HPC Methods for Computational Fluid Dynamics and Astrophysics
@Cineca

https://www.fisica.uniroma2.it/~biferale/HPC-LEAP.html

Pseudo-spectral methods for HPC of MHD turbulence

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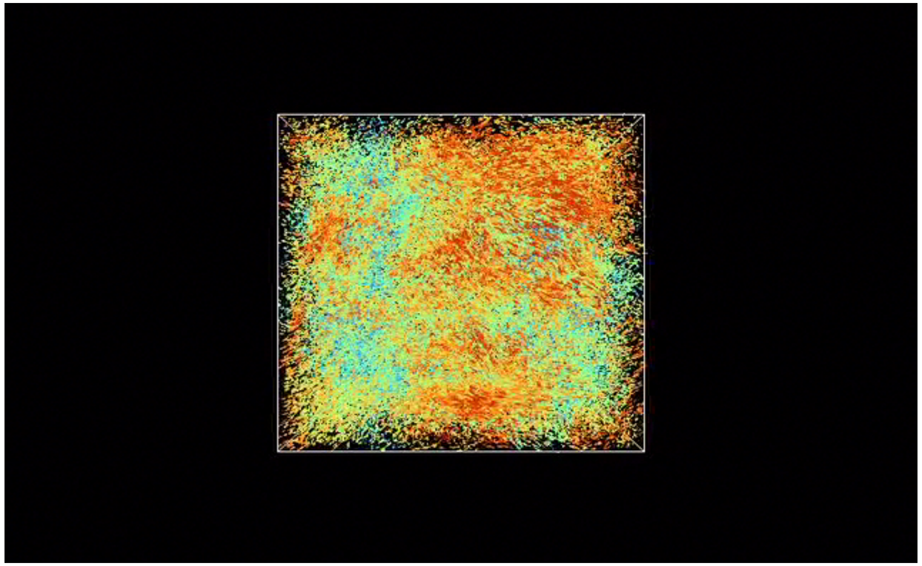

Tor Vergata
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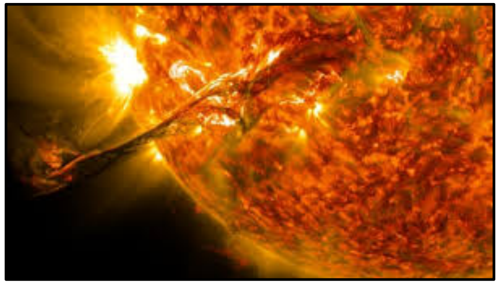
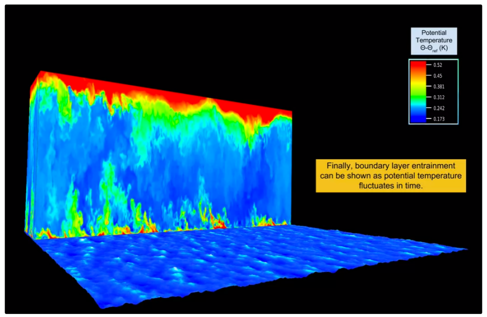
WHY PSEUDO-SPECTRAL?

1) MULTI-SCALES ACCURACY + EXPONENTIAL ACCURACY FOR DERIVATIVES
2) POTENTIAL TOOL TO PERFORM EXPERIMENTS IN SILICO: EXACT CONTROL OF THE DEGREES-OF-FREEDOM

- Too many turbulences?
- Can we disentangle universal from non-universal properties?
- Can we understand universal properties ?
- Does ‘computing’ mean ‘understanding’? (Computo ergo sum?)





Turbulence or Turbulences?

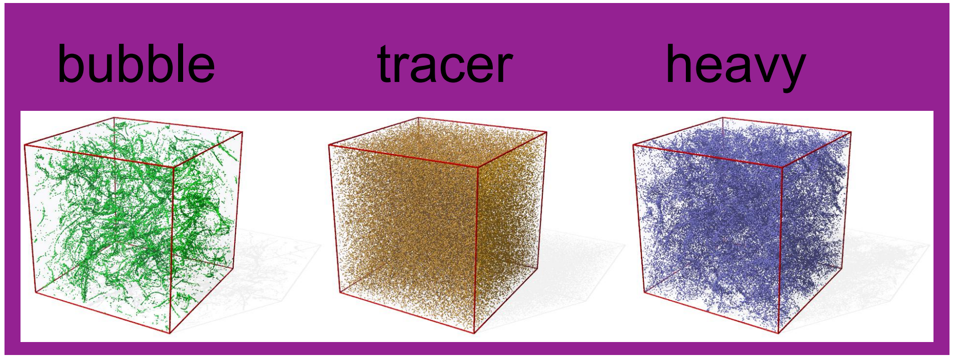
$$\begin{cases} \partial_t v + v \cdot \partial v = -\partial P + \nu \partial^2 v + F(\boldsymbol{B},\boldsymbol{B}) + \boldsymbol{g}\theta + \sum_i c_0(\boldsymbol{u}_i,\boldsymbol{v})\delta(\boldsymbol{r}-\boldsymbol{r}_i) + f \\ \partial_t \theta + \boldsymbol{v} \cdot \boldsymbol{\partial}\theta = \chi \partial^2 \theta \\ \partial_t \boldsymbol{B} + \boldsymbol{v} \cdot \boldsymbol{\partial B} = \boldsymbol{B} \cdot \boldsymbol{\partial v} + \chi \partial^2 \boldsymbol{B} \\ \Delta P = -\partial_i \partial_j v_i v_j \\ \text{+ boundary conditions} \end{cases}$$

temperature

magnetic field

$$\begin{cases} \dfrac{d\boldsymbol{u}_i(\boldsymbol{r}_i,t)}{dt} = -\rho_f|\boldsymbol{u}_i - \boldsymbol{v}|(\boldsymbol{u}_i - \boldsymbol{v}) \\ +\rho_f(\dfrac{D\mathbf{v}}{Dt} - \dfrac{D\boldsymbol{u}_i}{Dt}) + (\boldsymbol{u}_i - \boldsymbol{v}) \times \boldsymbol{\omega} \end{cases}$$

small particles: drag, added mass, lift force, etc...



Flows with additives:

Advection-diffusion-reaction of passive scalar/vectors (temperature, magnetic field, chemical reactions, etc...)
Advection-diffusion of active scalars/vectors (convection, magnetic dinamo)
Polymers (drag reduction)
Bubbles/Droplets (two phase flows, rain formation, etc...)
Swimmers (cooperative hydrodynamical interactions)

$$
\begin{cases}
\partial_t v + v \cdot \partial v = -\partial P + \nu \partial^2 v \;\; + f \; + F(B,B) \\
\partial_t \boldsymbol{B} + \boldsymbol{v} \cdot \boldsymbol{\partial B} = \boldsymbol{B} \cdot \boldsymbol{\partial v} + \chi \partial^2 \boldsymbol{B} \\
\Delta P = -\partial_i \partial_j v_i v_j \\
\text{+ periodic boundary conditions}
\end{cases}
$$

- -homogeneous
- -isotropic
- -Gaussian
- -white-noise in time
- -large-scale

HOMOGENEOUS & ISOTROPIC MHD TURBULENCE

3D CASE: MAINLY UNSOLVED!

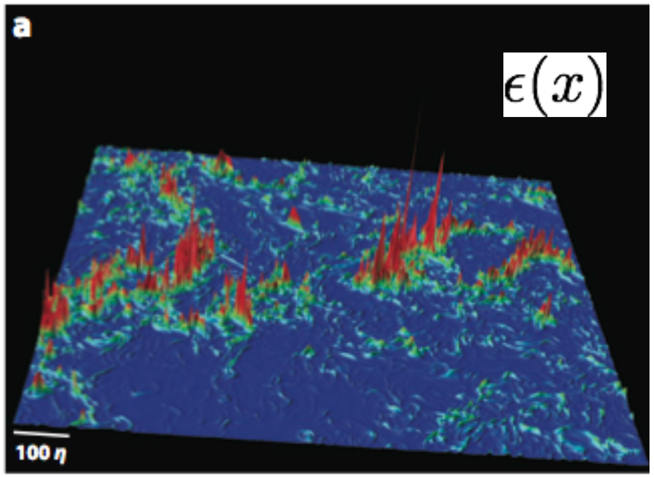
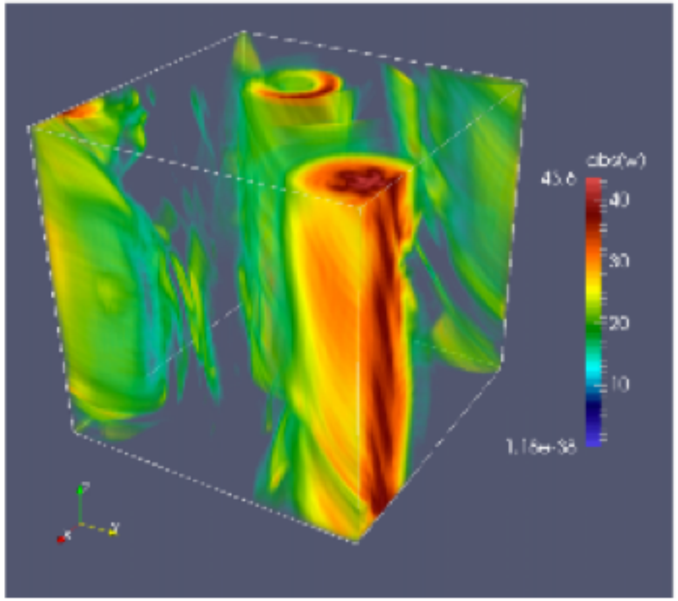


