Helical Fourier decomposition in magnetohydrodynamics

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69^{th} Annual Meeting of the APS DFD, Portland, USA

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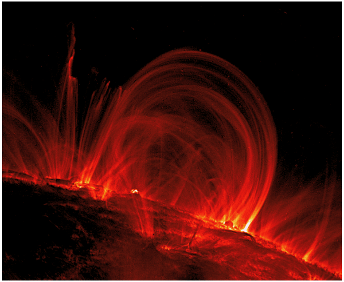
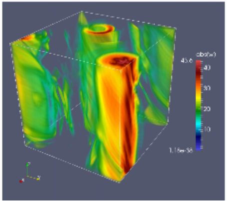
20/11/2016

Large-scale magnetic fields in MHD

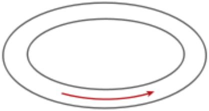


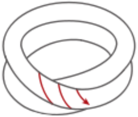
ⓒTRACE operation team,
Lockheed Martin



Dallas & Alexakis PoF 2015

Minnini, Annu. Rev. Fluid Mech. 2011





$$\partial_t \boldsymbol{u} = -\frac{1}{\rho}\nabla P - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}(\nabla \times \boldsymbol{b}) \times \boldsymbol{b} + \nu\Delta\boldsymbol{u}$$

$$\partial_t \boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} + \eta\Delta\boldsymbol{b}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{and} \quad \nabla\cdot\boldsymbol{b} = 0$$

$$H_m(t) = \int_V d\boldsymbol{x}\ \boldsymbol{a}(\boldsymbol{x},t)\cdot\boldsymbol{b}(\boldsymbol{x},t) \to \text{ inverse cascade}$$

$$H_k(t) = \int_V d\boldsymbol{x}\ \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\omega}(\boldsymbol{x},t) \to \text{ dynamo action (e.g. } \alpha\text{-effect)}$$

$$\partial_t \boldsymbol{u} = -\frac{1}{\rho}\nabla P - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}(\nabla\times\boldsymbol{b})\times\boldsymbol{b} + \nu\Delta\boldsymbol{u}$$

$$\partial_t \boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} + \eta\Delta\boldsymbol{b}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{and} \quad \nabla\cdot\boldsymbol{b} = 0$$

Theory

Fourier transform

helical decomposition

reduction to triads

stability analysis

Simulations

pseudospectral

helical decomposition

full system (all triads)

helical projection

Helical Fourier decomposition

$$(\partial_t + \cancel{\nu k^2})\hat{\mathbf{u}}_{\boldsymbol{k}} = - FT\left[\nabla\left(P + \frac{|\boldsymbol{u}|^2}{2}\right)\right] + \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0}\left(-(i\boldsymbol{p}\times\hat{\mathbf{u}}_{\boldsymbol{p}})^*\times\hat{\mathbf{u}}_{\boldsymbol{q}}^* + (i\boldsymbol{p}\times\hat{\mathbf{b}}_{\boldsymbol{p}})^*\times\hat{\mathbf{b}}_{\boldsymbol{q}}^*\right)$$

$$(\partial_t + \cancel{\eta k^2})\hat{\mathbf{b}}_{\boldsymbol{k}} = i\boldsymbol{k}\times\sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0}\hat{\mathbf{u}}_{\boldsymbol{p}}^*\times\hat{\mathbf{b}}_{\boldsymbol{q}}^*$$

Helical Fourier decomposition

$$(\partial_t + \cancel{\nu k^2})\hat{\mathbf{u}}_{\boldsymbol{k}} = -\, FT\left[\nabla\left(P + \frac{|\boldsymbol{u}|^2}{2}\right)\right] + \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0}\Big(-(i\boldsymbol{p}\times\hat{\mathbf{u}}_{\boldsymbol{p}})^* \times \hat{\mathbf{u}}^*_{\boldsymbol{q}} + (i\boldsymbol{p}\times\hat{\mathbf{b}}_{\boldsymbol{p}})^* \times \hat{\mathbf{b}}^*_{\boldsymbol{q}}\Big)$$

