Helical mode interactions in magnetohydrodynamics

Moritz Linkmann

Ganapati Sahoo, Luca Biferale
Department of Physics & INFN, University of Rome Tor Vergata, Italy.

Arjun Berera, Mairi McKay, Julia Jäger
School of Physics and Astronomy, University of Edinburgh, Scotland.

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linkmann@roma2.infn.it

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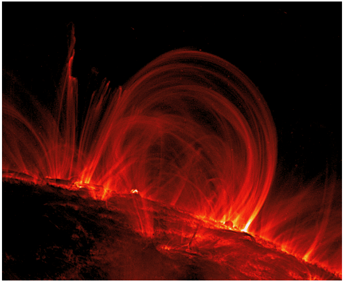
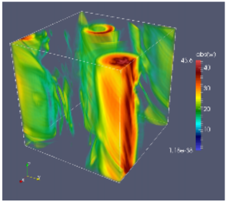
Large-scale magnetic fields in MHD

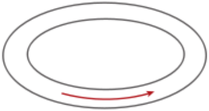


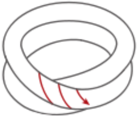
ⓒTRACE operation team,
Lockheed Martin



Dallas & Alexakis PoF 2015

Minnini, Annu. Rev. Fluid Mech. 2011





$$\partial_t \boldsymbol{u} = -\frac{1}{\rho}\nabla P - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}(\nabla\times\boldsymbol{b})\times\boldsymbol{b} + \nu\Delta\boldsymbol{u}$$

$$\partial_t \boldsymbol{b} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} + \eta\Delta\boldsymbol{b}$$

$$\nabla\cdot\boldsymbol{u} = 0 \quad \text{and} \quad \nabla\cdot\boldsymbol{b} = 0$$

$$H_m(t) = \int_V d\boldsymbol{x}\ \boldsymbol{a}(\boldsymbol{x},t)\cdot\boldsymbol{b}(\boldsymbol{x},t) \rightarrow \ \text{inverse cascade}$$

$$H_k(t) = \int_V d\boldsymbol{x}\ \boldsymbol{u}(\boldsymbol{x},t)\cdot\boldsymbol{\omega}(\boldsymbol{x},t) \rightarrow \ \text{dynamo action (e.g. } \alpha\text{-effect)}$$

$$\partial_t \boldsymbol{u} = -\frac{1}{\rho}\nabla P - (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \frac{1}{\rho}(\nabla\times\boldsymbol{b})\times\boldsymbol{b} + \nu\Delta\boldsymbol{u}$$
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Fourier transform (periodic BC)

helical decomposition

reduction of the convolution to triads

reduction to combinations of helical modes

stability analysis of low-dimensional dynamical system

Navier-Stokes: F. Waleffe PoF A, **4**, 350-363 (1992)

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Helical Fourier decomposition

$$\partial_t \hat{\mathbf{u}}_{\boldsymbol{k}} = - FT\left[\nabla\left(P + \frac{|\boldsymbol{u}|^2}{2}\right)\right] + \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0}\left(-(i\boldsymbol{p}\times\hat{\mathbf{u}}_{\boldsymbol{p}})^* \times \hat{\mathbf{u}}^*_{\boldsymbol{q}} + (i\boldsymbol{p}\times\hat{\mathbf{b}}_{\boldsymbol{p}})^* \times \hat{\mathbf{b}}^*_{\boldsymbol{q}}\right)$$

$$\partial_t \hat{\mathbf{b}}_{\boldsymbol{k}} = i\boldsymbol{k} \times \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} \hat{\mathbf{u}}^*_{\boldsymbol{p}} \times \hat{\mathbf{b}}^*_{\boldsymbol{q}}$$