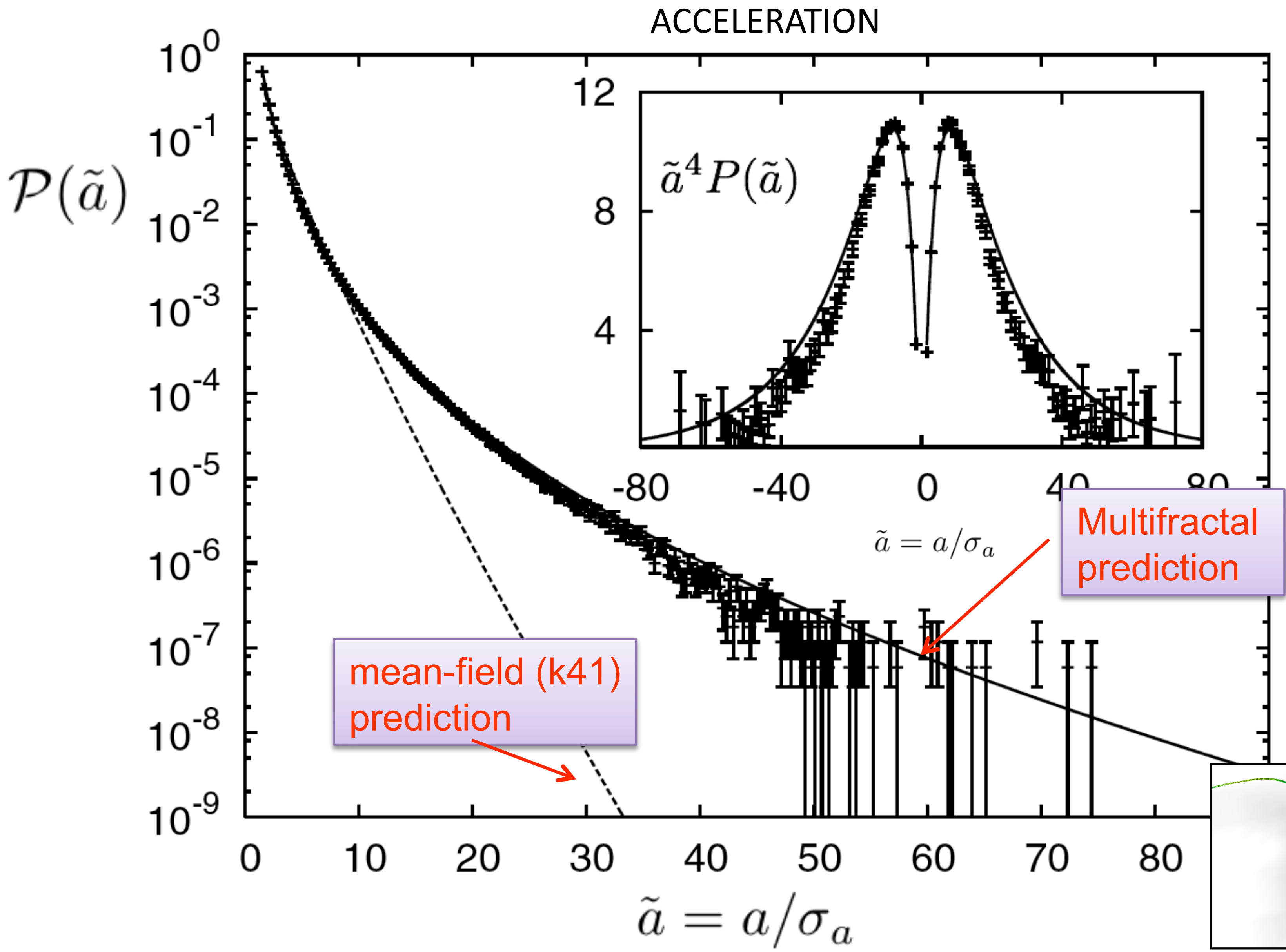
Figure 4

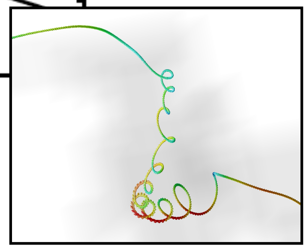
Snapshot of the intensity distributions of (*a*) the energy-dissipatio
DNS-ES at $R_\lambda = 675$ in arbitrary units.



Dallas & Alexakis PoF 2015

L. B., G. Boffetta, A. Celani, B. Devenish, A. Lanotte and F. Toschi *Phys. Rev. Lett.* **93**, 064502, 2004





$$\mathcal{P}(a) \sim \int_{h\in I} \mathrm{d}h\, a^{\frac{h-5+D(h)}{3}} \nu^{\frac{7-2h-2D(h)}{3}} L_0^{D(h)+h-3} \sigma_v^{-1} \times \quad \exp\left(- \frac{a^{\frac{2(1+h)}{3}} \nu^{\frac{2(1-2h)}{3}} L_0^{2h}}{2\sigma_v^2} \right)$$

COMPLEX PHYSICS WITH SIMPLE FLOWS

$$\partial_{\hat{t}}\hat{v} + \hat{v}\cdot\partial\hat{v} = -\hat{\partial}\hat{P} + \frac{1}{Re}\hat{\partial}^2\hat{v}$$

$$\begin{cases} \hat{t} = t/t_0 \\ \hat{x} = x/l_0 \\ \hat{v} = v/v_0 \end{cases}$$

$$Re \sim \frac{v\partial v}{\nu\partial^2 v} \qquad Re = \frac{l_0 v_0}{\nu}$$

Reynolds number ~ (Non-Linear)/(Linear terms)

$$Re \to \infty$$

Fully Developed Turbulence:

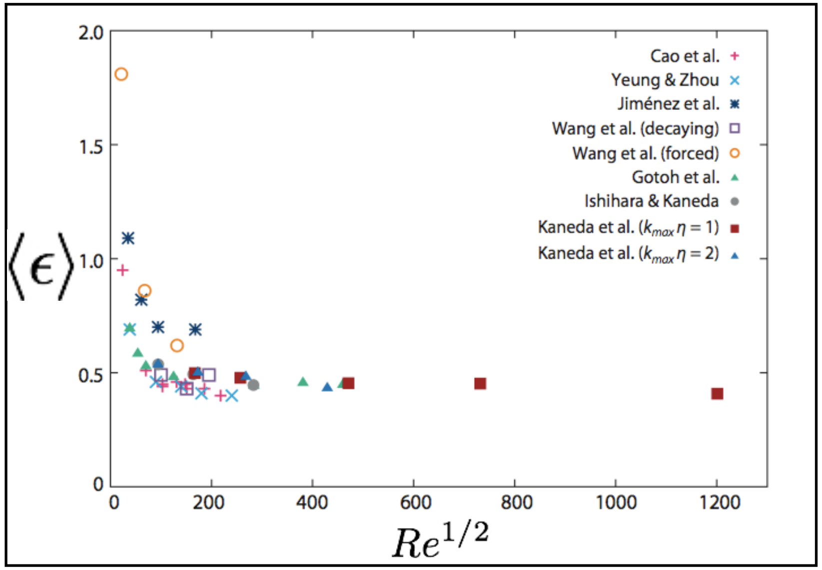
1. Strongly non-linear & non-perturbative system

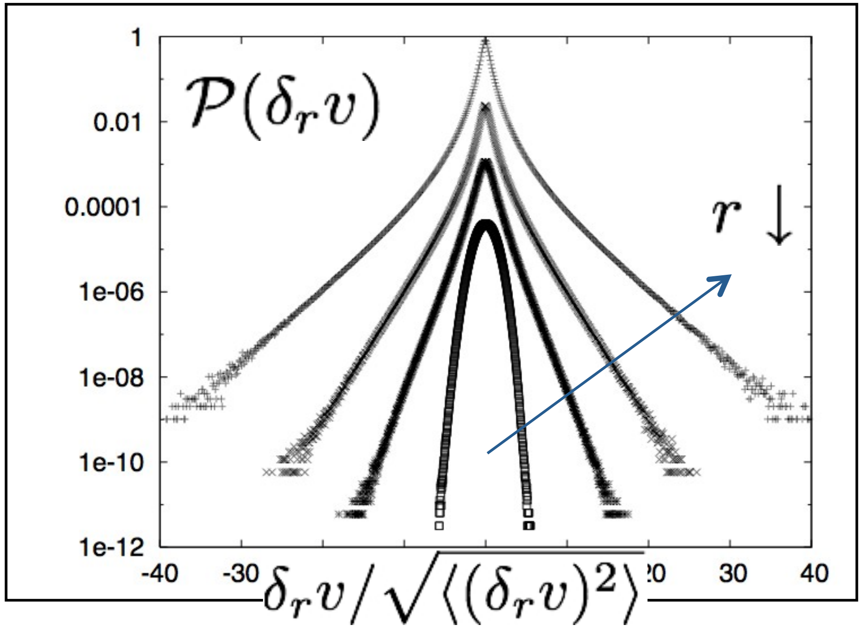
COMPLEX PHYSICS WITH SIMPLE FLOWS

$$\partial_{\hat{t}}\hat{v} + \hat{v}\cdot\hat{\partial}\hat{v} = -\hat{\partial}\hat{P} + \frac{1}{Re}\hat{\partial}^2\hat{v}$$

$$Re \to \infty$$

$$\langle\epsilon\rangle = \nu\langle(\partial v)^2\rangle \propto \frac{1}{Re}\langle(\partial v)^2\rangle \to const.$$





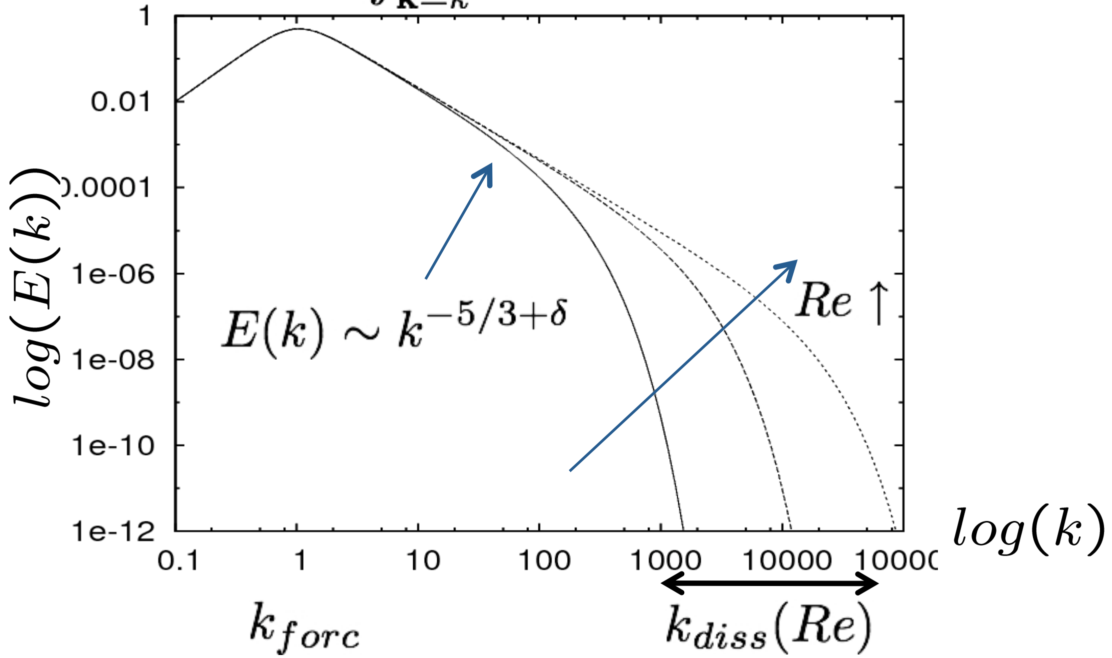
2. Out of Equilibrium (non perturbative)
Dissipative anomaly

3. Small-scales PDFs strongly non-Gaussian
Anomalous scaling

COMPLEX PHYSICS WITH SIMPLE FLOWS

THE ENERGY CASCADE:

$$E(k) = \int_{\mathbf{k}=k} d^3k \langle v_i(\mathbf{k}) v_i(-\mathbf{k}) \rangle$$



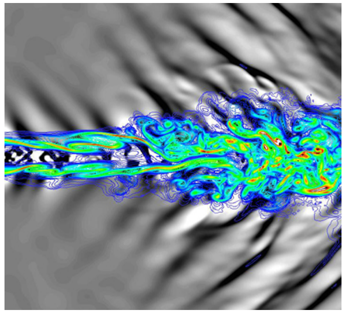
$$k_{forc} \ll k \ll k_{diss}(Re)$$

1. inertial range of scales: power law (anomalous)
2. extension increases with Reynolds!