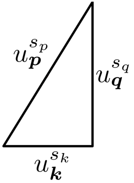
$$(\partial_t + \cancel{\eta k^2})\hat{\mathbf{b}}_{\boldsymbol{k}} = i\boldsymbol{k} \times \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} \hat{\mathbf{u}}^*_{\boldsymbol{p}} \times \hat{\mathbf{b}}^*_{\boldsymbol{q}}$$

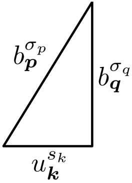
$$\hat{\mathbf{u}}_{\boldsymbol{k}}(t) = u^+_{\boldsymbol{k}}(t)\boldsymbol{h}^+_{\boldsymbol{k}} + u^-_{\boldsymbol{k}}(t)\boldsymbol{h}^-_{\boldsymbol{k}} = \sum_{s_k} u^{s_k}_{\boldsymbol{k}}(t)\boldsymbol{h}^{s_k}_{\boldsymbol{k}}$$

$$\hat{\mathbf{b}}_{\boldsymbol{k}}(t) = b^+_{\boldsymbol{k}}(t)\boldsymbol{h}^+_{\boldsymbol{k}} + b^-_{\boldsymbol{k}}(t)\boldsymbol{h}^-_{\boldsymbol{k}} = \sum_{s_k} b^{s_k}_{\boldsymbol{k}}(t)\boldsymbol{h}^{s_k}_{\boldsymbol{k}}$$

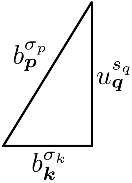
where $i\boldsymbol{k} \times \boldsymbol{h}^{s_k}_{\boldsymbol{k}} = s_k k \boldsymbol{h}^{s_k}_{\boldsymbol{k}}$

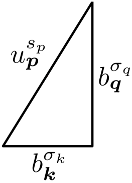
Minimal triadic interaction





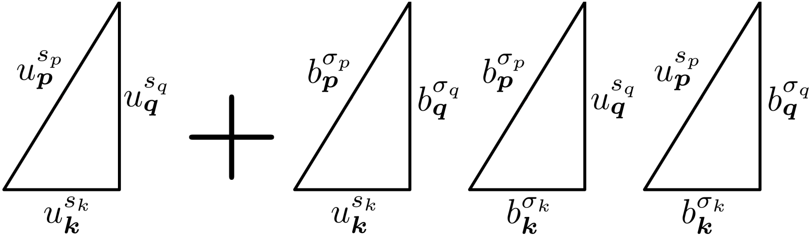
$$\partial_t u_{\boldsymbol{k}}^{s_k *} = \frac{1}{2}\left(g_{s_k s_p s_q}(s_p p - s_q q) u_{\boldsymbol{p}}^{s_p} u_{\boldsymbol{q}}^{s_q} - g_{s_k \sigma_p \sigma_q}(\sigma_p p - \sigma_q q) b_{\boldsymbol{p}}^{\sigma_p} b_{\boldsymbol{q}}^{\sigma_q}\right)$$





$$\partial_t b_{\boldsymbol{k}}^{\sigma_k *} = \frac{\sigma_k k}{2}\left(g_{\sigma_k \sigma_p s_q} b_{\boldsymbol{p}}^{\sigma_p} u_{\boldsymbol{q}}^{s_q} - g_{\sigma_k s_p \sigma_q} u_{\boldsymbol{p}}^{s_p} b_{\boldsymbol{q}}^{\sigma_q}\right)$$

Minimal triadic interaction

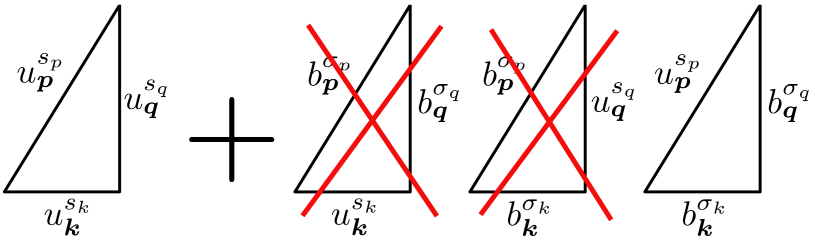


inertial — Lorentz — symmetrised magnetic

4 possibilities for (s_k, s_p, s_q) : $(+,+,+)$, $(-,+,+)$, $(+,-,+)$, $(+,+,-)$
4 possibilities for $(s_k, \sigma_p, \sigma_q)$
2 possibilities for σ_k.

