Chaotic and regular instantons in helical shell models of turbulence

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$$\begin{aligned}\dot{u}_n^+ &= i(ak_{n+1}u_{n+2}^{s_1}u_{n+1}^{s_2*} + bk_n u_{n+1}^{s_3}u_{n-1}^{s_4*} \\ &\quad + ck_{n-1}u_{n-1}^{s_5}u_{n-2}^{s_6}) + f_n^+ - \nu k_n^2 u_n^+ \,, \\ \dot{u}_n^- &= i(ak_{n+1}u_{n+2}^{-s_1}u_{n+1}^{-s_2*} + bk_n u_{n+1}^{-s_3}u_{n-1}^{-s_4*} \\ &\quad + ck_{n-1}u_{n-1}^{-s_5}u_{n-2}^{-s_6}) + f_n^- - \nu k_n^2 u_n^- \,,\end{aligned}$$

$$E = \sum_{n=1}^{\infty} E_n \,, \qquad H = \sum_{n=1}^{\infty} H_n \,,$$

where the Energy and Helicity spectra are

$$E_n = |u_n^+|^2 + |u_n^-|^2, \quad H_n = k_n(|u_n^+|^2 - |u_n^-|^2).$$

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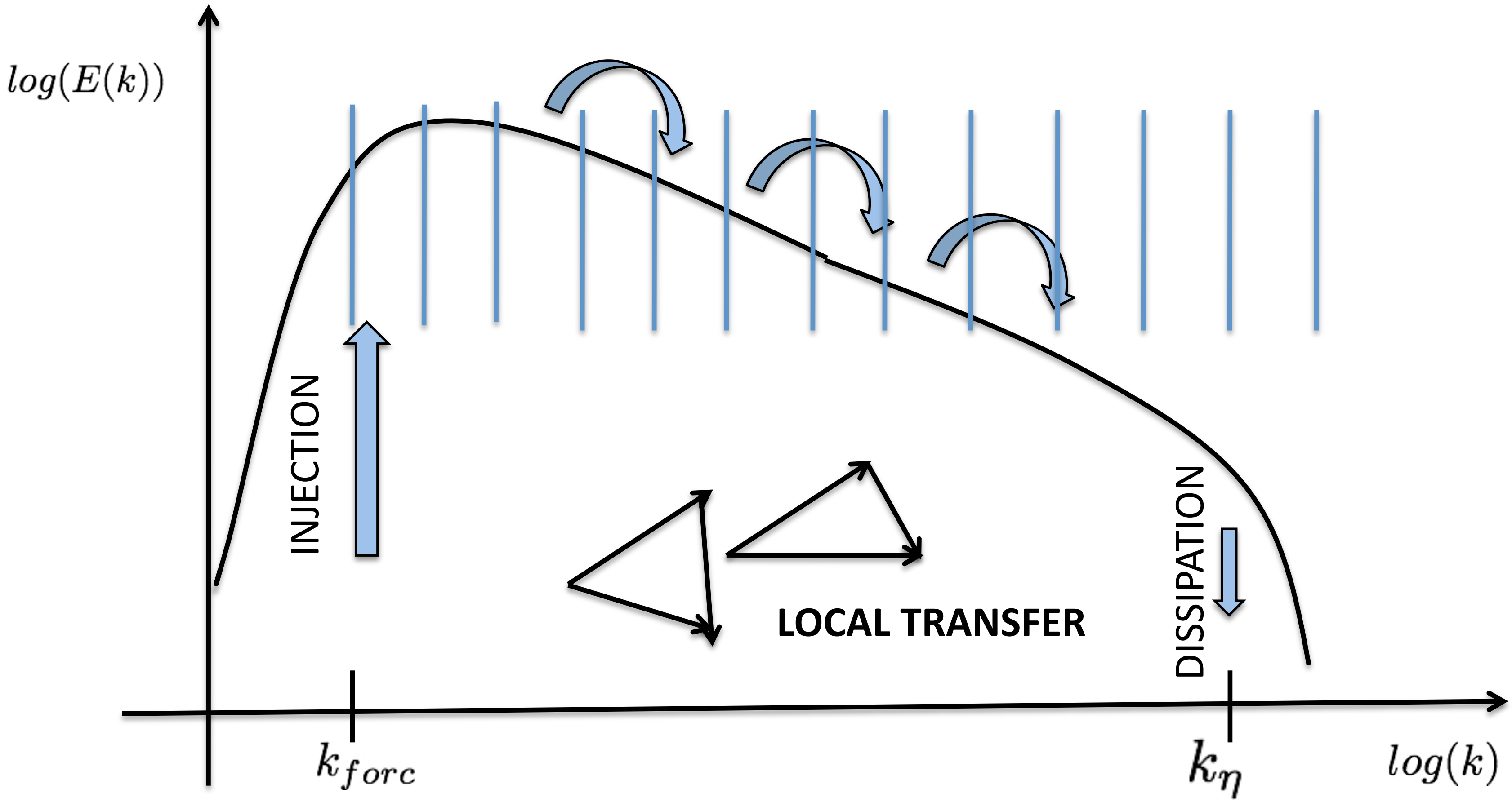