4. Many-body problem:

$$\#_{dof} = \left(\frac{k_{diss}}{k_{forc}}\right)^3 \sim Re^{9/4}$$

Number crunching approach: **computo ergo sum**.



laboratory flow

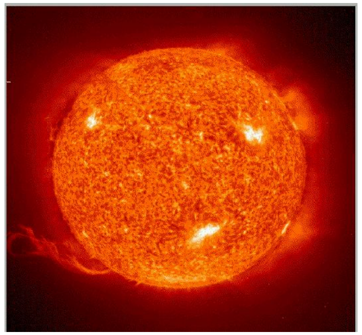
$$Re \sim 10^5 - 10^9$$

$$\#_{dof} \sim 10^{11} - 10^{20}$$

atmosph. flow

$$Re \sim 10^8 - 10^{12}$$

$$\#_{dof} \sim 10^{18} - 10^{30}$$



astrophys. flow

$$Re > 10^{15}$$

$$\#_{dof} \sim \infty$$

state-of-the-art DNS (Kaneda's group):

Isotropic, homogeneous Fully Periodic Flows
Pseudo-Spectral Methods.
Resolution 12000^3

Reynolds : 10^7,
Storage of 1 velocity configuration: 30 Tbyte

Moral: easy to saturate any computing power (present and/or future)

Pseudo-spectral method

$$\partial_t v + v \cdot \partial v = -\partial P + \nu \partial^2 v$$

- Spectral: Discretize the fields on a Fourier series
- Evaluate the terms local in Fourier space
- Pseudo-spectral: Evaluate the convolution term in real space, then move back to Fourier space

- *Chebyshev and Fourier Spectral Methods, Second Edition, John P. Boyd, DOVER Publications, Inc. (2000)*
- *Canuto et al. Spectral Methods in Fluid Dynamics. Book (1988)*
- *Rogallo. Numerical Experiments in Homogeneous Turbulence. NASA Tech. Memo. (1981) vol. 81315* pp. 1-92

c

Treatment of Pressure

Helmholtz Theorem $\nabla^2 p = -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})$

$\mathbf{u} = \nabla \phi + \nabla \times \xi$ Vector potential

Transversal component $\boldsymbol{\nabla} \times \xi$

Longitudinal component $\boldsymbol{\nabla} \phi$

$\nabla \cdot \mathbf{u} = 0 \Rightarrow \boldsymbol{\nabla} \phi = 0$

$\omega = \nabla \times \mathbf{u}$ $\omega = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \xi = -\Delta \xi$

$\xi = -\Delta^{-1} \omega$

Treatment of Pressure (Fourier)

Helmholtz Theorem

$$\mathbf{u} = \nabla\phi + \nabla \times \xi \longrightarrow \boldsymbol{u}(\boldsymbol{k}) = -i\boldsymbol{k} \times \xi(\boldsymbol{k})$$

$$\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{u}^*(\boldsymbol{r}) \qquad \hat{\boldsymbol{u}}(\boldsymbol{k}) = \hat{\boldsymbol{u}}^*(-\boldsymbol{k})$$

Implication on memory storage requirements

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} \qquad \boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \xi = -\Delta\xi$$

$$\xi = -\Delta^{-1}\boldsymbol{\omega}$$

Equation in Fourier space

$$\nabla^2 p = -\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}$$

$$\partial_t \xi = \frac{i\mathbf{k}}{|\mathbf{k}|^2} \times \mathcal{F}(\mathbf{u} \times \boldsymbol{\omega})_{\mathbf{k}} - \nu |\mathbf{k}|^2 \xi$$

Equation in Fourier space

$$\partial_t \xi = \frac{i\mathbf{k}}{|\mathbf{k}|^2} \times \mathcal{F}(\mathbf{u} \times \boldsymbol{\omega})_{\mathbf{k}} - \nu |\mathbf{k}|^2 \xi$$

Non linear term (NLT) is a convolution in Fourier space

$\longrightarrow$ in real space NTL is a scalar product

$$\mathcal{F}(\mathbf{u} \times \boldsymbol{\omega})_{\mathbf{k}}$$

Fourier		Real		Fourier
$\boldsymbol{k}$	$\longrightarrow$	$\boldsymbol{x}$	$\longrightarrow$	$\boldsymbol{k}$

Pseudo-spectral method

VISCOUS TERM EXACTLY INTEGRATED
ADAMS-BASHFORTH 2° ORDER

Time marching

$$\partial_t \xi = \frac{i\mathbf{k}}{|\mathbf{k}|^2} \times \mathcal{F}(\mathbf{u} \times \boldsymbol{\omega})_{\mathbf{k}} - \nu |\mathbf{k}|^2 \xi$$

$$\mathbf{N}_{\mathbf{k}} = (N_{1\mathbf{k}}, N_{2\mathbf{k}}, N_{3\mathbf{k}}) = \frac{i\mathbf{k}}{|\mathbf{k}|^2} \times \mathcal{F}(\mathbf{u} \times \boldsymbol{\omega})_{\mathbf{k}}$$

$$\partial_t \xi + \nu k^2 \xi = N_k$$

$$G(t) = exp(\nu k^2 t)$$

$$\xi_k^{n+1} = \xi_k^n + \frac{dt}{2}[3G(t^n)N_k^n - G(t^{n-1})N_k^{n-1}]$$

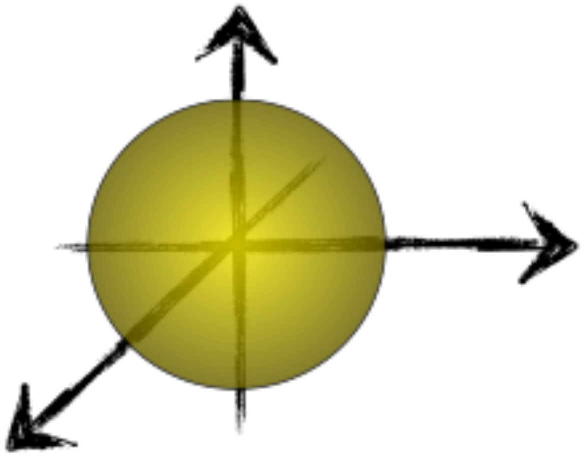
Forcing schemes

- Many possibilities... non unique choice, i.e. non universality at large scales.
- *Inertial range turbulence* is universal with respect to the forcing **but**:
  - Different forcing may affect more or less directly and severely the extension of the inertial range
  - Different forcing may make the large scales more or less isotropic

WHAT ABOUT MHD????

Frozen amplitude

$\boldsymbol{K}_f : |\boldsymbol{k}| \leq k_t$ (e.g. $k_t = 2$)

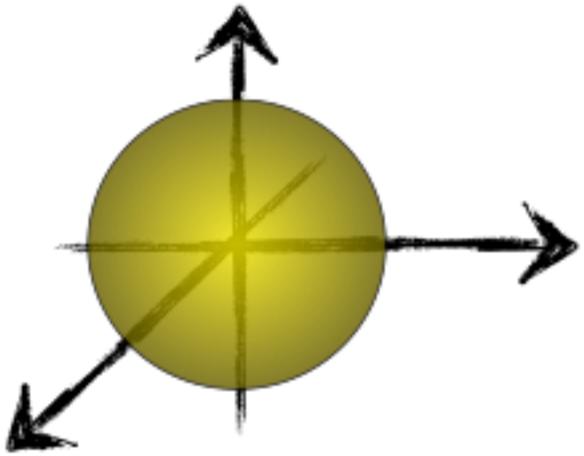


shell of small wavenumbers

$$\boldsymbol{u}(\boldsymbol{k}, t) \equiv \boldsymbol{u}_0(\boldsymbol{k}) \qquad \boldsymbol{k} \in \boldsymbol{K}_f$$

Constant energy input

$$\boldsymbol{K}_f : |\boldsymbol{k}| \leq k_t \qquad \text{(e.g. } k_t = 2 \text{)}$$



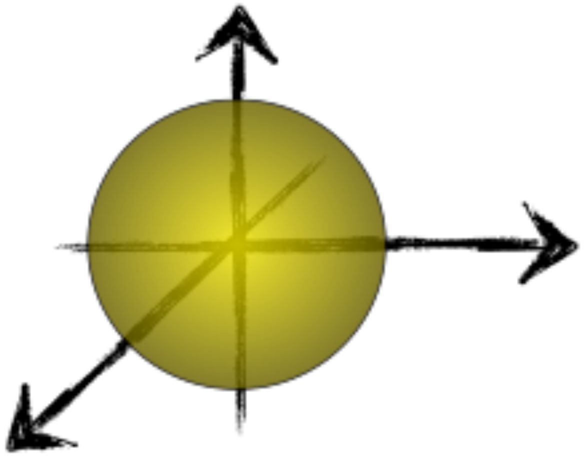
shell of small wavenumbers

$$\boldsymbol{f}(\boldsymbol{k},t) = \varepsilon \cdot \frac{\boldsymbol{u}(\boldsymbol{k},t)}{\sum_{\boldsymbol{k}\in\boldsymbol{K}_f} |\boldsymbol{u}(\boldsymbol{k},t)|^2} \qquad \boldsymbol{k} \in \boldsymbol{K}_f$$

$$\boldsymbol{u}\boldsymbol{f}(\boldsymbol{k},t) = \varepsilon \cdot \frac{|\boldsymbol{u}(\boldsymbol{k},t)|^2}{\sum_{\boldsymbol{k}\in\boldsymbol{K}_f} |\boldsymbol{u}(\boldsymbol{k},t)|^2} \qquad \sum_{\boldsymbol{k}\in\boldsymbol{K}_f} \boldsymbol{u}\boldsymbol{f}(\boldsymbol{k},t) = \varepsilon \cdot \frac{\sum_{\boldsymbol{k}\in\boldsymbol{K}_f} |\boldsymbol{u}(\boldsymbol{k},t)|^2}{\sum_{\boldsymbol{k}\in\boldsymbol{K}_f} |\boldsymbol{u}(\boldsymbol{k},t)|^2} = \varepsilon$$

Appended modes

$\boldsymbol{K}_f : |\boldsymbol{k}| \leq k_t$ (e.g. $k_t = 2$)



shell of small wavenumbers

Rescale amplitude to keep energy fixed in a shell of small wavenumbers

$$\boldsymbol{u}_k \longrightarrow \alpha \boldsymbol{u}_k \qquad \boldsymbol{k} \in \boldsymbol{K}_f$$

$$\mathcal{E}_{k_c} \equiv \sum_{|\boldsymbol{k}|<k_c} |\boldsymbol{u}_k|^2 \qquad \overline{\mathcal{E}}_{k_c} \equiv \alpha^2 \sum_{|\boldsymbol{k}|<k_c} |\boldsymbol{u}_k|^2$$

Aliasing

"Blowup of an aliased, non-energy-conserving model is God's way of protecting you from believing a bad simulation." J. P. Boyd

- from P. Boyd. Chebyshev and Fourier spectral methods. Second edition (Revised). (2001) pp. 668

Spectral blocking & 2/3 rule

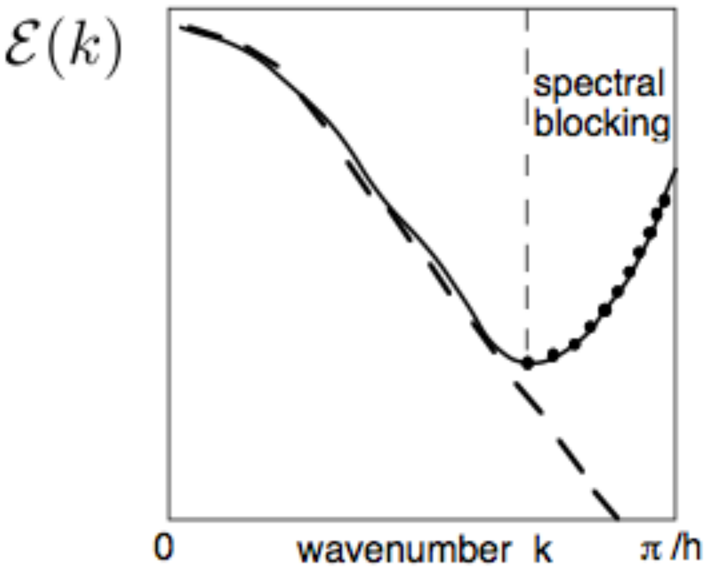
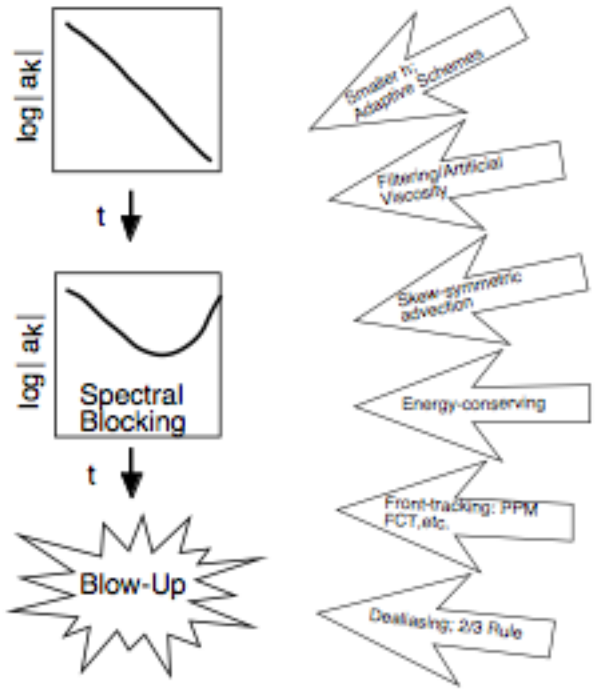


Figure 11.3: Schematic of "spectral blocking". Dashed line: logarithm of the absolute values of Fourier coefficients ("spectrum") at $t = 0$. Solid: spectrum at a later time. The dashed vertical dividing line is the boundary in wavenumber between the decreasing part of the spectrum and the unphysical region where the amplitude increases with k due to numerical noise and aliasing. The corrupted coefficients are marked with disks.



