Small-scale anisotropy in Random-Kolmogorov Flows

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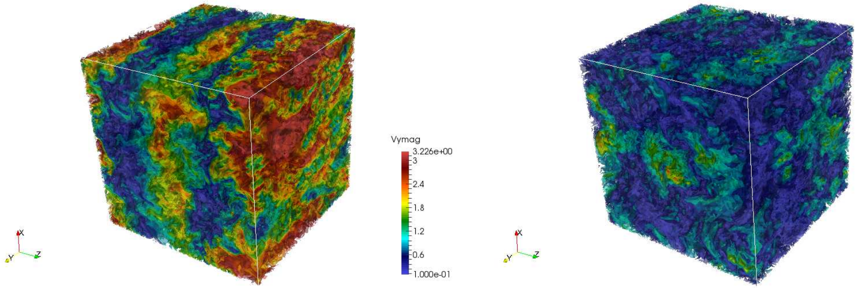
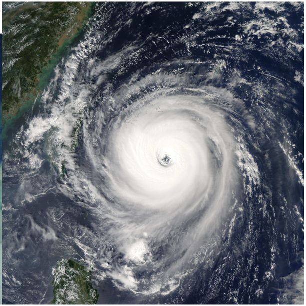


source: Wikipedia

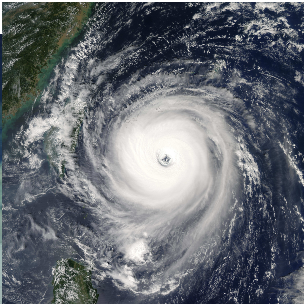
- Incompressible NSE invariant under Rotation + Translation

At large scales:
forcing, B.C
break rotation invariance

$$\begin{cases} \partial_t \mathbf{v} + \mathbf{v} \cdot \partial \mathbf{v} = -\partial p + \nu \partial^2 \mathbf{v} + \mathbf{f} \\ \partial \cdot \mathbf{v} = 0 \\ + \text{ boundary conditions} \end{cases}$$

- Is breaking of rotational symmetry passed down-scale ?

source: Wikipedia



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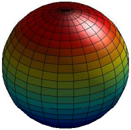
Universal signatures in small-scale fluctuations?

Longitudinal structure function: $S^{(n)}(\mathbf{r}) \equiv \left\langle \left[\left(\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x}) \right) \cdot \hat{\mathbf{r}} \right]^n \right\rangle$ (0-rank tensor)

$$\boxed{S^{(n)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{m=j} S_{jm}^{(n)}(r) Y_{jm}(\hat{\mathbf{r}})}$$

Arad et. al. PRL'98

$Y_{00} = 1/\sqrt{4\pi}$

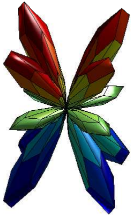


$Y_{20} = \sqrt{\frac{5}{16\pi}} (3cos^2\theta - 1)$

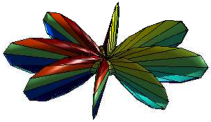
$Y_{22} = \sqrt{\frac{15}{16\pi}} (\sin^2\theta \cos 2\phi)$

Y_{40}

Y_{42}



Y_{44}



rotational invariant operators

$$\boxed{\partial_t S^{(2)}(\mathbf{r}) + \Gamma^{(3)} S^{(3)}(\mathbf{r}) - 2\nu \nabla^2 S^{(2)}(\mathbf{r}) = f^{(2)}(\mathbf{r})}$$

$r \ll L_f$

Univerality at small scales

$$\partial_t S^{(2)}(\mathbf{r}) + \Gamma^{(3)} S^{(3)}(\mathbf{r}) - 2\nu \nabla^2 S^{(2)}(r) \sim 0$$