$$\frac{d}{dt}u(k_n) = k_n[a\,u(k_{n+2})u(k_{n+1}) + b\,u(k_{n+1})u(k_{n-1}) + c\,u(k_{n-2})u(k_{n-1})] - \nu k_n^2 u(k_n)$$

Bohr T., Jensen M. H., Paladin G. and Vulpiani A., *Dynamical Systems Approach to Turbulence*, Cambridge, in press (1998)

L.B. Annu. Rev. Fluid. Mech. 35, 441, 2003

The nature of triad interactions in homogeneous turbulence

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$$\boldsymbol{u}(\boldsymbol{k}) = u^+(\boldsymbol{k})\boldsymbol{h}^+(\boldsymbol{k}) + u^-(\boldsymbol{k})\boldsymbol{h}^-(\boldsymbol{k})$$

$$\boldsymbol{h}^\pm = \hat{\boldsymbol{\nu}} \times \hat{\boldsymbol{k}} \pm i\hat{\boldsymbol{\nu}}$$

$$\hat{\boldsymbol{\nu}} = \boldsymbol{z} \times \boldsymbol{k}/||\boldsymbol{z} \times \boldsymbol{k}||$$

$$i\boldsymbol{k} \times \boldsymbol{h}^\pm = \pm k\boldsymbol{h}^\pm$$

$$\begin{cases} E = \sum_{\boldsymbol{k}} |u^+(\boldsymbol{k})|^2 + |u^-(\boldsymbol{k})|^2; \\ H = \sum_{\boldsymbol{k}} k(|u^+(\boldsymbol{k})|^2 - |u^-(\boldsymbol{k})|^2). \end{cases}$$

$$u^{s_k}(\mathbf{k},t) \quad (s_k = \pm 1)$$

$$\frac{d}{dt} u^{s_k}(\mathbf{k}) + \nu k^2 u^{s_k}(\mathbf{k}) = \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} \sum_{s_p, s_q} g_{\mathbf{k},\mathbf{p},\mathbf{q}} (s_p p - s_q q) \times [u^{s_p}(\mathbf{p}) u^{s_q}(\mathbf{q})]^*. \tag{15}$$