Solution

pad to zero amplitudes for modes $k \in \left[\frac{2}{3}k_{max}, k_{max}\right]$

P. Boyd. Chebyshev and Fourier spectral methods. Second edition (Revised). (2001) pp. 668

Memory issues

- Memory occupancy:
- 3 (arrays) x SizeX x SizeY x SizeZ x 3 (velocity components) x (4 or 8 bytes, single or double)
- + eventual work array needed by FFT
- (in place FFTs)
- Eventual arrays for additional measurements

e.g. $A_{ij} = (\partial_i v_j)$

MPI - Message Passing Interface

Np processors, each need to allocate M/Np

Data structure

u[x][y][z].{vx,vy,vz}

cu[x][y][z].{vx,vy,vz}.{re,im}

Direction ***x*** *split on processors*
Direction ***z*** *complexified by the FFT*

*The inverse transform (from x to k space) has to be normalized dividing by Nx*Ny*Nz.*

Computational cost

- Mflops = 5 N $\log_2(N)$ / (time for one FFT in microseconds) / 2 for real-data FFTs
- Rule of the thumb: In spectral code FFTs takes “roughly” 50% of full computational time
- Checkpoint and restart negligible
- Heavy I/O can have an impact but usually when it hits on performance, hits on disk space first.
- Lagrangian integration usually negligible as long as particle density if much smaller than grid point density
- Additional “innocent” measurements which imply extra FFTs hit hard, but usually diluted as not performed at each time step

FFT: fftw

- http://www.fftw.org/
- Features
- FFTW 3.2.2 is the latest official version of FFTW (refer to the release notes to find out what is new). Subscribe to the fftw-announce mailing list to receive announcements of future updates. Here is a list of some of FFTW's more interesting features:

Speed. (Supports SSE/SSE2/3dNow!/Altivec, since version 3.0.)
Both one-dimensional and multi-dimensional transforms.
Arbitrary-size transforms. (Sizes with small prime factors are best, but FFTW uses O(N log N) algorithms even for prime sizes.)
Fast transforms of purely real input or output data.
Transforms of real even/odd data: the discrete cosine transform (DCT) and the discrete sine transform (DST), types I-IV. (Version 3.0 or later.)
Efficient handling of multiple, strided transforms. (This lets you do things like transform multiple arrays at once, transform one dimension of a multi-dimensional array, or transform one field of a multi-component array.)
Parallel transforms: parallelized code for platforms with Cilk or for SMP machines with some flavor of threads (e.g. POSIX). An MPI version for distributed-memory transforms is also available, currently only as part of FFTW 2.1.5. FFTW 3.2.2 includes support for Cell processors.
Portable to any platform with a C compiler. Documentation in HTML and other formats.
Both C and Fortran interfaces.
Free software, released under the GNU General Public License (GPL, see FFTW license). (Non-free licenses may also be purchased from MIT, for users who do not want their programs protected by the GPL. Contact us for details.) (Also see the FAQ.)

HTTP://WWW.P3DFFT.NET

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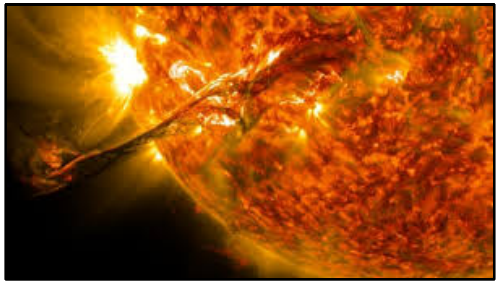
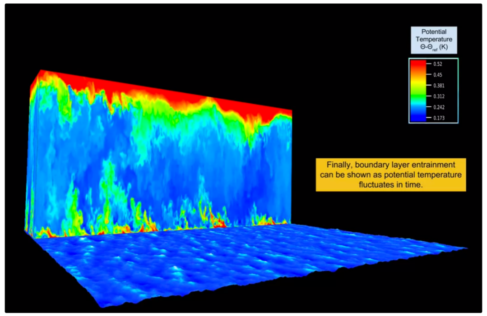
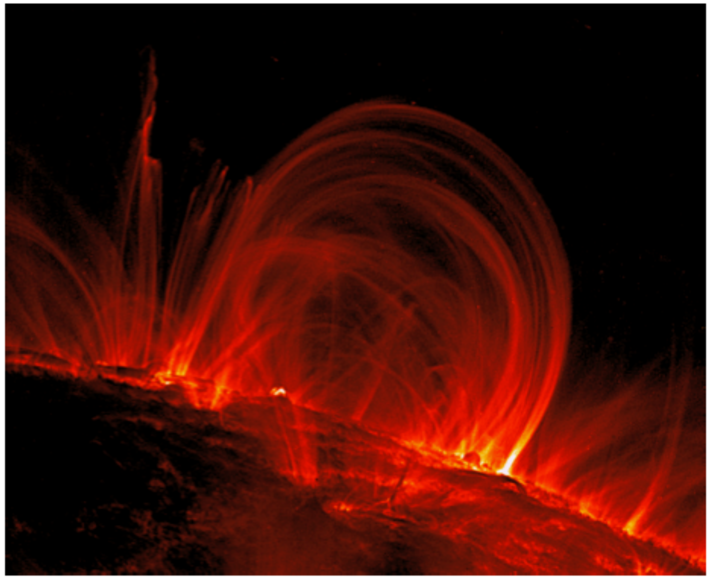






Large-scale magnetic fields in MHD



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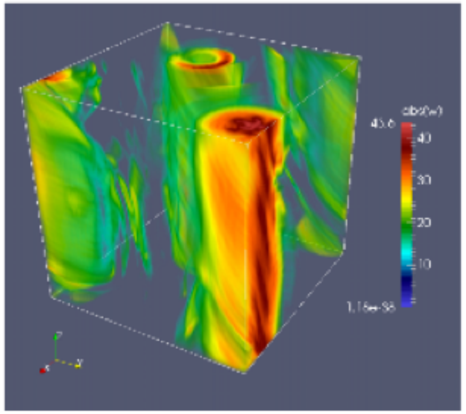

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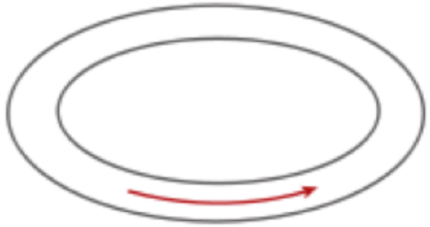
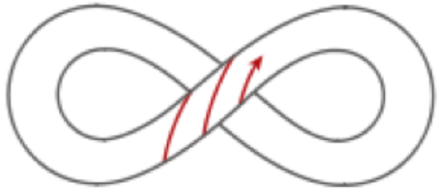
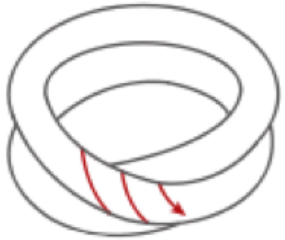






$$\partial_t \boldsymbol{u} = -\frac{1}{\rho}\nabla P - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}(\nabla\times\boldsymbol{b})\times\boldsymbol{b} + \nu\Delta\boldsymbol{u}$$

$$\partial_t \boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} + \eta\Delta\boldsymbol{b}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{and} \quad \nabla\cdot\boldsymbol{b} = 0$$

$$H_m(t) = \int_V d\boldsymbol{x}\ \boldsymbol{a}(\boldsymbol{x},t)\cdot\boldsymbol{b}(\boldsymbol{x},t) \rightarrow \text{ inverse cascade}$$

$$H_k(t) = \int_V d\boldsymbol{x}\ \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\omega}(\boldsymbol{x},t) \rightarrow \text{ dynamo action (e.g. } \alpha\text{-effect)}$$

Helical Fourier decomposition

$$(\partial_t + \nu k^2)\hat{\mathbf{u}}_{\boldsymbol{k}} = \left(\mathbb{I} - \frac{\boldsymbol{k} \otimes \boldsymbol{k}}{k^2}\right) \times \left[\sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} \left(-(i\boldsymbol{p} \times \hat{\mathbf{u}}_{\boldsymbol{p}})^* \times \hat{\mathbf{u}}_{\boldsymbol{q}}^* + (i\boldsymbol{p} \times \hat{\mathbf{b}}_{\boldsymbol{p}})^* \times \hat{\mathbf{b}}_{\boldsymbol{q}}^*\right)\right]$$

$$(\partial_t + \eta k^2)\hat{\mathbf{b}}_{\boldsymbol{k}} = i\boldsymbol{k} \times \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} \hat{\mathbf{u}}_{\boldsymbol{p}}^* \times \hat{\mathbf{b}}_{\boldsymbol{q}}^*$$

$$\hat{\mathbf{u}}_{\boldsymbol{k}}(t) = u_{\boldsymbol{k}}^+(t)\boldsymbol{h}_{\boldsymbol{k}}^+ + u_{\boldsymbol{k}}^-(t)\boldsymbol{h}_{\boldsymbol{k}}^- = \sum_{s_k} u_{\boldsymbol{k}}^{s_k}(t)\boldsymbol{h}_{\boldsymbol{k}}^{s_k}$$

$$\hat{\mathbf{b}}_{\boldsymbol{k}}(t) = b_{\boldsymbol{k}}^+(t)\boldsymbol{h}_{\boldsymbol{k}}^+ + b_{\boldsymbol{k}}^-(t)\boldsymbol{h}_{\boldsymbol{k}}^- = \sum_{s_k} b_{\boldsymbol{k}}^{s_k}(t)\boldsymbol{h}_{\boldsymbol{k}}^{s_k}$$

where $i\boldsymbol{k} \times \boldsymbol{h}_{\boldsymbol{k}}^{s_k} = s_k k \boldsymbol{h}_{\boldsymbol{k}}^{s_k}$

Helical Fourier decomposition

$$(\partial_t + \nu k^2) u_{\boldsymbol{k}}^{s_k *} = \frac{1}{2} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} \Bigg(\sum_{s_p, s_q} g_{s_p s_q}^{s_k} (s_p p - s_q q) u_{\boldsymbol{p}}^{s_p} u_{\boldsymbol{q}}^{s_q} - \sum_{\sigma_p, \sigma_q} g_{\sigma_p \sigma_q}^{s_k} (\sigma_p p - \sigma_q q) b_{\boldsymbol{p}}^{\sigma_p} b_{\boldsymbol{q}}^{\sigma_q} \Bigg) ,$$