Navier-Stokes: F. Waleffe PoF A, **4**, 350-363 (1992)
homochiral system: Lessinnes et al., Theor. Comput. Fluid Dyn. **23**, 439450 (2009)

Kinematic dynamo



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4 possibilities for (s_k, s_p, s_q) : $(+,+,+)$, $(-,+,+)$, $(+,-,+)$, $(+,+,-)$
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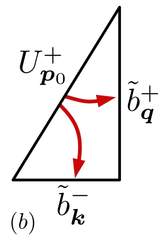
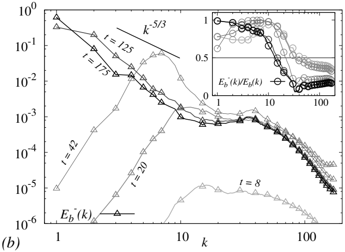
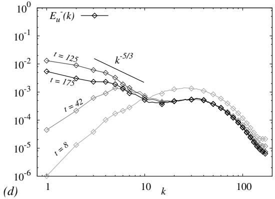
Navier-Stokes: F. Waleffe PoF A, **4**, 350-363 (1992)
homochiral system: Lessinnes et al., Theor. Comput. Fluid Dyn. **23**, 439450 (2009)

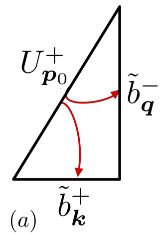
Large-scale dynamo: DNS - laminar flow ($Re_\lambda = 15$)

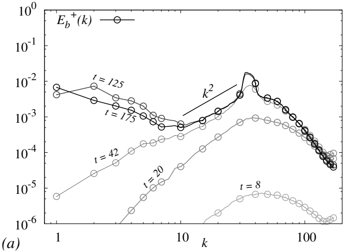
Magnetic field | **Velocity field**

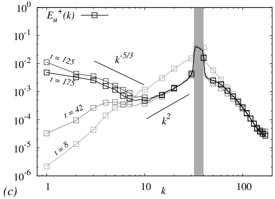


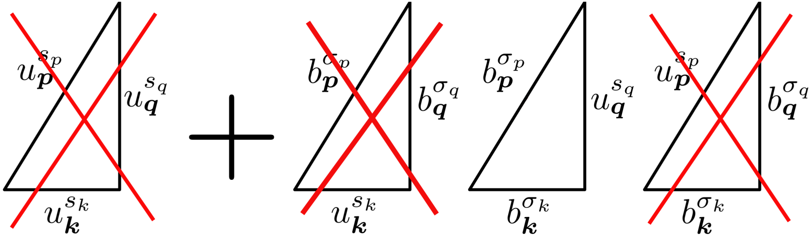








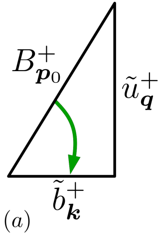


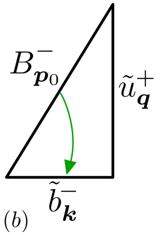


4 possibilities for (s_k, s_p, s_q) : $(+,+,+)$, $(-,+,+)$, $(+,-,+)$, $(+,+,-)$
4 possibilities for $(\sigma_k, \sigma_p, s_q)$
2 possibilities for σ_k.