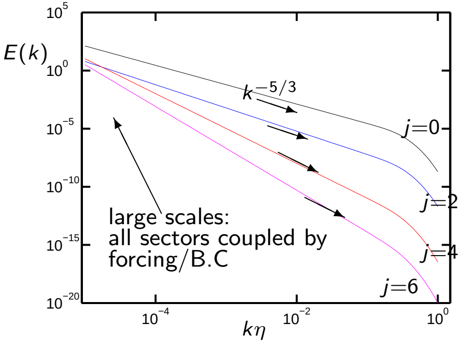
+ SO(3)

Weak Anisotropy

$$S_j^{(2)}(\mathbf{r}) = \sum_{m=-j}^{m=+j} Y_{jm}(\hat{\mathbf{r}}) \int S_{jm}^{(2)}(r) Y_{jm}(\hat{\mathbf{r}}) d\hat{\mathbf{r}}$$

$$\boxed{\partial_t S_j^{(2)}(\mathbf{r}) + \Gamma_j^{(3)} S_j^{(3)}(\mathbf{r}) - 2\nu \nabla^2 S_j^{(2)}(\mathbf{r}) \sim 0} \quad j = 0, 1, 2, \ldots$$

FOLIATION ???

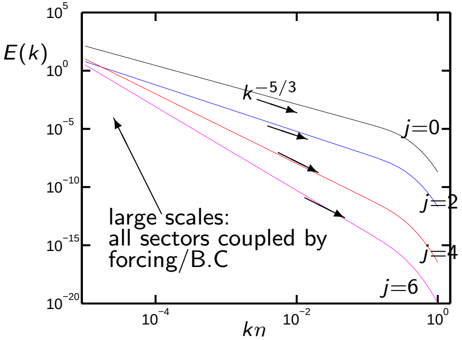


(1) Univerality: leading isotropic sector

(2) Foliation of j-sectors for $k \gg k_F$

(3) different physics in different sectors

(4) return-to-isotropy



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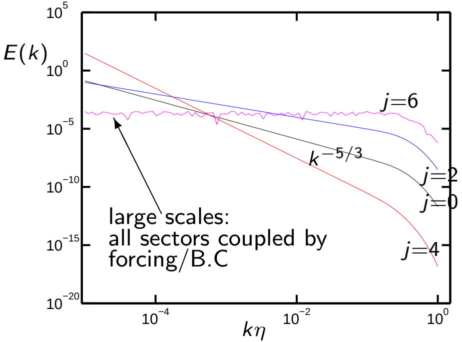


(1) NO Univerality:
sub-leading isotropic sector

(2) NO Foliation:
j-sectors coupled at $k \gg k_F$

(3) Isotropy not recovered at small-scales

$$S^{(n)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{m=j} S^{(n)}_{jm}(r) Y_{jm}(\hat{\mathbf{r}})$$

Working Hypothesis

$$S^{(n)}_{jm}(r) = A^{(n)}_{jm} \left(\frac{r}{L}\right)^{\varsigma^n_j}$$

- Projection on sector-j has universal scaling exponent ς^n_j in inertial range depending on that sector **only**
- Power law behavior **only** in each separated sector
- Prefactors depend on large scale physics

$$S^{(n)}(\mathbf{r}) \sim A_0 \left(\frac{r}{L}\right)^{\varsigma^n_0} + A_1 \left(\frac{r}{L}\right)^{\varsigma^n_1} + A_2 \left(\frac{r}{L}\right)^{\varsigma^n_2} + + \dots$$

$$S^{(n)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{m=j} S^{(n)}_{jm}(r) Y_{jm}(\hat{\mathbf{r}})$$

$$S^{(n)}(\mathbf{r}) \sim A_0 \Big(\frac{r}{L}\Big)^{\varsigma_0^n} + A_1 \Big(\frac{r}{L}\Big)^{\varsigma_1^n} + A_2 \Big(\frac{r}{L}\Big)^{\varsigma_2^n} + + \dots$$

$$S^{(n)}(\mathbf{L}) \sim A_0 + A_1 + A_2 + \dots$$

Prefactors cannot be universal!

$$S^{(n)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{m=j} S_{jm}^{(n)}(r) Y_{jm}(\hat{\mathbf{r}})$$

$$S^{(n)}(\mathbf{r}) \sim A_0 \left(\frac{r}{L}\right)^{\varsigma_0^n} + A_1 \left(\frac{r}{L}\right)^{\varsigma_1^n} + A_2 \left(\frac{r}{L}\right)^{\varsigma_2^n} + + \dots$$

$$S^{(n)}(\mathbf{L}) \sim A_0 + A_1 + A_2 + \dots$$

Prefactors cannot be universal!