LORENTZ

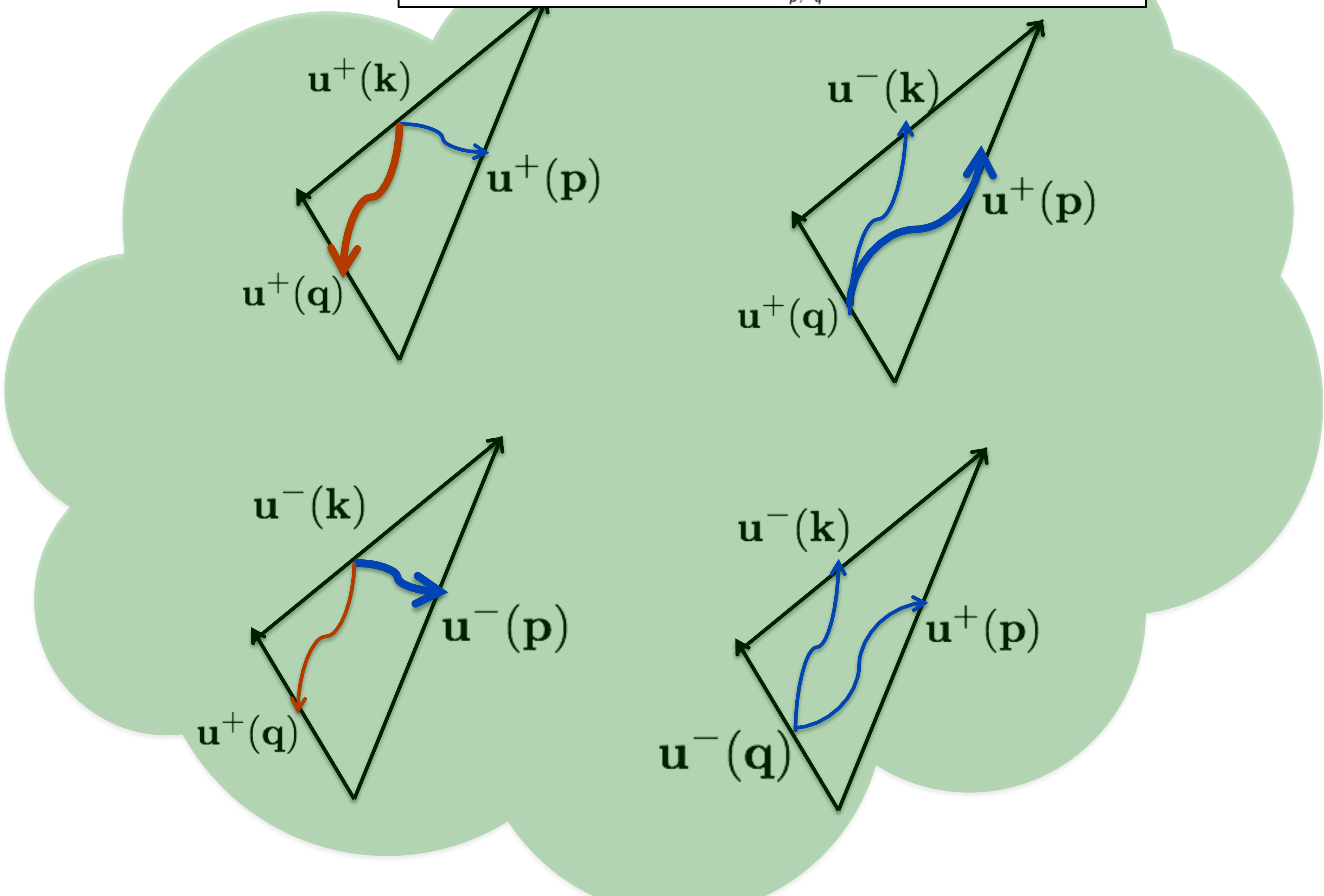
$$(\partial_t + \eta k^2) b_{\boldsymbol{k}}^{\sigma_k *} = \frac{\sigma_k k}{2} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} \Bigg(\sum_{\sigma_p, s_q} g_{\sigma_p s_q}^{\sigma_k} b_{\boldsymbol{p}}^{\sigma_p} u_{\boldsymbol{q}}^{s_q} - \sum_{s_p, \sigma_q} g_{s_p \sigma_q}^{\sigma_k} u_{\boldsymbol{p}}^{s_p} b_{\boldsymbol{q}}^{\sigma_q} \Bigg)$$

ADVECTION + STRETCHING

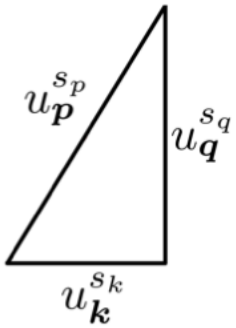
$$E_{kin} = \sum_k |u_k^+|^2 + |u_k^-|^2 \qquad H_{kin} = \sum_k k(|u_k^+|^2 - |u_k^-|^2)$$

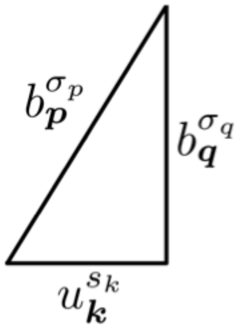
$$E_{mag} = \sum_k (|b_k^+|^2 + |b_k^-|^2) \qquad H_{mag} = \sum_k k^{-1}(|b_k^+|^2 - |b_k^-|^2)$$

$$(\partial_t + \nu k^2) u_{\boldsymbol{k}}^{s_k *} = \frac{1}{2}\sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0}\Big(\sum_{s_p,s_q} g^{s_k}_{s_p s_q}(s_p p - s_q q)\, u_{\boldsymbol{p}}^{s_p} u_{\boldsymbol{q}}^{s_q} - \sum_{\sigma_p,\sigma_q} g^{s_k}_{\sigma_p\sigma_q}(\sigma_p p - \sigma_q q)\, b_{\boldsymbol{p}}^{\sigma_p} b_{\boldsymbol{q}}^{\sigma_q}\Big)\,,$$



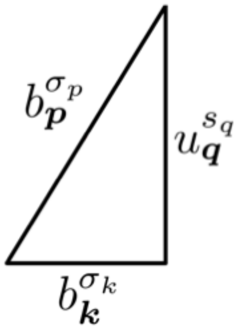
Generic two-triads system

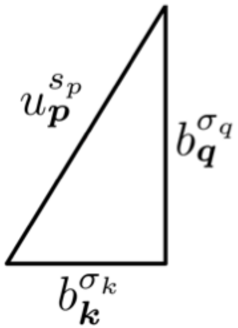




LORENTZ

$$\partial_t u_{\boldsymbol{k}}^{s_k *} = \frac{1}{2}\left(g_{s_p s_q}^{s_k}(s_p p - s_q q) u_{\boldsymbol{p}}^{s_p} u_{\boldsymbol{q}}^{s_q} - g_{\sigma_p \sigma_q}^{s_k}(\sigma_p p - \sigma_q q) b_{\boldsymbol{p}}^{\sigma_p} b_{\boldsymbol{q}}^{\sigma_q} \right)$$





$$\partial_t b_{\boldsymbol{k}}^{\sigma_k *} = \frac{\sigma_k k}{2}\left(g_{\sigma_p s_q}^{\sigma_k} b_{\boldsymbol{p}}^{\sigma_p} u_{\boldsymbol{q}}^{s_q} - g_{s_p \sigma_q}^{\sigma_k} u_{\boldsymbol{p}}^{s_p} b_{\boldsymbol{q}}^{\sigma_q} \right)$$

STRETCHING+ADVECTION

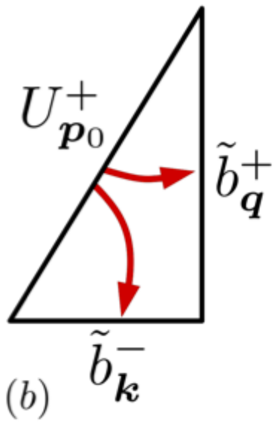
Stability analysis

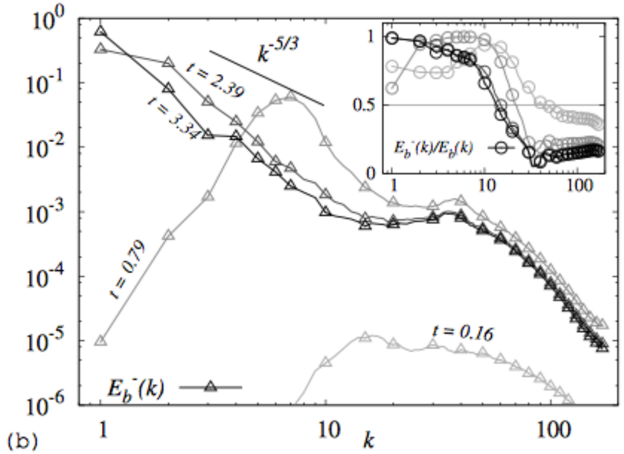
$$
\begin{aligned}
\partial_t u_{\boldsymbol{k}}^{s_k *} &= g^{s_k}_{s_p s_q}(s_p p - s_q q)\, u_{\boldsymbol{p}}^{s_p} u_{\boldsymbol{q}}^{s_q} - g^{s_k}_{\sigma_p \sigma_q}(\sigma_p p - \sigma_q q)\, b_{\boldsymbol{p}}^{\sigma_p} b_{\boldsymbol{q}}^{\sigma_q} \\
\partial_t u_{\boldsymbol{p}}^{s_p *} &= g^{s_p}_{s_q s_k}(s_q q - s_k k)\, u_{\boldsymbol{q}}^{s_q} u_{\boldsymbol{k}}^{s_k} - g^{s_p}_{\sigma_q \sigma_k}(\sigma_q q - \sigma_k k)\, b_{\boldsymbol{q}}^{\sigma_q} b_{\boldsymbol{k}}^{\sigma_k} \\
\partial_t u_{\boldsymbol{q}}^{s_q *} &= g^{s_q}_{s_k s_q}(s_k k - s_p p)\, u_{\boldsymbol{k}}^{s_k} u_{\boldsymbol{q}}^{s_q} - g^{s_q}_{\sigma_k \sigma_p}(\sigma_k k - \sigma_p p)\, b_{\boldsymbol{k}}^{\sigma_k} b_{\boldsymbol{p}}^{\sigma_p} \\
\partial_t b_{\boldsymbol{k}}^{\sigma_k *} &= \sigma_k k\, \left(g^{\sigma_k}_{\sigma_p s_q} b_{\boldsymbol{p}}^{\sigma_p} u_{\boldsymbol{q}}^{s_q} - g^{\sigma_k}_{s_p \sigma_q} u_{\boldsymbol{p}}^{s_p} b_{\boldsymbol{q}}^{\sigma_q} \right) \\
\partial_t b_{\boldsymbol{p}}^{\sigma_p *} &= \sigma_p p\, \left(g^{\sigma_p}_{\sigma_q s_k} b_{\boldsymbol{q}}^{\sigma_q} u_{\boldsymbol{k}}^{s_k} - g^{\sigma_p}_{s_q \sigma_k} u_{\boldsymbol{q}}^{s_q} b_{\boldsymbol{k}}^{\sigma_k} \right) \\
\partial_t b_{\boldsymbol{q}}^{\sigma_q *} &= \sigma_q q\, \left(g^{\sigma_q}_{\sigma_k s_p} b_{\boldsymbol{k}}^{\sigma_k} u_{\boldsymbol{p}}^{s_p} - g^{\sigma_q}_{s_k \sigma_p} u_{\boldsymbol{k}}^{s_k} b_{\boldsymbol{p}}^{\sigma_p} \right)
\end{aligned}
$$

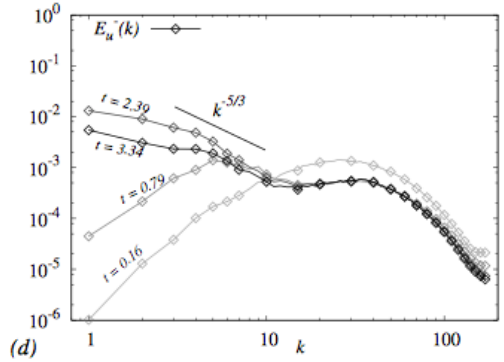
Large-scale dynamo: DNS - laminar flow ($Re_\lambda = 15$)