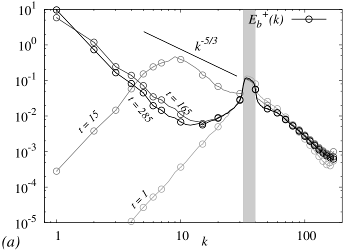
Navier-Stokes: F. Waleffe PoF A, **4**, 350-363 (1992)
homochiral system: Lessinnes et al., Theor. Comput. Fluid Dyn. **23**, 439450 (2009)

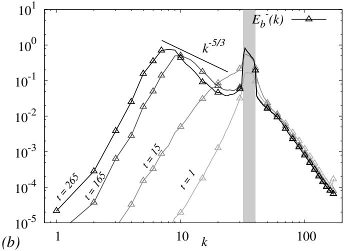
Inverse cascade of magnetic helicity: DNS ($Re_\lambda = 140$)





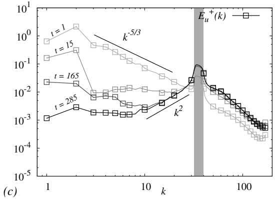
Magnetic field

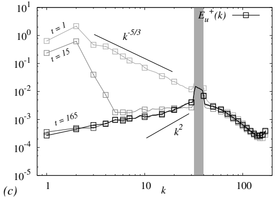




Velocity field

$\boldsymbol{u} = \boldsymbol{u}^+$





Conclusions

The triadic systems give qualitatively correct descriptions of the dynamics.

1. STF-like dynamo on the triad level:
   - α-like triadic dynamo is the dominant large-scale instability.
   - 'anti-α' triadic dynamo is the dominant small-scale instability.
2. The α-like triadic dynamo becomes more dominant with larger scale separation.
3. Inverse cascade of magnetic helicity is most efficient if H_k and H_m are of the same sign.

Thank you

M. Linkmann, G. Sahoo, M. McKay, A. Berera, L. Biferale, ApJ (in press), arXiv:1609.01781

M. Linkmann, A. Berera, M. McKay, J. Jäger, JFM **791**, 61-96 (2016)

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The nature of triad interactions in homogeneous turbulence
Phys. Fluids A 4, 350

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Helical shell models for MHD
Theor. Comput. Fluid Dyn. 23:439-450

L. Biferale, S. Musacchio and F. Toschi (2013)
Split energy-helicity cascades in three-dimensional homogeneous and isotropic turbulence
J. Fluid Mech. 730:309-327

G. Sahoo, F. Bonaccorso and L. Biferale (2015)
On the role of helicity for large- and small-scale turbulent fluctuations
Phys. Rev. E 92, 051002

M. F. Linkmann, A. Berera, M. E. McKay and Julia Jäger (2016)
Helical mode interactions and spectral transfer processes in magnetohydrodynamic turbulence
J. Fluid Mech. 791:61-96

Stability analysis

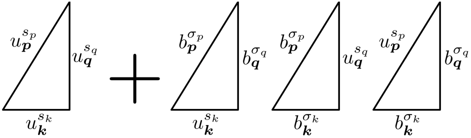
Large wavenumber (small-scale) equilibria:

$$U_{\boldsymbol{p}_0}^{s_{p_0}} \text{ and } B_{\boldsymbol{p}_0}^{s_{p_0}} \text{ with } k < q < p_0$$

Small wavenumber (large-scale) perturbations:

$$
\begin{aligned}
\partial_t \tilde{u}_{\boldsymbol{k}}^{s_k *} &= g^{s_k}_{s_{p_0} s_q} \left(s_p p_0 - s_q q\right) U_{\boldsymbol{p}_0}^{s_{p_0}} \tilde{u}_{\boldsymbol{q}}^{s_q} - g^{s_k}_{\sigma_{p_0} \sigma_q} \left(\sigma_{p_0} p_0 - \sigma_q q\right) B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \tilde{b}_{\boldsymbol{q}}^{\sigma_q} \ , \\
\partial_t \tilde{u}_{\boldsymbol{q}}^{s_q *} &= g^{s_q}_{s_k s_{p_0}} \left(s_k k - s_{p_0} p_0\right) \tilde{u}_{\boldsymbol{k}}^{s_k} U_{\boldsymbol{p}_0}^{s_{p_0}} - g^{s_q}_{\sigma_k \sigma_{p_0}} \left(\sigma_k k - \sigma_{p_0} p_0\right) \tilde{b}_{\boldsymbol{k}}^{\sigma_k} B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \ , \\
\partial_t \tilde{b}_{\boldsymbol{k}}^{\sigma_k *} &= \sigma_k k \left(g^{\sigma_k}_{\sigma_{p_0} s_q} B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \tilde{u}_{\boldsymbol{q}}^{s_q} - g^{\sigma_k}_{s_{p_0} \sigma_q} U_{\boldsymbol{p}_0}^{s_{p_0}} \tilde{b}_{\boldsymbol{q}}^{\sigma_q} \right) \ , \\
\partial_t \tilde{b}_{\boldsymbol{q}}^{\sigma_q *} &= \sigma_q q \left(g^{\sigma_q}_{\sigma_k s_{p_0}} \tilde{b}_{\boldsymbol{k}}^{\sigma_k} U_{\boldsymbol{p}_0}^{s_{p_0}} - g^{\sigma_q}_{s_k \sigma_{p_0}} \tilde{u}_{\boldsymbol{k}}^{s_k} B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \right) \ ,
\end{aligned}
$$

$U_{\boldsymbol{p}_0}^{+} \longrightarrow$ kinematic dynamo
$B_{\boldsymbol{p}_0}^{+} \longrightarrow$ magnetic self-interaction



Stability analysis of single-type equilibria

Large wavenumber (small-scale) equilibria:

$$U_{\boldsymbol{p}_0}^{s_{p_0}} \text{ and } B_{\boldsymbol{p}_0}^{s_{p_0}} \text{ with } k < q < p_0$$

Small wavenumber (large-scale) perturbations:

$$\partial_t \tilde{u}_{\boldsymbol{k}}^{s_k *} = g_{s_{p_0} s_q}^{s_k} (s_p p_0 - s_q q) \, U_{\boldsymbol{p}_0}^{s_{p_0}} \tilde{u}_{\boldsymbol{q}}^{s_q} - g_{\sigma_{p_0} \sigma_q}^{s_k} (\sigma_{p_0} p_0 - \sigma_q q) \, B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \tilde{b}_{\boldsymbol{q}}^{\sigma_q} \ ,$$

$$\partial_t \tilde{u}_{\boldsymbol{q}}^{s_q *} = g_{s_k s_{p_0}}^{s_q} (s_k k - s_{p_0} p_0) \, \tilde{u}_{\boldsymbol{k}}^{s_k} U_{\boldsymbol{p}_0}^{s_{p_0}} - g_{\sigma_k \sigma_{p_0}}^{s_q} (\sigma_k k - \sigma_{p_0} p_0) \, \tilde{b}_{\boldsymbol{k}}^{\sigma_k} B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \ ,$$

$$\partial_t \tilde{b}_{\boldsymbol{k}}^{\sigma_k *} = \sigma_k k \left(g_{\sigma_{p_0} s_q}^{\sigma_k} B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \tilde{u}_{\boldsymbol{q}}^{s_q} - g_{s_{p_0} \sigma_q}^{\sigma_k} U_{\boldsymbol{p}_0}^{s_{p_0}} \tilde{b}_{\boldsymbol{q}}^{\sigma_q} \right) \ ,$$

$$\partial_t \tilde{b}_{\boldsymbol{q}}^{\sigma_q *} = \sigma_q q \left(g_{\sigma_k s_{p_0}}^{\sigma_q} \tilde{b}_{\boldsymbol{k}}^{\sigma_k} U_{\boldsymbol{p}_0}^{s_{p_0}} - g_{s_k \sigma_{p_0}}^{\sigma_q} \tilde{u}_{\boldsymbol{k}}^{s_k} B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \right) \ ,$$

