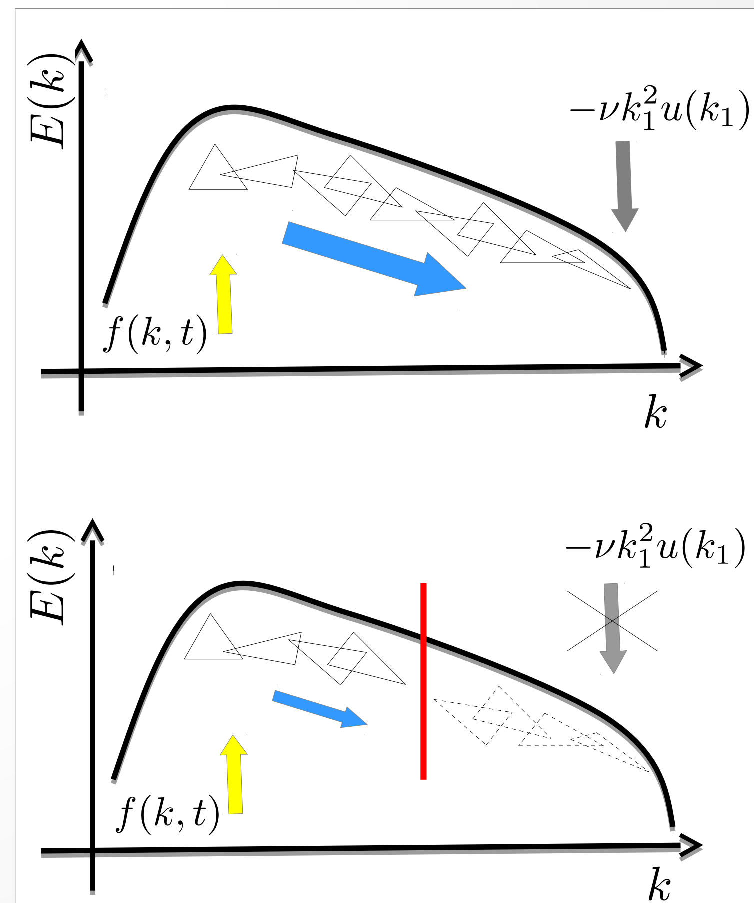


..Non Self-Averaging

Total energy evolution: **different masks**

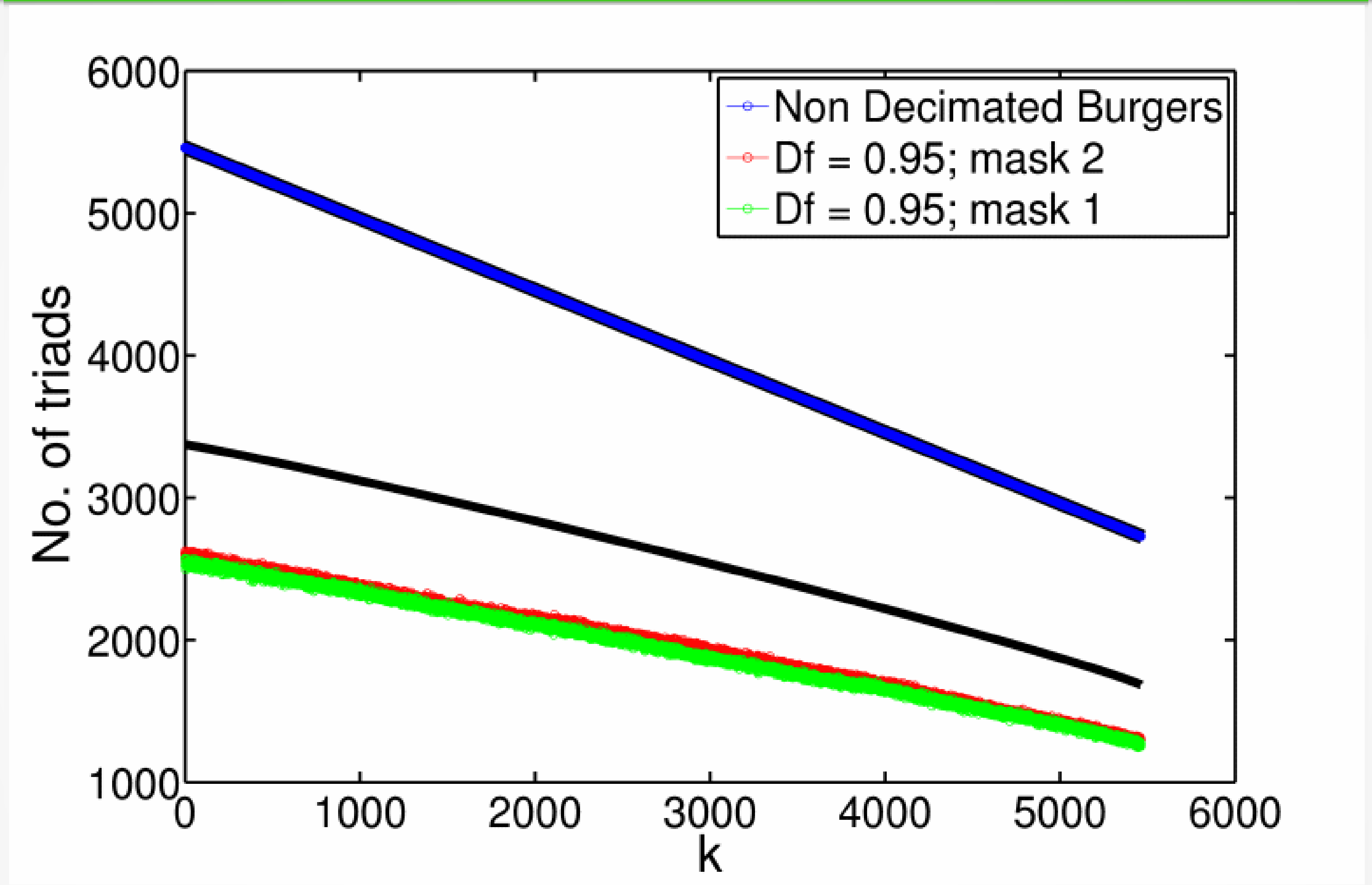


$$\frac{\partial u(k_1,t)}{\partial t} = \sum_{k_2+k_3=k_1} \Pi(u(k_2),u(k_3)) - \nu k_1^2 u(k_1) + f(k,t)$$