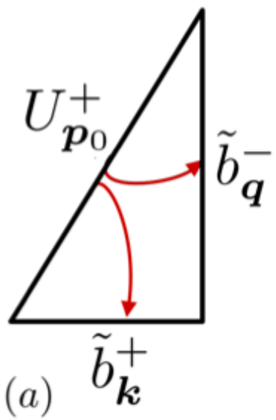
Magnetic field

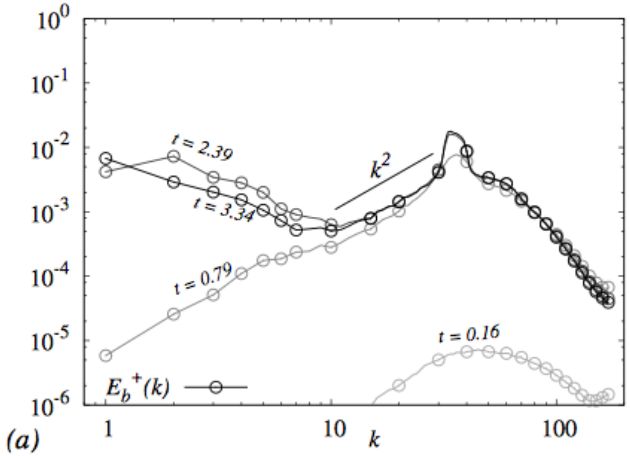
Velocity field

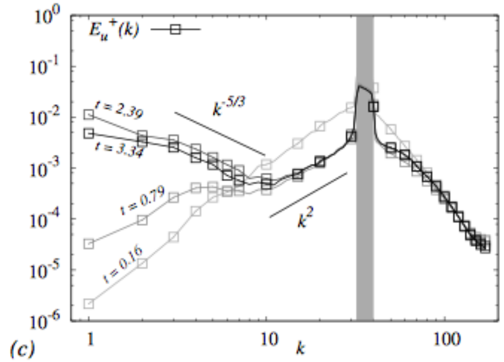








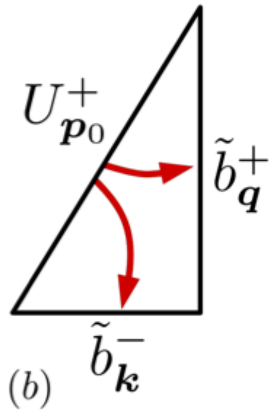


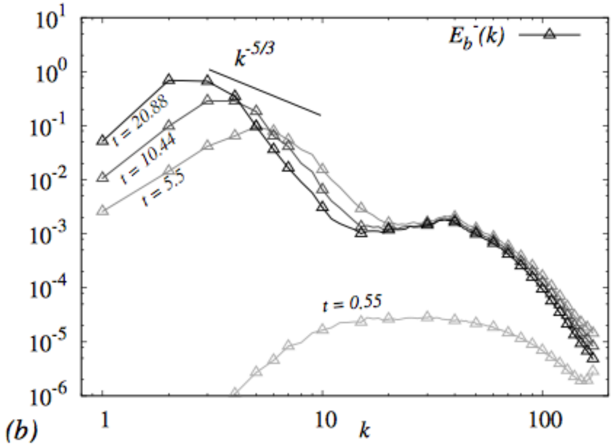


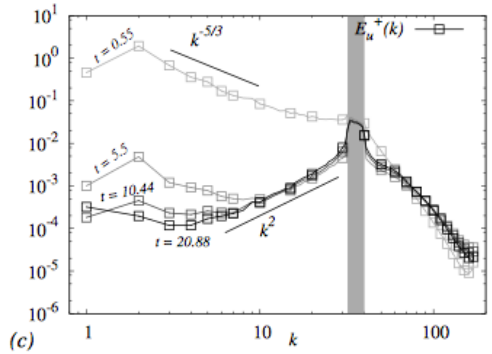
Large-scale dynamo: DNS - turbulent flow ($Re_\lambda = 140$)

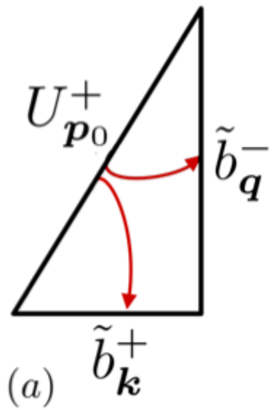
Magnetic field

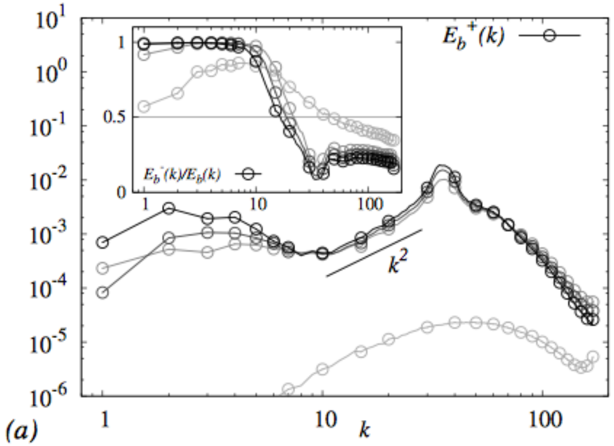
Velocity field $\boldsymbol{u} = \boldsymbol{u}^+$







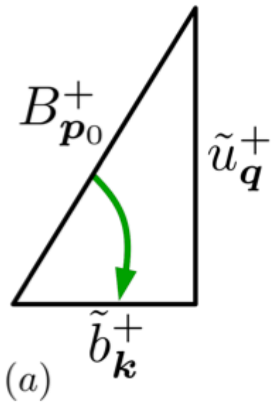


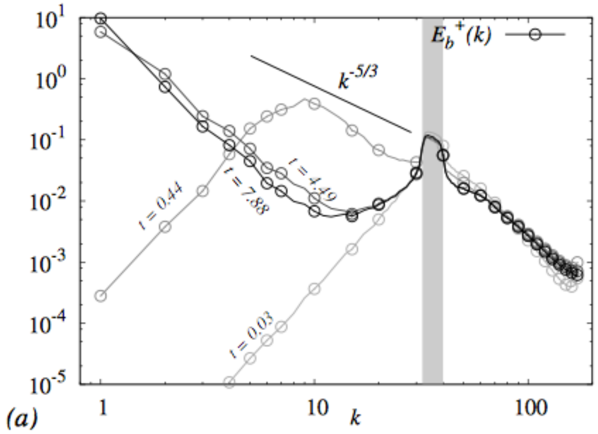


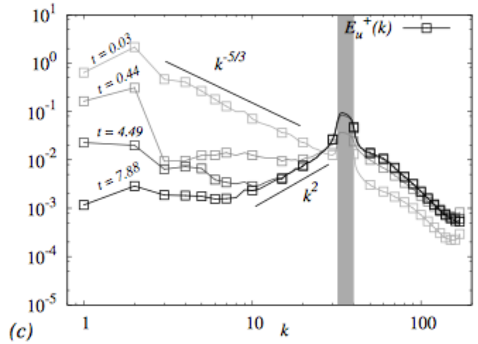
Inverse cascade of magnetic helicity: DNS

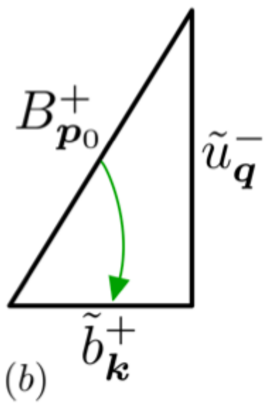
Magnetic field

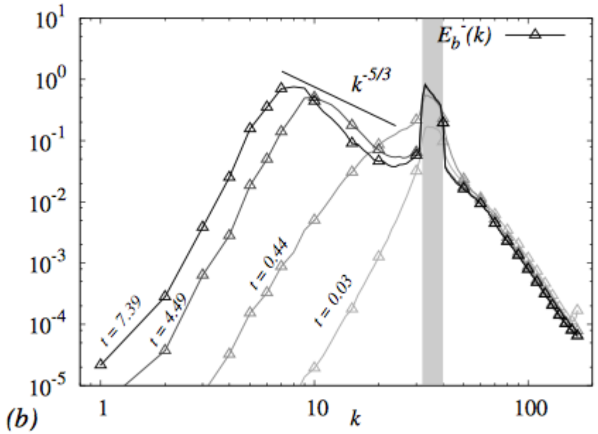
Velocity field
$\boldsymbol{u} = \boldsymbol{u}^{+}$

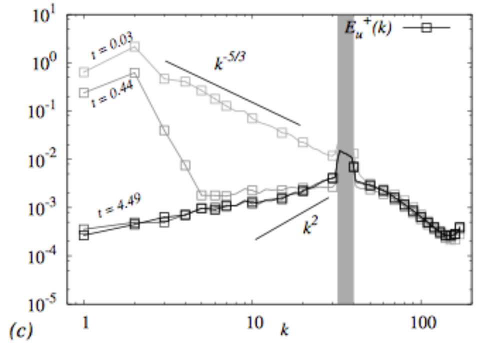












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J. Fluid Mech. 791:61-96

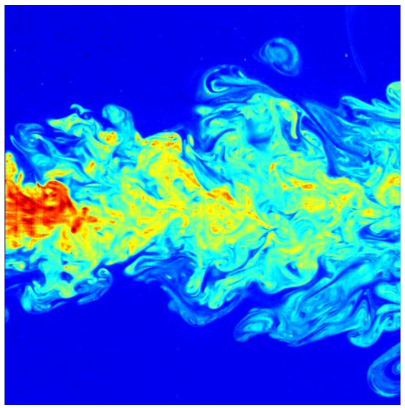
M. Linkmann, G. Sahoo, M. McKay, A. Berera, L. Biferale, arXiv:1609.01781

$$
\begin{cases}
\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\partial} \boldsymbol{v} = -\boldsymbol{\partial} P + \nu \partial^2 \boldsymbol{v} + \boldsymbol{f} \\
\boldsymbol{\partial} \cdot \boldsymbol{v} = 0 \\
+ \text{ boundary conditions}
\end{cases}
$$

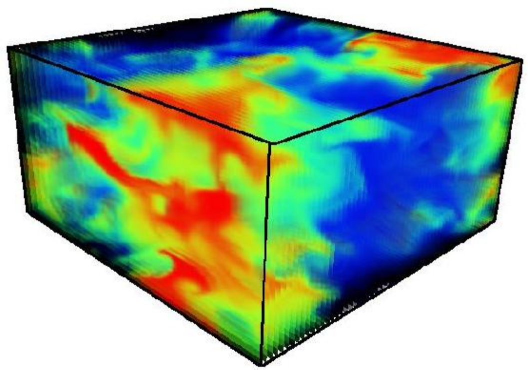
Kinematics + Dissipation are invariant under Rotation+Translation