Open Questions

- Are scaling exponents ς_j^n in j-sector m-independent ?
- Are scaling exponents ς_j^n universal ?
- Return-to-Isotropy ?

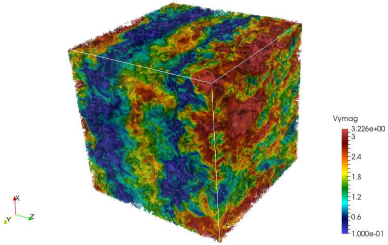
$$\varsigma_0^n \leq \varsigma_2^n \leq \varsigma_4^n \leq \dots$$

- No rigorous inferences from NSE at least thus far ...

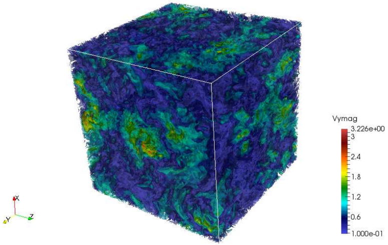
Random Kolmogorov Flow *(Biferale, Toschi PRL'01)*

- RKF is stationary, homogeneous on average and anisotropic

$$1024^3, R_\lambda = 280$$



t/T_E=1



t/T_E=3

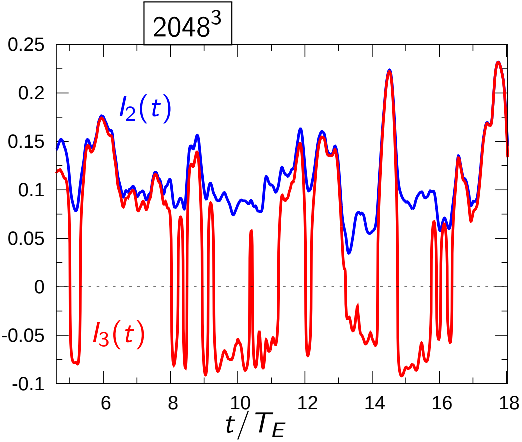
- Anisotropic forcing:

$$\boxed{f_i(\mathbf{k}_{1,2}) = \delta_{i,2} f_{1,2}(t) e^{i\theta_{1,2}(t)}}, \mathbf{k}_1 = (1,0,0), \mathbf{k}_2 = (2,0,0)$$

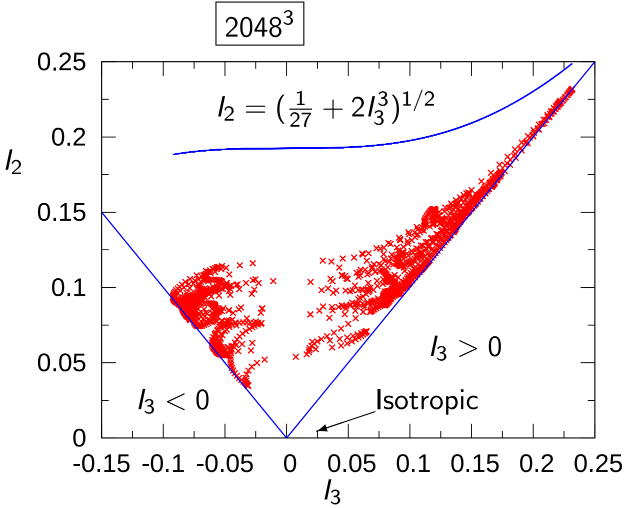
- $f_i(t)$: time-dependent amplitude
- $\theta_i(t)$: RANDOM time-varying phases