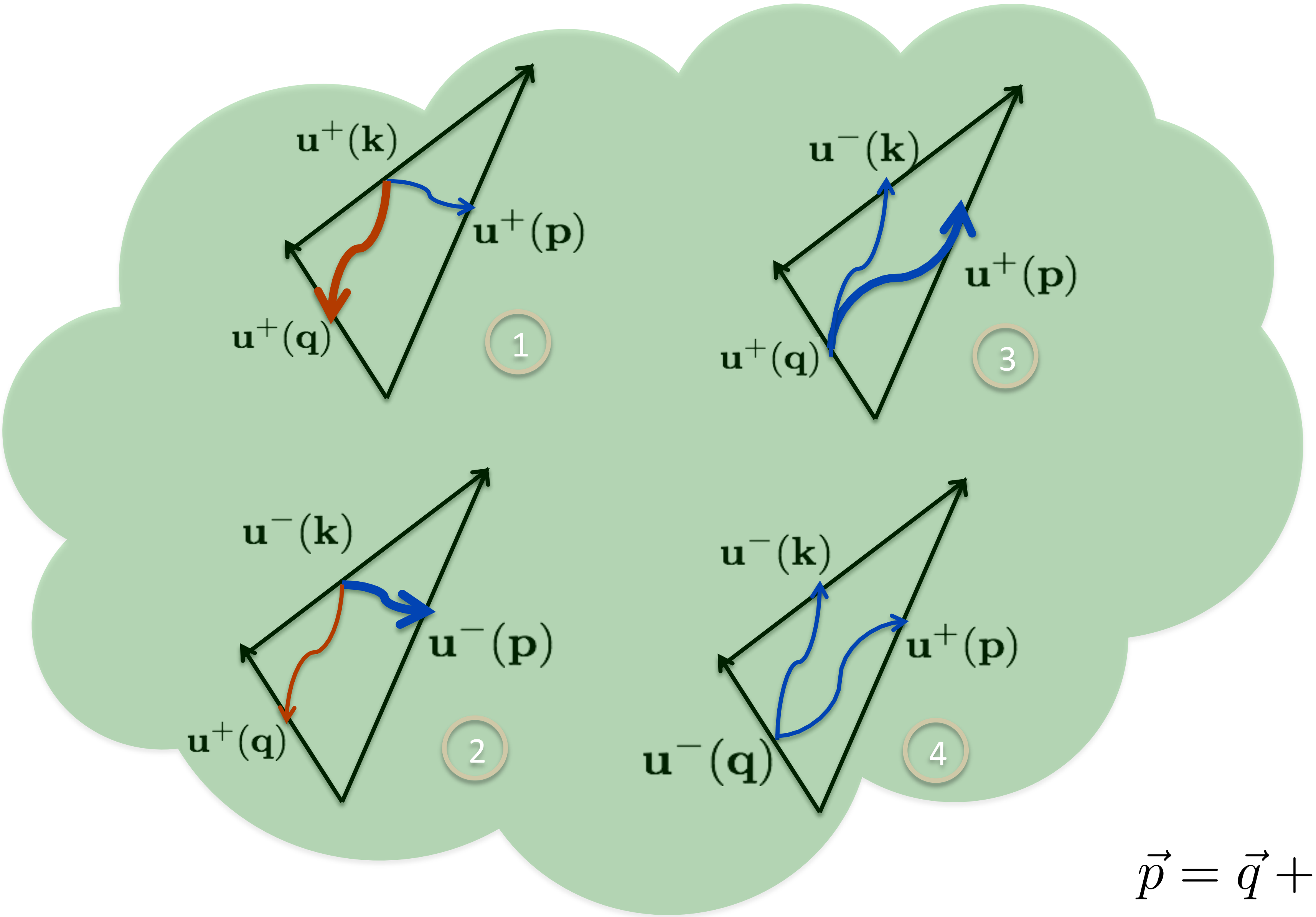
Eight different types of interaction between three modes $u^{s_k}(\mathbf{k})$, $u^{s_p}(\mathbf{p})$, and $u^{s_q}(\mathbf{q})$ with $|\mathbf{k}| < |\mathbf{p}| < |\mathbf{q}|$ are allowed according to the value of the triplet (s_k, s_p, s_q)

$$\dot{u}^{s_k} = r(s_p p - s_q q) \frac{s_k k + s_p p + s_q q}{p} (u^{s_p} u^{s_q})^*,$$

$$\dot{u}^{s_p} = r(s_q q - s_k k) \frac{s_k k + s_p p + s_q q}{p} (u^{s_q} u^{s_k})^*,$$

$$\dot{u}^{s_q} = r(s_k k - s_p p) \frac{s_k k + s_p p + s_q q}{p} (u^{s_k} u^{s_p})^*.$$