Helical Fourier decomposition

$$\partial_t \hat{\mathbf{u}}_{\boldsymbol{k}} = -\,FT\left[\nabla\left(P + \frac{|\boldsymbol{u}|^2}{2}\right)\right] + \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0}\Big(-(i\boldsymbol{p}\times\hat{\mathbf{u}}_{\boldsymbol{p}})^*\times\hat{\mathbf{u}}^*_{\boldsymbol{q}} + (i\boldsymbol{p}\times\hat{\mathbf{b}}_{\boldsymbol{p}})^*\times\hat{\mathbf{b}}^*_{\boldsymbol{q}}\Big)$$

$$\partial_t \hat{\mathbf{b}}_{\boldsymbol{k}} = i\boldsymbol{k}\times\sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0}\hat{\mathbf{u}}^*_{\boldsymbol{p}}\times\hat{\mathbf{b}}^*_{\boldsymbol{q}}$$

$$\hat{\mathbf{u}}_{\boldsymbol{k}}(t) = u^+_{\boldsymbol{k}}(t)\boldsymbol{h}^+_{\boldsymbol{k}} + u^-_{\boldsymbol{k}}(t)\boldsymbol{h}^-_{\boldsymbol{k}} = \sum_{s_k} u^{s_k}_{\boldsymbol{k}}(t)\boldsymbol{h}^{s_k}_{\boldsymbol{k}}$$

$$\hat{\mathbf{b}}_{\boldsymbol{k}}(t) = b^+_{\boldsymbol{k}}(t)\boldsymbol{h}^+_{\boldsymbol{k}} + b^-_{\boldsymbol{k}}(t)\boldsymbol{h}^-_{\boldsymbol{k}} = \sum_{s_k} b^{s_k}_{\boldsymbol{k}}(t)\boldsymbol{h}^{s_k}_{\boldsymbol{k}}$$

where $i\boldsymbol{k}\times\boldsymbol{h}^{s_k}_{\boldsymbol{k}} = s_k k\,\boldsymbol{h}^{s_k}_{\boldsymbol{k}}$

Helical Fourier decomposition

$$\partial_t u_{\mathbf{k}}^{s_k *} = \frac{1}{2} \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} \Bigg(\sum_{s_p, s_q} g_{s_p s_q}^{s_k} (s_p p - s_q q) u_{\mathbf{p}}^{s_p} u_{\mathbf{q}}^{s_q} - \sum_{\sigma_p, \sigma_q} g_{\sigma_p \sigma_q}^{s_k} (\sigma_p p - \sigma_q q) b_{\mathbf{p}}^{\sigma_p} b_{\mathbf{q}}^{\sigma_q} \Bigg) \ ,$$

$$\partial_t b_{\mathbf{k}}^{\sigma_k *} = \frac{\sigma_k k}{2} \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} \Bigg(\sum_{\sigma_p, s_q} g_{\sigma_p s_q}^{\sigma_k} b_{\mathbf{p}}^{\sigma_p} u_{\mathbf{q}}^{s_q} - \sum_{s_p, \sigma_q} g_{s_p \sigma_q}^{\sigma_k} u_{\mathbf{p}}^{s_p} b_{\mathbf{q}}^{\sigma_q} \Bigg)$$

$$g_{s_p s_q}^{s_k} = -\tfrac{1}{2} \mathbf{h}_{\mathbf{k}}^{s_k *} \cdot \left(\mathbf{h}_{\mathbf{p}}^{s_p *} \times \mathbf{h}_{\mathbf{p}}^{s_p *} \right)$$

Navier-Stokes: F. Waleffe PoF A, **4**, 350-363 (1992)
homochiral system: Lessinnes et al., Theor. Comput. Fluid Dyn. **23**, 439450 (2009)

Helical Fourier decomposition

$$\partial_t u_{\boldsymbol{k}}^{s_k *} = \frac{1}{2} \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} \Bigg(\sum_{s_p, s_q} g^{s_k}_{s_p s_q} (s_p p - s_q q) u^{s_p}_{\boldsymbol{p}} u^{s_q}_{\boldsymbol{q}} - \sum_{\sigma_p, \sigma_q} g^{s_k}_{\sigma_p \sigma_q} (\sigma_p p - \sigma_q q) b^{\sigma_p}_{\boldsymbol{p}} b^{\sigma_q}_{\boldsymbol{q}} \Bigg) \ ,$$

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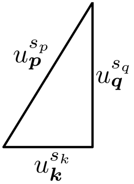
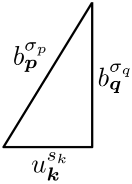
$$g_{s_p s_q}^{s_k} = -\tfrac{1}{2} \boldsymbol{h}_{\boldsymbol{k}}^{s_k *} \cdot \left(\boldsymbol{h}_{\boldsymbol{p}}^{s_p *} \times \boldsymbol{h}_{\boldsymbol{p}}^{s_p *} \right)$$