Block in the **energy transfer**

1D and Decimated Triads



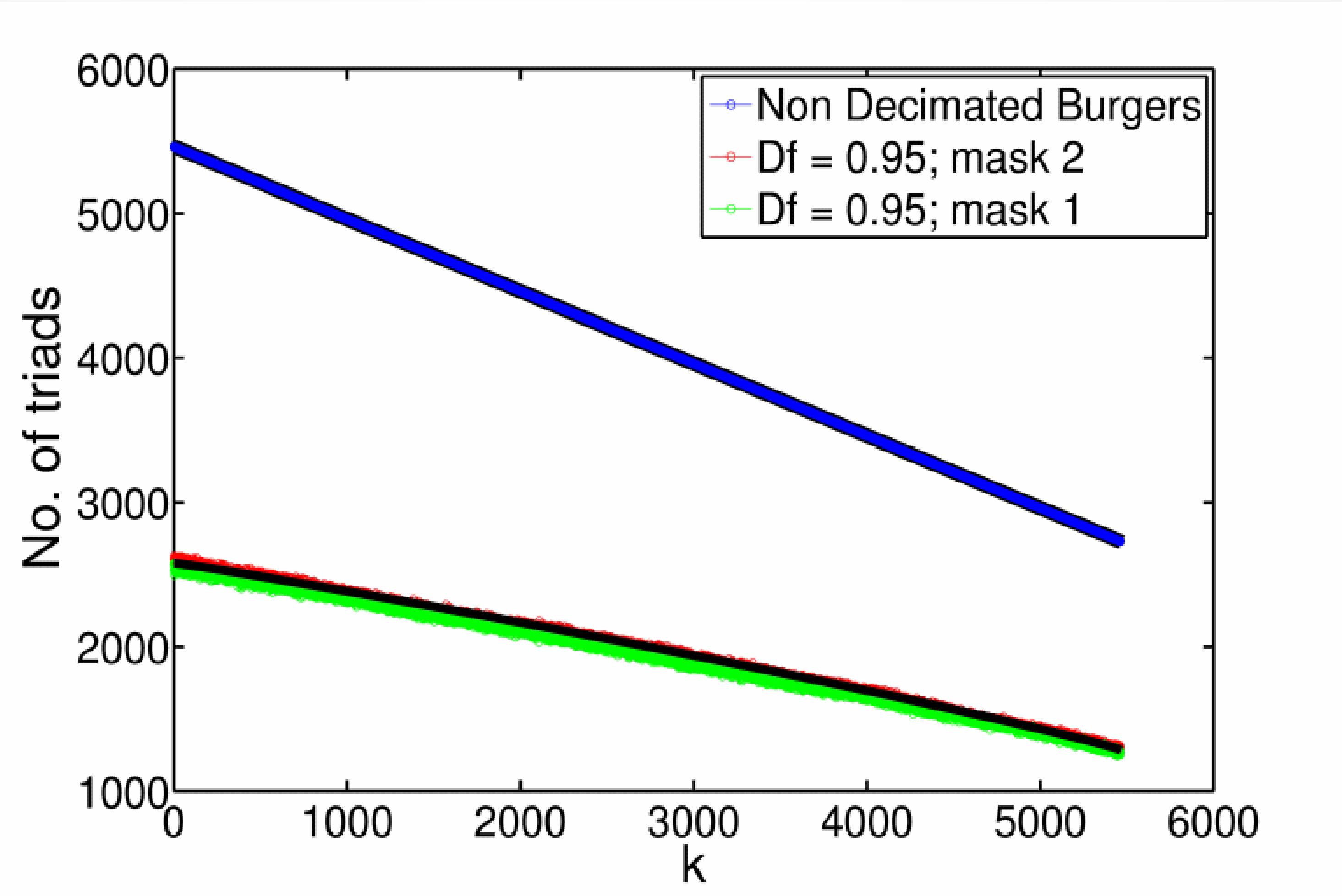
$$\frac{k}{2} + (N - k)$$

No. of triads in 1-D

$$\frac{k^D}{2} D + (N - k)^D D$$

No. of triads in D-Dim.

1D and Decimated Triads



$$\frac{k^D}{2}D + (N-k)^D D \quad \rightarrow \quad \left(\frac{k^D}{2}D + (N-k)^D D\right)\ \frac{N - N^D D}{N/2}$$