$U_{\boldsymbol{p}_0}^{+} \longrightarrow$ kinematic dynamo

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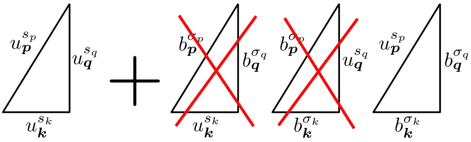

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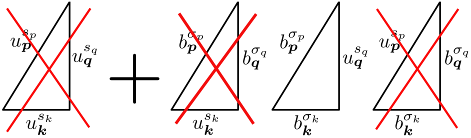
Small wavenumber (large-scale) perturbations:

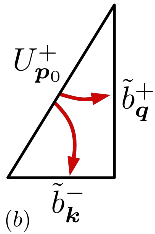
$$\partial_t \tilde{u}_{\boldsymbol{k}}^{s_k *} = g_{s_{p_0} s_q}^{s_k} (s_p p_0 - s_q q) \, U_{\boldsymbol{p}_0}^{s_{p_0}} \tilde{u}_{\boldsymbol{q}}^{s_q} - g_{\sigma_{p_0} \sigma_q}^{s_k} (\sigma_{p_0} p_0 - \sigma_q q) \, B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \tilde{b}_{\boldsymbol{q}}^{\sigma_q} \ ,$$
$$\partial_t \tilde{u}_{\boldsymbol{q}}^{s_q *} = g_{s_k s_{p_0}}^{s_q} (s_k k - s_{p_0} p_0) \, \tilde{u}_{\boldsymbol{k}}^{s_k} U_{\boldsymbol{p}_0}^{s_{p_0}} - g_{\sigma_k \sigma_{p_0}}^{s_q} (\sigma_k k - \sigma_{p_0} p_0) \, \tilde{b}_{\boldsymbol{k}}^{\sigma_k} B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \ ,$$
$$\partial_t \tilde{b}_{\boldsymbol{k}}^{\sigma_k *} = \sigma_k k \left(g_{\sigma_{p_0} s_q}^{\sigma_k} B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \tilde{u}_{\boldsymbol{q}}^{s_q} - g_{s_{p_0} \sigma_q}^{\sigma_k} U_{\boldsymbol{p}_0}^{s_{p_0}} \tilde{b}_{\boldsymbol{q}}^{\sigma_q} \right) \ ,$$
$$\partial_t \tilde{b}_{\boldsymbol{q}}^{\sigma_q *} = \sigma_q q \left(g_{\sigma_k s_{p_0}}^{\sigma_q} \tilde{b}_{\boldsymbol{k}}^{\sigma_k} U_{\boldsymbol{p}_0}^{s_{p_0}} - g_{s_k \sigma_{p_0}}^{\sigma_q} \tilde{u}_{\boldsymbol{k}}^{s_k} B_{\boldsymbol{p}_0}^{\sigma_{p_0}} \right) \ ,$$

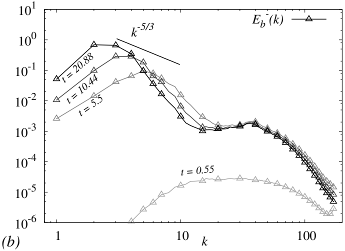
$U_{\boldsymbol{p}_0}^{+} \longrightarrow$ kinematic dynamo
$B_{\boldsymbol{p}_0}^{+} \longrightarrow$ magnetic self-interaction

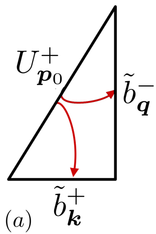


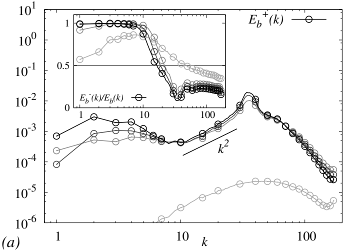
Large-scale dynamo: DNS - turbulent flow ($Re_\lambda = 140$)

Magnetic field









Velocity field

$\boldsymbol{u} = \boldsymbol{u}^+$