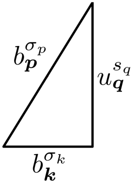
Navier-Stokes: F. Waleffe PoF A, **4**, 350-363 (1992)
homochiral system: Lessinnes et al., Theor. Comput. Fluid Dyn. **23**, 439450 (2009)

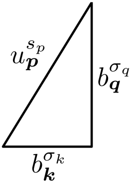
Generic minimal triadic interaction



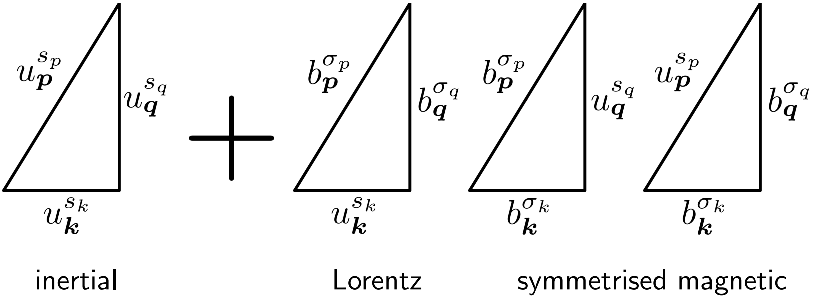


$$\partial_t u_{\boldsymbol{k}}^{s_k\,*} = \frac{1}{2}\left(g^{s_k}_{s_p s_q}(s_p p - s_q q)\,u_{\boldsymbol{p}}^{s_p} u_{\boldsymbol{q}}^{s_q} - g^{s_k}_{\sigma_p \sigma_q}(\sigma_p p - \sigma_q q)\,b_{\boldsymbol{p}}^{\sigma_p} b_{\boldsymbol{q}}^{\sigma_q}\right)$$





$$\partial_t b_{\boldsymbol{k}}^{\sigma_k\,*} = \frac{\sigma_k k}{2}\left(g^{\sigma_k}_{\sigma_p s_q}\,b_{\boldsymbol{p}}^{\sigma_p} u_{\boldsymbol{q}}^{s_q} - g^{\sigma_k}_{s_p \sigma_q}\,u_{\boldsymbol{p}}^{s_p} b_{\boldsymbol{q}}^{\sigma_q}\right)$$



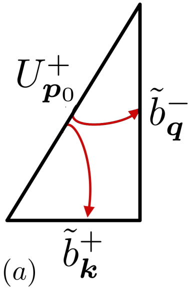
4 possibilities for (s_k, s_p, s_q) : $(+,+,+)$, $(-,+,+)$, $(+,-,+)$, $(+,+,-)$
4 possibilities for $(s_k, \sigma_p, \sigma_q)$
2 possibilities for σ_k.