Non-universal statistical behaviour <-> Anisotropy

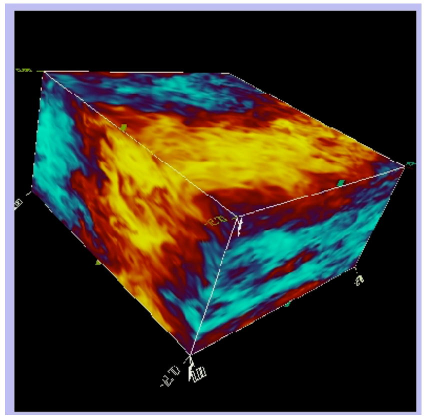
Small scales vs large scales



Turbulent jet



3d Convective Cell



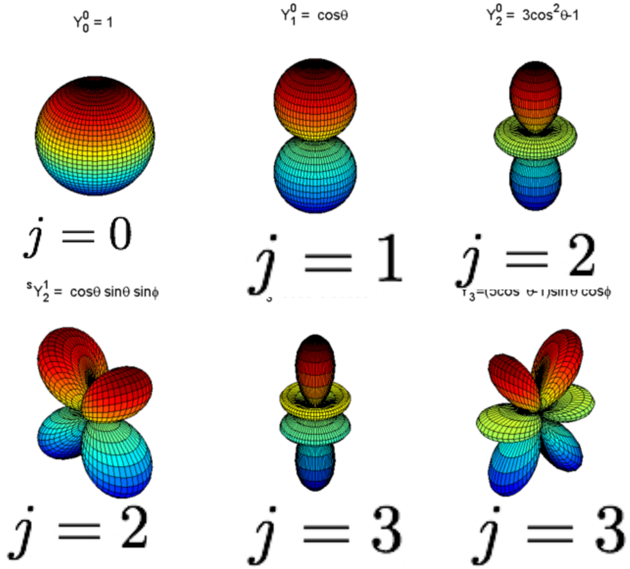
Shear Flow

The simplest set of 0-rank tensor (SCALAR) observable:

Longitudinal Structure Functions

$$S^{(n)}(\boldsymbol{r}) = \langle [(\boldsymbol{v}(\mathbf{x}+\boldsymbol{r}) - \boldsymbol{v}(\mathbf{x})) \cdot \hat{\boldsymbol{r}}]^n \rangle \, .$$

$$S^{(n)}(\boldsymbol{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \mathcal{S}^{(n)}_{jm}(r) Y_{jm}(\hat{\boldsymbol{r}}).$$



Arad, V. L'Vov I. Procaccia PRE 59, 6753 (1999).
Arad et al. PRL 82, 5040 (1999).
Arad et al. PRL 81, 5330 (1998).

$$S_n^{\alpha_1\cdots\alpha_n}(\boldsymbol{r}) \stackrel{\text{def}}{=} \langle \delta v^{\alpha_1}(\boldsymbol{x},\boldsymbol{r},t)\cdots\delta v^{\alpha_n}(\boldsymbol{x},\boldsymbol{r},t)\rangle \ ,$$
$$\delta\boldsymbol{v}(\boldsymbol{x},\boldsymbol{r},t) \stackrel{\text{def}}{=} \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r},t)-\boldsymbol{v}(\boldsymbol{x},t) \ ,$$

3d rotation

$$x'_\alpha = \Lambda_{\alpha,\beta}x_\beta$$

Decomposition in terms of (irreducible) invariant subset -labelled by q,j=0,1,2,…

Set of 3n*(2j+1) Eigenfunctions of group of rotations in 3d: $B_{q,jm}^{\alpha_1\ldots\alpha_n}(\mathbf{r})$

$$S_n^{\alpha_1\cdots\alpha_n}(\boldsymbol{r}) = \boxed{\text{n-rank tensor which depends on a 3d vector}} = \sum_{qjm}\mathcal{S}_{qjm}(r)B_{qjm}^{\alpha_1\cdots\alpha_n}(\hat{\boldsymbol{r}}) \ .$$

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\boldsymbol{\partial}\boldsymbol{v} = -\boldsymbol{\partial}P + \nu\partial^2\boldsymbol{v} + \boldsymbol{f}$$

$$\partial_t v_i + \Gamma_{ijk}(v_j v_k) - \nu\Delta v_i = f_i$$

$$\partial_t S^n + \Gamma^{n+1}S^{n+1} - \nu D^n S^n = \langle \delta f_1 \delta v_2 \cdots \delta v_{n-1}\rangle + perm.$$

rotational invariant operator

$r << L_f$

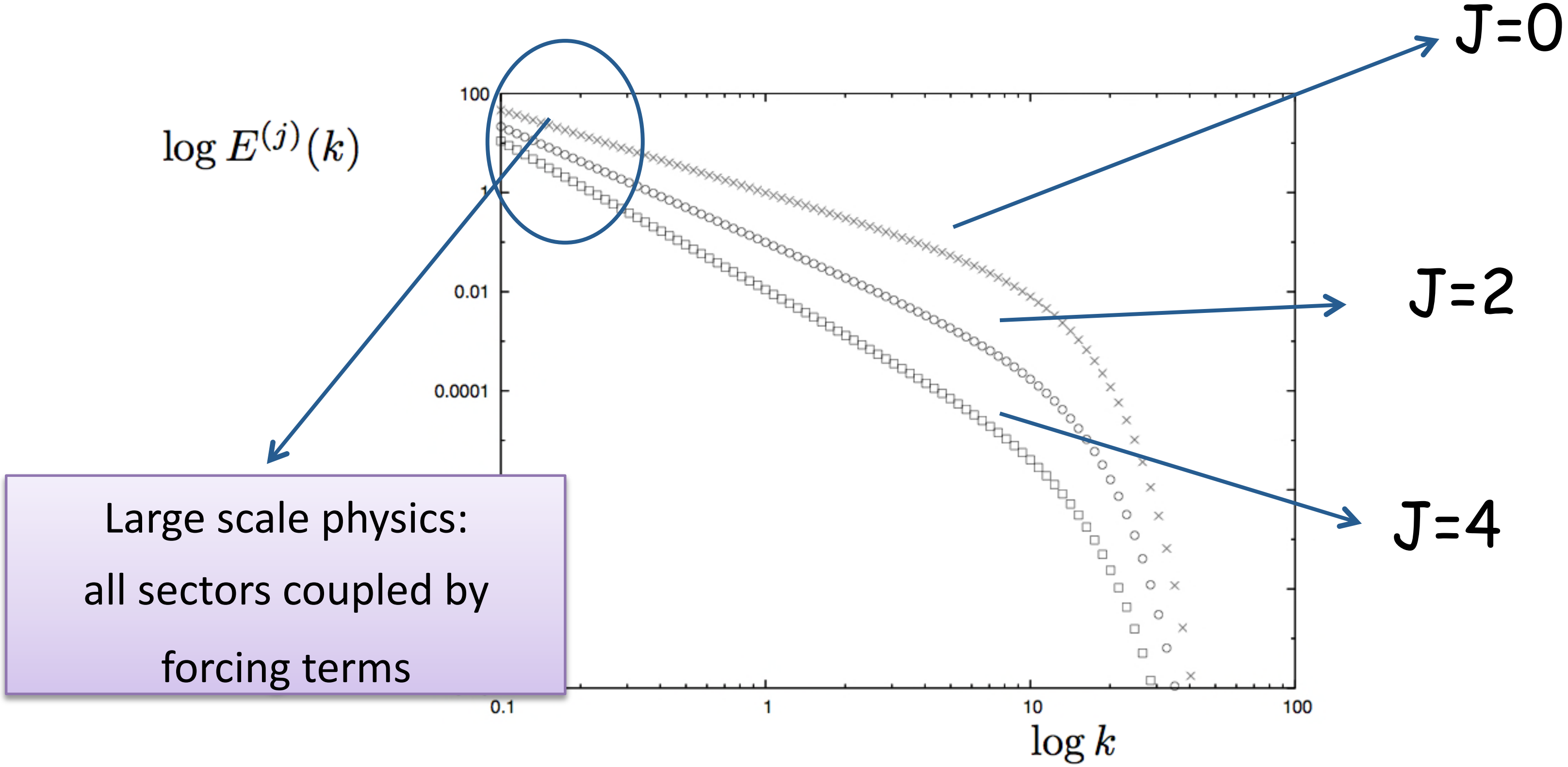
$$\partial_t S^n + \Gamma^{n+1}S^{n+1} - \nu D^n S^n \sim 0$$

+ so(3) -> $S^{(n)}_{\alpha_1\ldots\alpha_n}(\mathbf{r}) = \sum_{jmq} \mathcal{S}^{(n)}_{jmq}(r) B^{jmq}_{\alpha_1\ldots\alpha_n}(\hat{r})$

$$\partial_t \mathcal{S}^n_{jmq} + \sum_{q'} \Gamma^{n+1}_{jmq'}\mathcal{S}^{n+1}_{jmq'} - \nu D^n_{jmq}\mathcal{S}^n_{jmq} = 0$$

FOLIATION !!!

$$\partial_t \mathcal{S}^{(n)}_{jq} + \sum_{q'} \Gamma^{(n+1)}_{jq'} \mathcal{S}^{(n+1)}_{jq'} - \nu D^{(n)}_{jq} \mathcal{S}^{(n)}_{jq} \sim 0$$



$\nu \to 0$ $$\partial_t \mathcal{S}^{(n)}_{jq} + \sum_{q'} \Gamma^{(n+1)}_{jq'} \mathcal{S}^{(n+1)}_{jq'} - \nu D^{(n)}_{jq} \mathcal{S}^{(n)}_{jq} \sim 0$$

$$\mathcal{S}^{(n)}_{jmq}(r) \propto a_{jmq} (\tfrac{r}{L})^{\zeta^j_n}$$ scaling ?

$$S^{(n)}_{jqm}(r) \sim A_{jmq}(\frac{r}{L})^{\xi^j_n}$$

Working Hypothesis

projection on each sector has a **universal** scaling exponent, ξ^j_n depending on that sector **only**.

Dependency on large scale physics shows up only in **prefactors**

Pure power laws **only** in each separeted sector:

$$S^{(n)}(\mathbf{r}) \sim \sum_j A_j(\frac{r}{L})^{\zeta^j_n} \longrightarrow S^{(n)}(\mathbf{r}) \sim A_0(\frac{r}{L})^{\zeta^0_n} + A_1(\frac{r}{L})^{\zeta^1_n} + \cdots$$

$$S^{(n)}(\boldsymbol{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} S_{jm}^{(n)}(r) Y_{jm}(\hat{\boldsymbol{r}}).$$

$$S^{(n)}(\mathbf{r}) \sim A_0 (\frac{r}{L})^{\zeta_n^0} + A_1 (\frac{r}{L})^{\zeta_n^1} + \cdots$$

Matching Infra-Red boundary conditions:

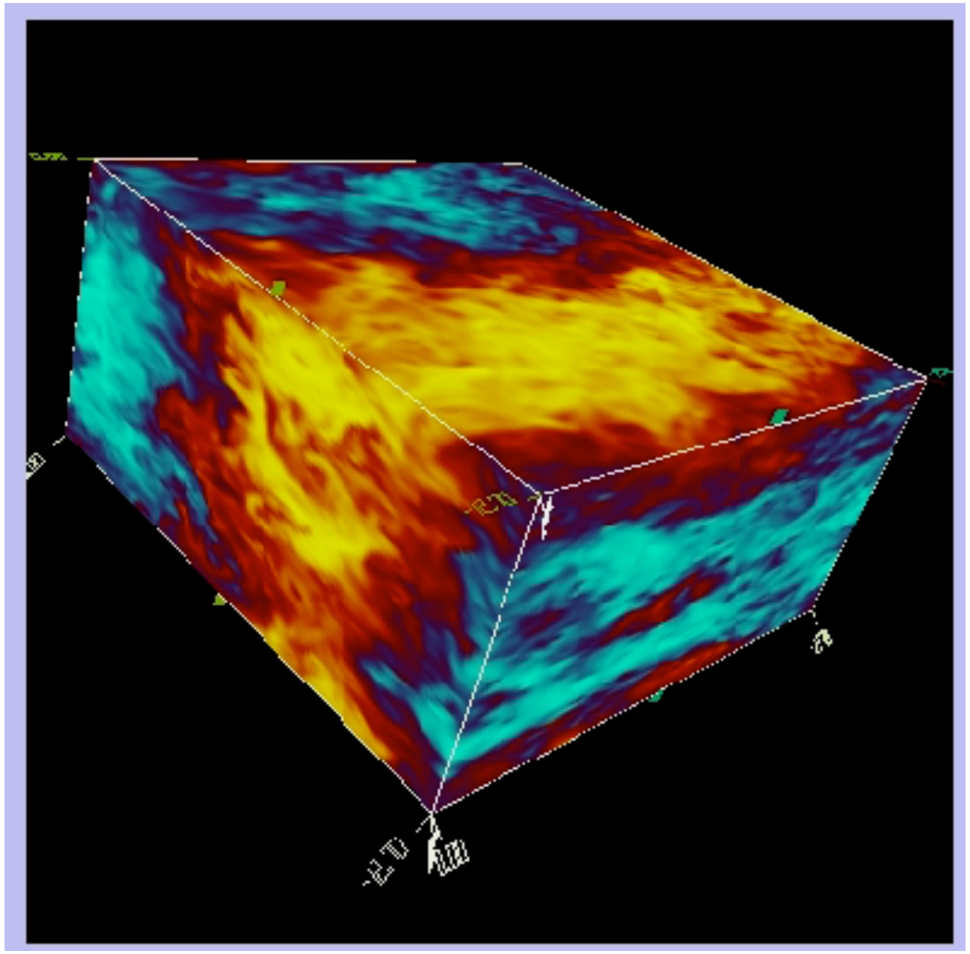
$$S^{(n)}(\mathbf{L}) \sim A_0 + A_1 + A_2 + \cdots$$

prefactor <u>cannot be universal</u>

About universality of scaling exponents nothing can be said rigorously, at least for the NS eqs.