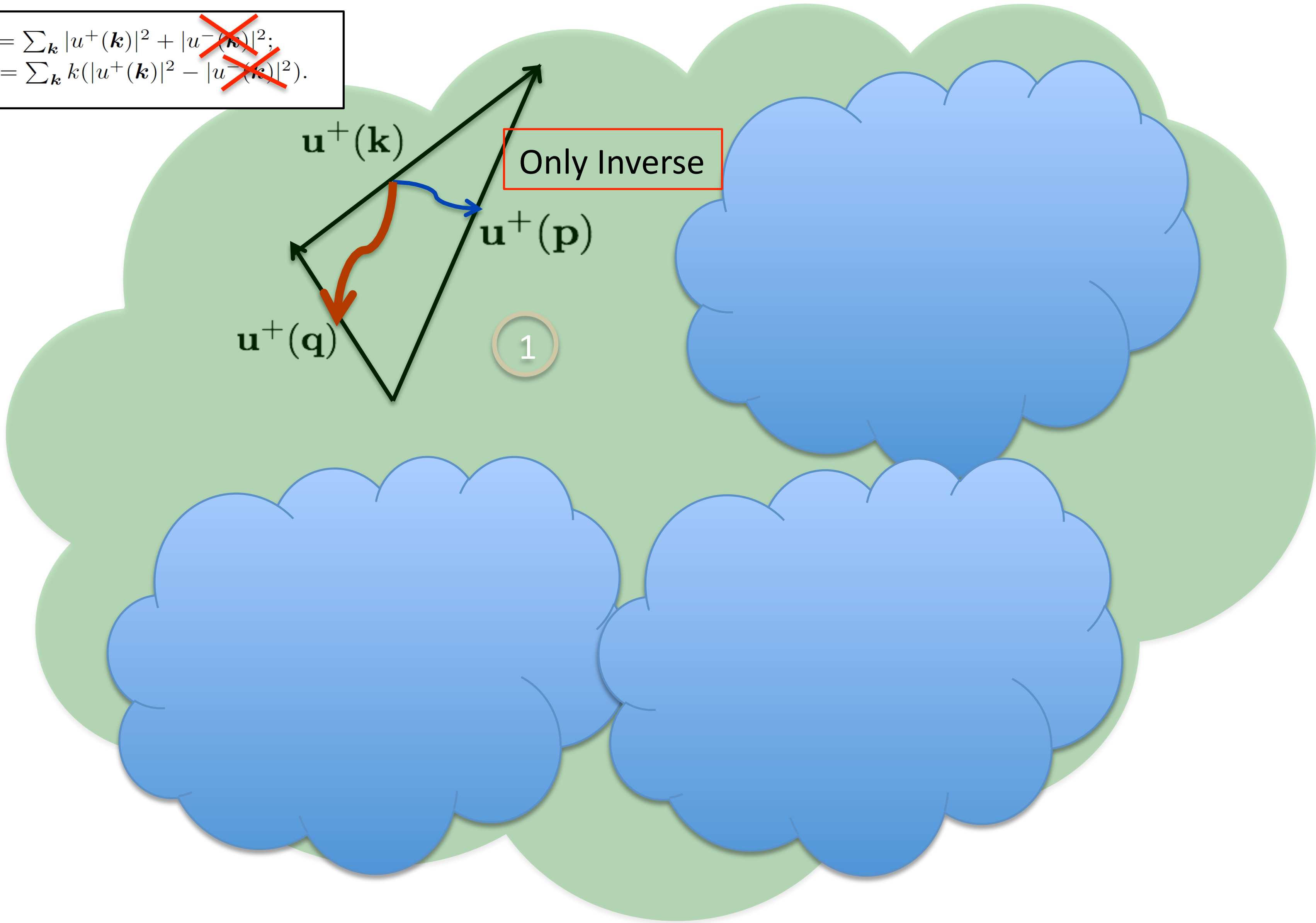
Triadic interactions in the 3D Navier-Stokes equations