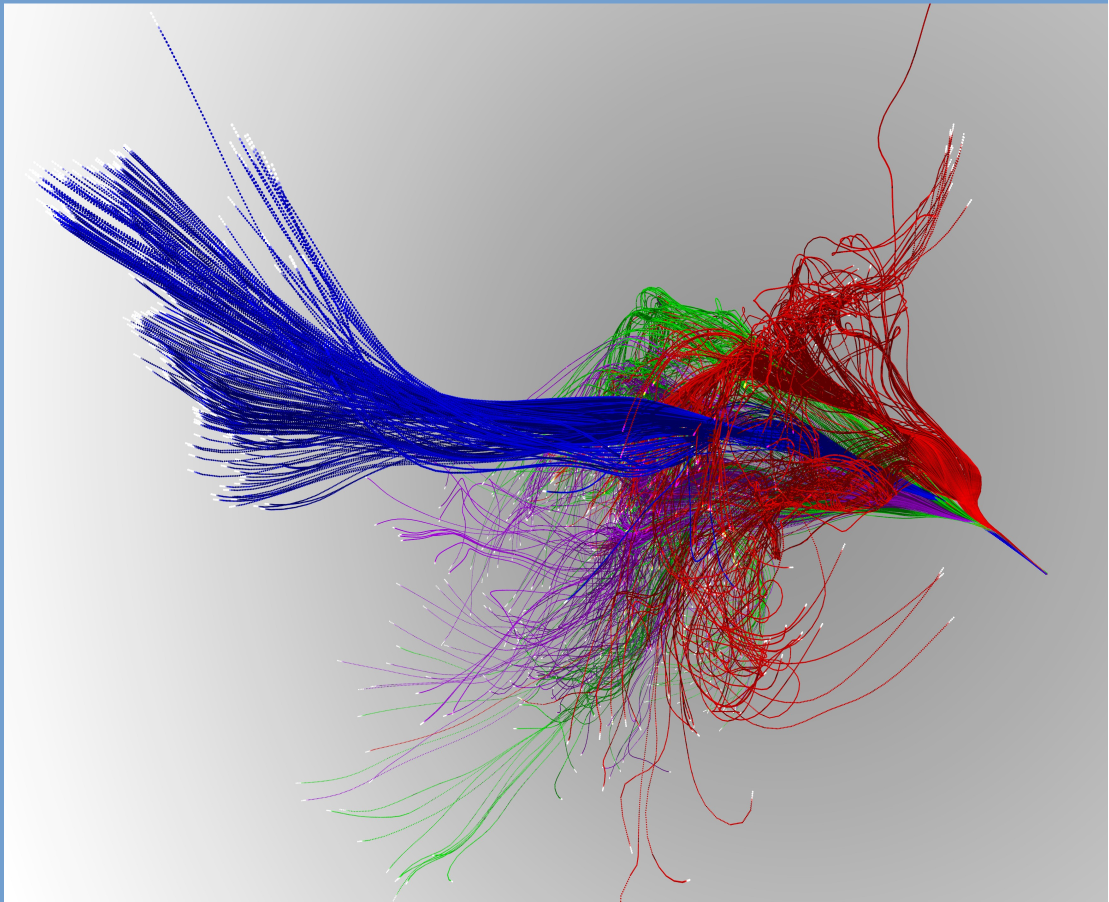
Conclusions:

1) Differences in the statistics arise on changing the fractal dimension D

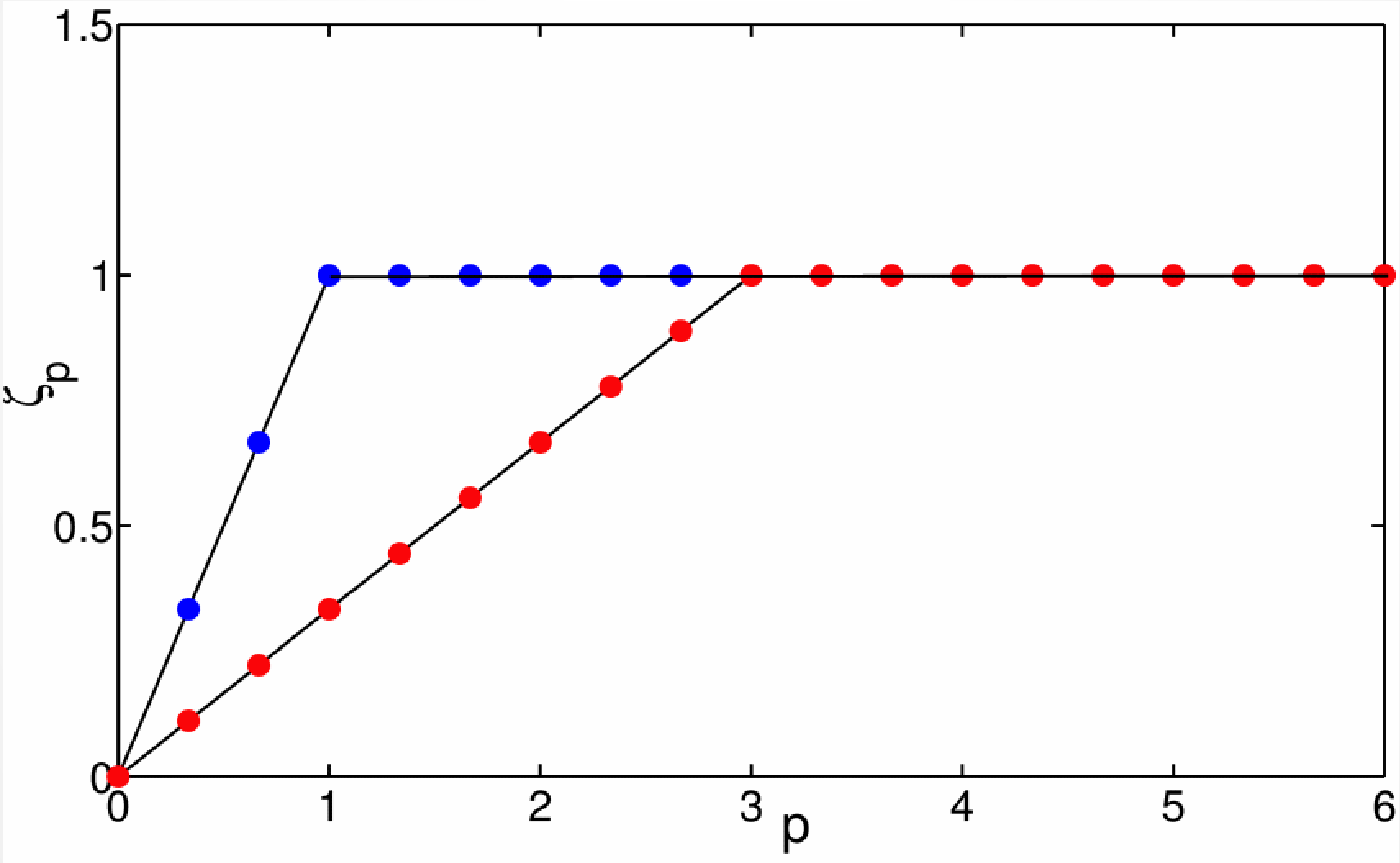
2) The system begins to be chaotic for D values very close to 1 (D ~ 0.98)

3) Non-Ergodicity

4) Problem of Non Self-Averaging



.. Navier-Stokes?



1.5
1
0.5
0
ζp
0
1
2
3
4
5
6
p

Intermittecy on Burgers' equation

..BIFRACTAL MODEL

$$\frac{\delta v_\ell(r)}{v_0} \sim \begin{cases} \left(\frac{\ell}{\ell_0}\right)^{h_1}, & r \in \mathscr{S}_1,\ \dim \mathscr{S}_1 = D_1 \\ \left(\frac{\ell}{\ell_0}\right)^{h_2}, & r \in \mathscr{S}_2,\ \dim \mathscr{S}_2 = D_2 \end{cases}$$

$$\frac{\langle \delta v_\ell^p \rangle}{v_0^p} \propto \left(\frac{\ell}{\ell_0}\right)^{ph_1} \left(\frac{\ell}{\ell_0}\right)^{1-D_1} + \left(\frac{\ell}{\ell_0}\right)^{ph_2} \left(\frac{\ell}{\ell_0}\right)^{1-D_2}$$

Probability *to be within a distance l of the set* $\mathscr{S}$

$$\begin{cases} D_1 = 0 \ ; \ h_1 = 0 & \leftarrow \text{isolated shock} \\ D_2 = 1 \ ; \ h_2 = 1 & \leftarrow \text{smooth ramps} \end{cases}$$

$$\frac{\langle \delta v_\ell^p \rangle}{v_0^p} \propto \left(\frac{\ell}{\ell_0}\right)^{1} + \left(\frac{\ell}{\ell_0}\right)^{p}$$