Stability analysis

$$
\begin{aligned}
\partial_t u_{\boldsymbol{k}}^{s_k\,*} &= g^{s_k}_{s_p s_q}(s_p p - s_q q)\, u_{\boldsymbol{p}}^{s_p} u_{\boldsymbol{q}}^{s_q} - g^{s_k}_{\sigma_p \sigma_q}(\sigma_p p - \sigma_q q)\, b_{\boldsymbol{p}}^{\sigma_p} b_{\boldsymbol{q}}^{\sigma_q} \\
\partial_t u_{\boldsymbol{p}}^{s_p\,*} &= g^{s_p}_{s_q s_k}(s_q q - s_k k)\, u_{\boldsymbol{q}}^{s_q} u_{\boldsymbol{k}}^{s_k} - g^{s_p}_{\sigma_q \sigma_k}(\sigma_q q - \sigma_k k)\, b_{\boldsymbol{q}}^{\sigma_q} b_{\boldsymbol{k}}^{\sigma_k} \\
\partial_t u_{\boldsymbol{q}}^{s_q\,*} &= g^{s_q}_{s_k s_q}(s_k k - s_p p)\, u_{\boldsymbol{k}}^{s_k} u_{\boldsymbol{q}}^{s_q} - g^{s_q}_{\sigma_k \sigma_p}(\sigma_k k - \sigma_p p)\, b_{\boldsymbol{k}}^{\sigma_k} b_{\boldsymbol{p}}^{\sigma_p} \\
\partial_t b_{\boldsymbol{k}}^{\sigma_k\,*} &= \sigma_k k\, \left(g^{\sigma_k}_{\sigma_p s_q} b_{\boldsymbol{p}}^{\sigma_p} u_{\boldsymbol{q}}^{s_q} - g^{\sigma_k}_{s_p \sigma_q} u_{\boldsymbol{p}}^{s_p} b_{\boldsymbol{q}}^{\sigma_q} \right) \\
\partial_t b_{\boldsymbol{p}}^{\sigma_p\,*} &= \sigma_p p\, \left(g^{\sigma_p}_{\sigma_q s_k} b_{\boldsymbol{q}}^{\sigma_q} u_{\boldsymbol{k}}^{s_k} - g^{\sigma_p}_{s_q \sigma_k} u_{\boldsymbol{q}}^{s_q} b_{\boldsymbol{k}}^{\sigma_k} \right) \\
\partial_t b_{\boldsymbol{q}}^{\sigma_q\,*} &= \sigma_q q\, \left(g^{\sigma_q}_{\sigma_k s_p} b_{\boldsymbol{k}}^{\sigma_k} u_{\boldsymbol{p}}^{s_p} - g^{\sigma_q}_{s_k \sigma_p} u_{\boldsymbol{k}}^{s_k} b_{\boldsymbol{p}}^{\sigma_p} \right)
\end{aligned}
$$

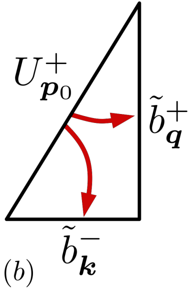
$U^+_{\boldsymbol{p}_0} \longrightarrow$ kinematic dynamo
$B^+_{\boldsymbol{p}_0} \longrightarrow$ magnetic self-interaction

Navier-Stokes: F. Waleffe PoF A, **4**, 350-363 (1992)
homochiral system: ML et al., JFM **791**, 61-96 (2016)



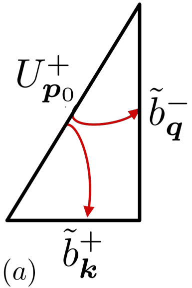
$$\partial_t^2 \tilde{b}^{+}_{\boldsymbol{k}} = |g^{++-}|^2 \, kq |U^{+}_{\boldsymbol{p}_0}|^2 \tilde{b}^{+}_{\boldsymbol{k}} \qquad U^{+}_{\boldsymbol{p}_0} \xrightarrow{\tilde{b}^{-}_{\boldsymbol{q}}} \tilde{b}^{+}_{\boldsymbol{k}}$$

$$\partial_t^2 \tilde{b}^{-}_{\boldsymbol{q}} = |g^{++-}|^2 \, kq |U^{+}_{\boldsymbol{p}_0}|^2 \tilde{b}^{-}_{\boldsymbol{q}} \qquad U^{+}_{\boldsymbol{p}_0} \xrightarrow{\tilde{b}^{+}_{\boldsymbol{k}}} \tilde{b}^{-}_{\boldsymbol{q}}$$



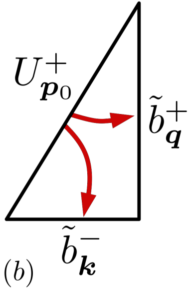
$$\partial_t^2 \tilde{b}^{-}_{\boldsymbol{k}} = |g^{-++}|^2 \, kq |U^{+}_{\boldsymbol{p}_0}|^2 \tilde{b}^{-}_{\boldsymbol{k}} \qquad U^{+}_{\boldsymbol{p}_0} \xrightarrow{\tilde{b}^{+}_{\boldsymbol{q}}} \tilde{b}^{-}_{\boldsymbol{k}}$$

$$\partial_t^2 \tilde{b}^{+}_{\boldsymbol{q}} = |g^{-++}|^2 \, kq |U^{+}_{\boldsymbol{p}_0}|^2 \tilde{b}^{+}_{\boldsymbol{q}} \qquad U^{+}_{\boldsymbol{p}_0} \xrightarrow{\tilde{b}^{-}_{\boldsymbol{k}}} \tilde{b}^{+}_{\boldsymbol{q}}$$