RKF: test-bed for anisotropic turbulence

- Reynolds Stress Tensor $b_{ij} \equiv \langle u_i u_j \rangle / \langle u_k u_k \rangle - \delta_{ij}/3$
- $I_2 \equiv b_{ii}^2/6$ $I_3 \equiv (b_{ij} b_{jk} b_{ki})^{1/3}/6$



Lumley Triangle

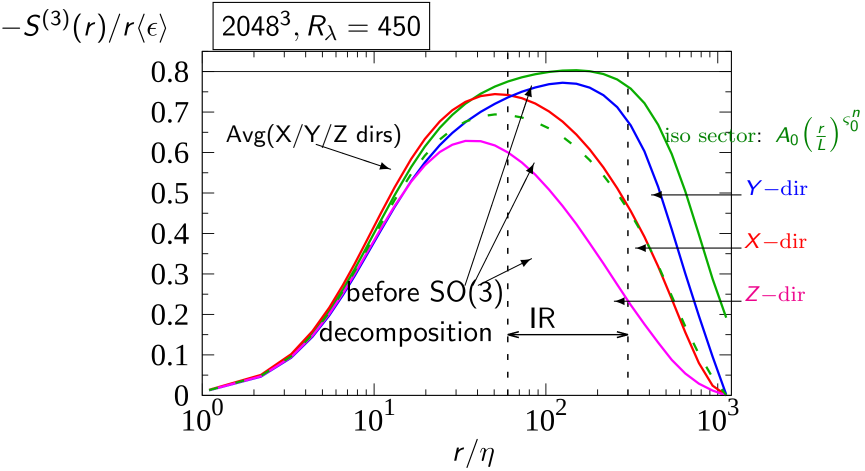


- Different large scale configurations (I_2-I_3) possible in RKF

Isotropic sector vs undecomposed structure function

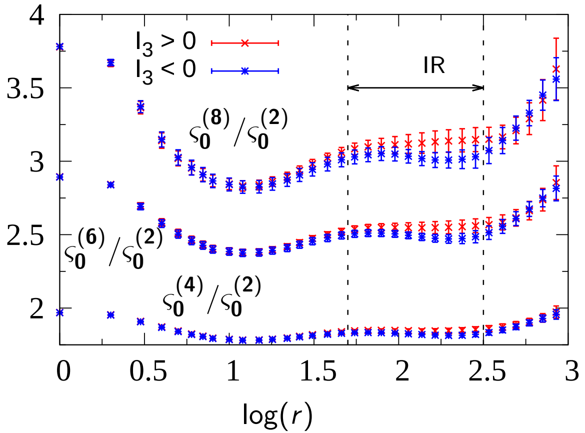
$$S^{(n)}(\mathbf{r}) = A_0 \left(\frac{r}{L}\right)^{\varsigma_0^n} + A_1 \left(\frac{r}{L}\right)^{\varsigma_1^n} + \dots$$

Isotropy in Inertial Range ($\eta \ll r \ll L$) $S^{(3)}(r) = -\frac{4}{5}\langle\epsilon\rangle r$



Universality of scaling exponent in isotropic sector

$$S^{(n)}(\mathbf{r}) = \boxed{A_0\left(\frac{r}{L}\right)^{\varsigma_0^{(n)}}} + A_1\left(\frac{r}{L}\right)^{\varsigma_n^{(1)}} + \dots \implies \text{Is } \varsigma_0^{(n)} \text{ universal?}$$

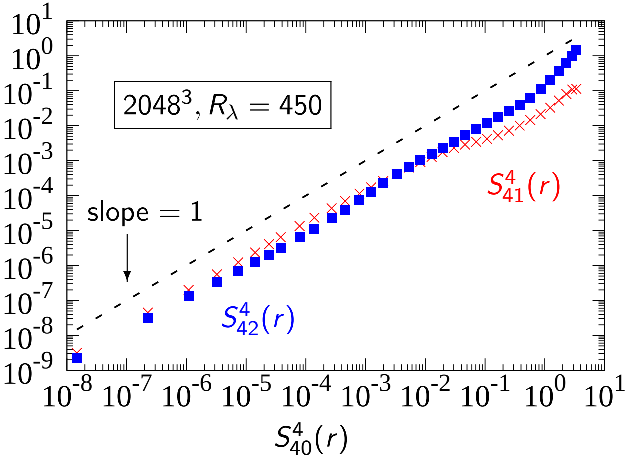


$2048^3, R_\lambda = 450$

$k_{max}\eta \approx 2$

- Different I2-I3 confs in Lumley triangle have similar Inertial Range scaling at least up to order 8