Recovery of Isotropy
Small-Scales Universality

$$\zeta^{j=0}(n) \leq \zeta^{j=1}(n) \leq \zeta^{j=2}(n) < \ldots$$



We performed a DNS
of a Random-Kolmogorov Flow

Periodic boundary conditions

256x256x256

Hyperviscosity

Homogeneous but Anisotropic

$$f_z = cos(z + \phi(t))$$
$$\langle \phi(t)\phi(t') \rangle = \delta(t - t')$$

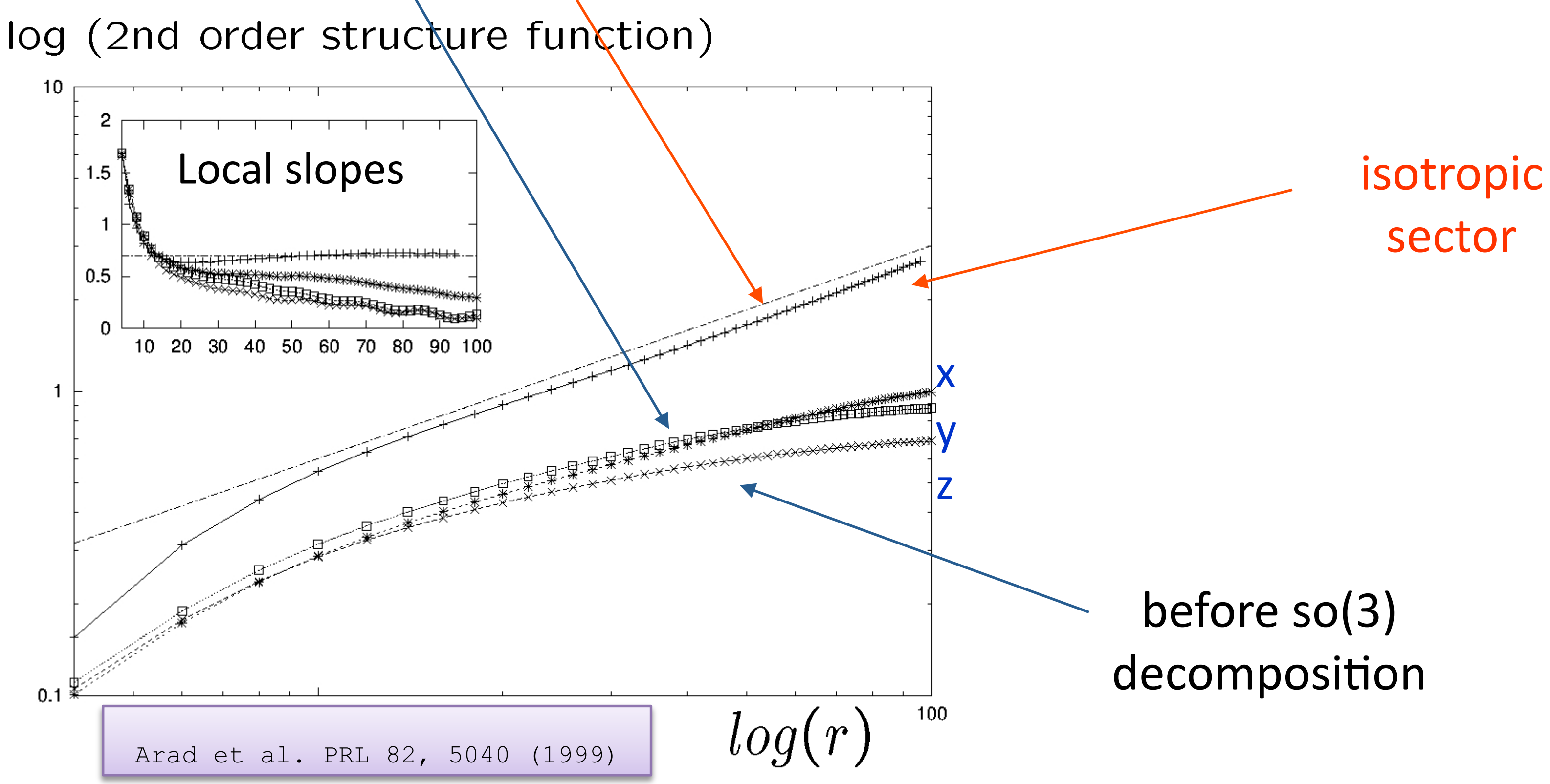
L.B. and F. Toschi, PRL 86, 4831 (2001)

L.B. I. Daumont, A. Lanotte and F. Toschi. PRE. 66, 056306 (2002)

Comparison of scaling properties:
isotropic sector (j=0,m=0) vs undecomposed structure function

$$S_n(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} S_n^{jm}(|r|) Y_{jm}(\hat{r})$$

$$S_n(\mathbf{r}) = a_0 r^{\zeta_n^{j=0}} + a_2 r^{\zeta_n^{j=2}} + a_4 r^{\zeta_n^{j=4}} + \cdots$$



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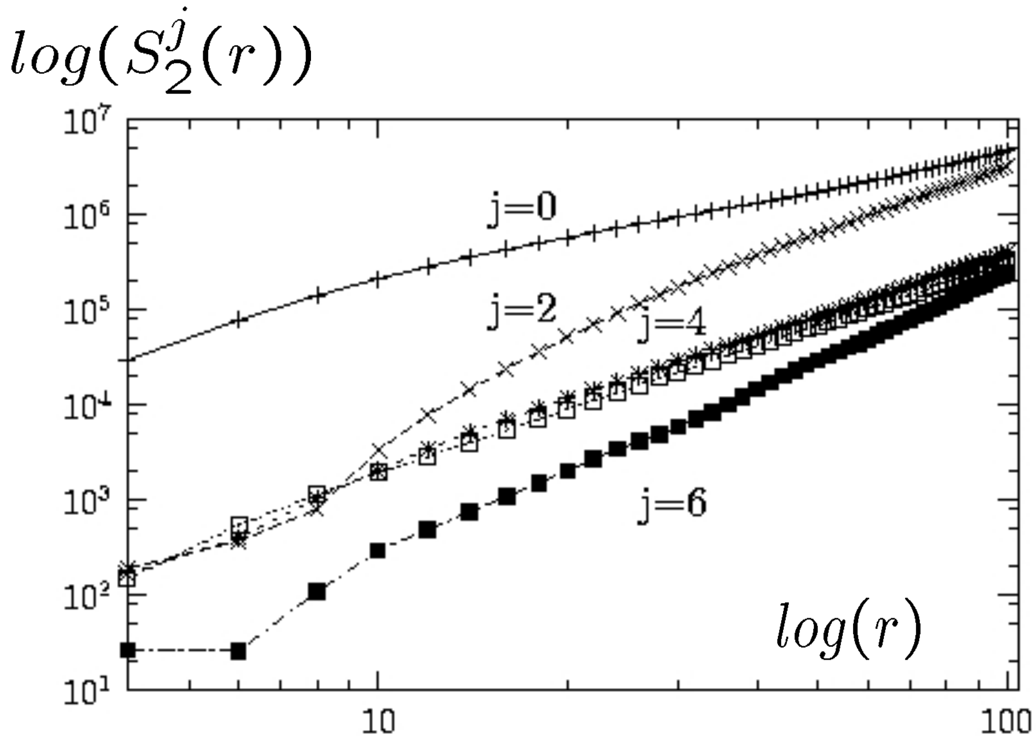
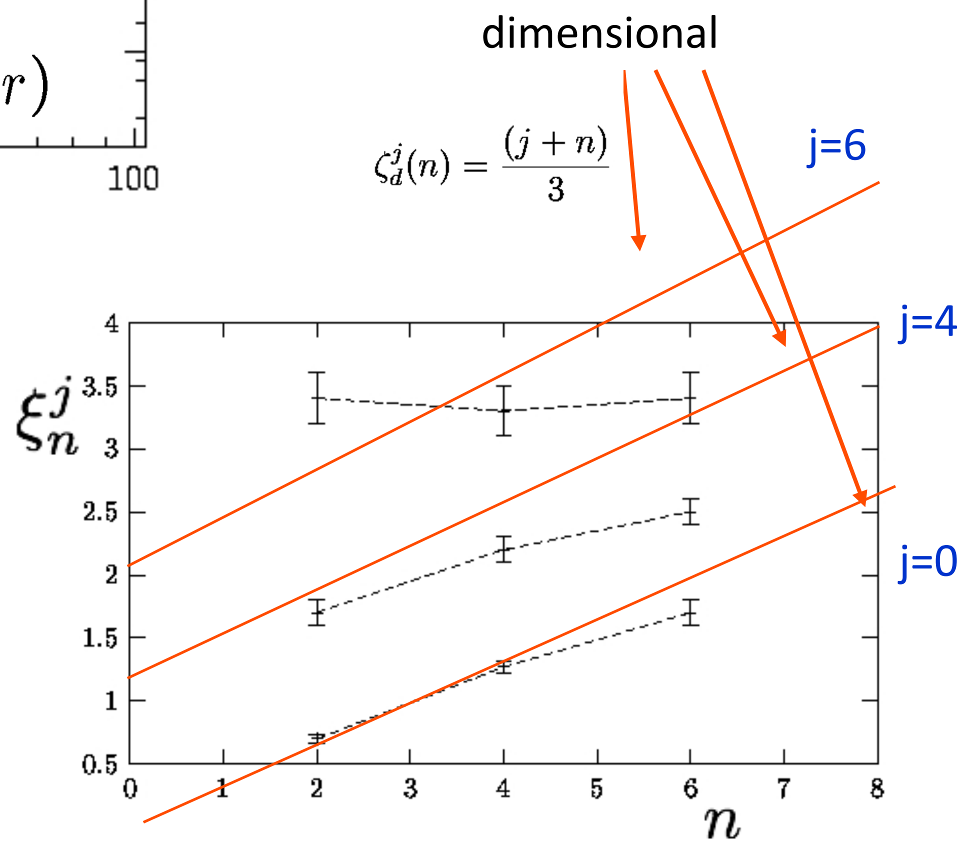
Extending Lumley's anisotropic theory



scaling is m-independent

$$\zeta_n^{j=0} < \zeta_n^{j=1} < \zeta_n^{j=2} < \cdots$$

Recovery of isotropy



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