$$\partial_t^2 \tilde{b}^{+}_{\boldsymbol{k}} = |g^{++-}|^2\, kq |U^{+}_{\boldsymbol{p}_0}|^2 \tilde{b}^{+}_{\boldsymbol{k}} \qquad U^{+}_{\boldsymbol{p}_0} \xrightarrow{\tilde{b}^{-}_{\boldsymbol{q}}} \tilde{b}^{+}_{\boldsymbol{k}}$$

$$\partial_t^2 \tilde{b}^{-}_{\boldsymbol{q}} = |g^{++-}|^2\, kq |U^{+}_{\boldsymbol{p}_0}|^2 \tilde{b}^{-}_{\boldsymbol{q}} \qquad U^{+}_{\boldsymbol{p}_0} \xrightarrow{\tilde{b}^{+}_{\boldsymbol{k}}} \tilde{b}^{-}_{\boldsymbol{q}}$$

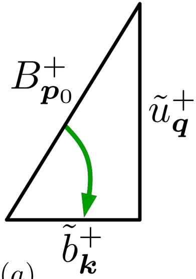


$$\partial_t^2 \tilde{b}^{-}_{\boldsymbol{k}} = |g^{-++}|^2\, kq |U^{+}_{\boldsymbol{p}_0}|^2 \tilde{b}^{-}_{\boldsymbol{k}} \qquad U^{+}_{\boldsymbol{p}_0} \xrightarrow{\tilde{b}^{+}_{\boldsymbol{q}}} \tilde{b}^{-}_{\boldsymbol{k}}$$

$$\partial_t^2 \tilde{b}^{+}_{\boldsymbol{q}} = |g^{-++}|^2\, kq |U^{+}_{\boldsymbol{p}_0}|^2 \tilde{b}^{+}_{\boldsymbol{q}} \qquad U^{+}_{\boldsymbol{p}_0} \xrightarrow{\tilde{b}^{-}_{\boldsymbol{k}}} \tilde{b}^{+}_{\boldsymbol{q}}$$

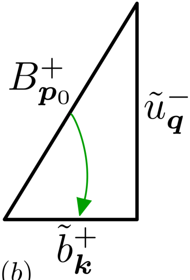
$$\partial_t \boldsymbol{B}_0 = \alpha \nabla \times \boldsymbol{B}_0 \qquad \alpha = -\tfrac{1}{3}\langle \boldsymbol{u} \cdot \boldsymbol{\omega} \rangle$$

Inverse cascade of magnetic helicity



(a)

$$\partial_t^2 \tilde{b}^+_{\boldsymbol{k}} = -|g^{+++}|^2\, k(k-p_0)\, |B^+_{\boldsymbol{p}_0}|^2 \tilde{b}^+_{\boldsymbol{k}} \qquad B^+_{\boldsymbol{p}_0} \xrightarrow{\tilde{u}^+_{\boldsymbol{q}}} \tilde{b}^+_{\boldsymbol{k}}$$



(b)

$$\partial_t^2 \tilde{b}^+_{\boldsymbol{k}} = -|g^{++-}|^2\, k(k-p_0) |B^+_{\boldsymbol{p}_0}|^2 \tilde{b}^+_{\boldsymbol{k}} \qquad B^+_{\boldsymbol{p}_0} \xrightarrow{\tilde{u}^-_{\boldsymbol{q}}} \tilde{b}^+_{\boldsymbol{k}}$$

Conclusions

1. STF-like dynamo on the triad level:
   - α-like triadic dynamo is the dominant large-scale instability.
   - 'anti-α' triadic dynamo is the dominant small-scale instability.
2. The α-like triadic dynamo becomes more dominant with larger scale separation.
3. Inverse cascade of magnetic helicity is most efficient if H_k and H_m are of the same sign.
4. The effect of the Lorentz force on the flow is most prominent if H_k and H_m are of the same sign.

Thank you

M. Linkmann, A. Berera, M. McKay, J. Jäger, JFM **791**, 61-96 (2016)
M. Linkmann, G. Sahoo, M. McKay, A. Berera, L. Biferale, arXiv:1609.01781

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