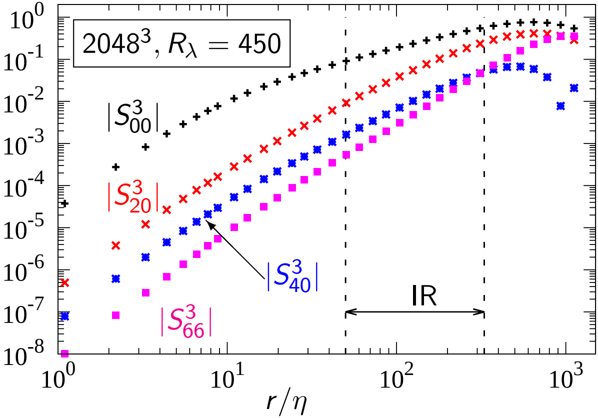
Scaling in anisotropic sectors



- Scaling of $S^n_{jm}(r)$ within same j-sector is m-independent

Ansatz for projection coeffs: $S^n_{jm}(r) \sim A^n_{jm} r^{\varsigma^n_j}$

Dimensional scaling exponent (*Biferale et. al. PRE'02*): $\varsigma^n_j = (j+n)/3$



$\varsigma^3_{00} = 0.97 \quad (1.00)$

$\varsigma^3_{20} = 1.67 \quad (1.67)$

$\varsigma^3_{40} = 1.76 \quad (2.33)$

$\varsigma^3_{66} = 2.49 \quad (3.00)$

- $\varsigma^n_0 < \varsigma^n_{20} < \varsigma^n_{40} < \varsigma^n_{66} \Rightarrow (r/L)^{\varsigma^n_0} > (r/L)^{\varsigma^n_{jm}}$ for $r/L \ll 1$
- Isotropic sector dominant. Anisotropic part: sub-leading

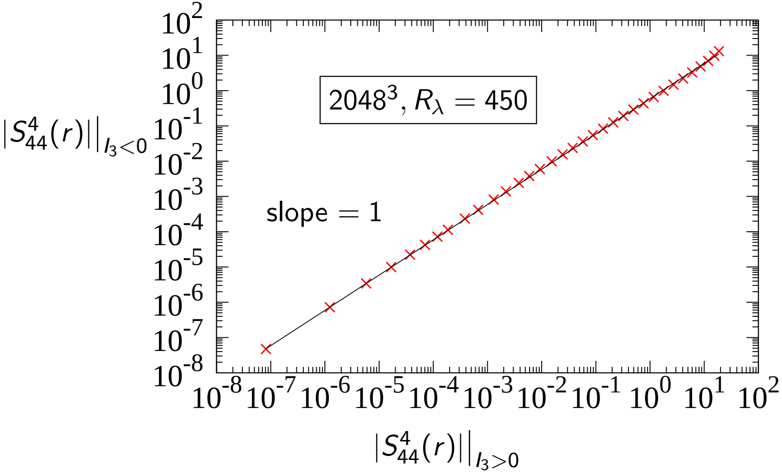
Conclusions

- SO(3): useful tool for a systematic investigation of anisotropic effects on scaling properties
- Isotropic sector: Universal power law behavior
- Scaling in anisotropic sectors (j, m) is intermittent and m-independent
- Scaling exponents in higher j-sectors might be Universal

Supported by ERC Grant No. 339032

Supercomputing resources at CINECA

Univerality of scaling exponents in anisotropic sectors



- *j*-sectors of different large-scale configurations scale similarly in inertial range