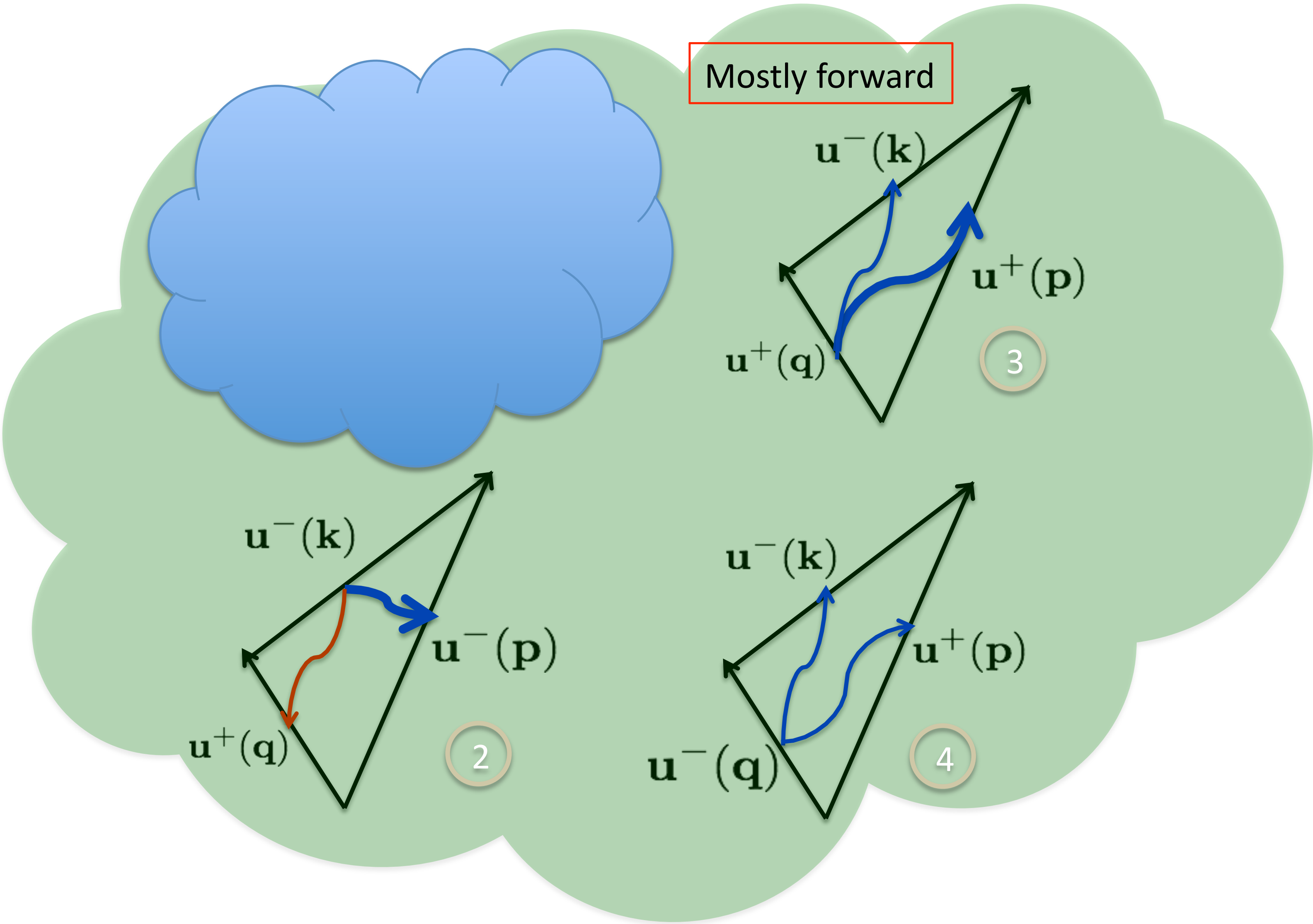
$$\vec{p} = \vec{q} + \vec{k}$$

Triadic interactions in the Decimated 3D Navier-Stokes equations

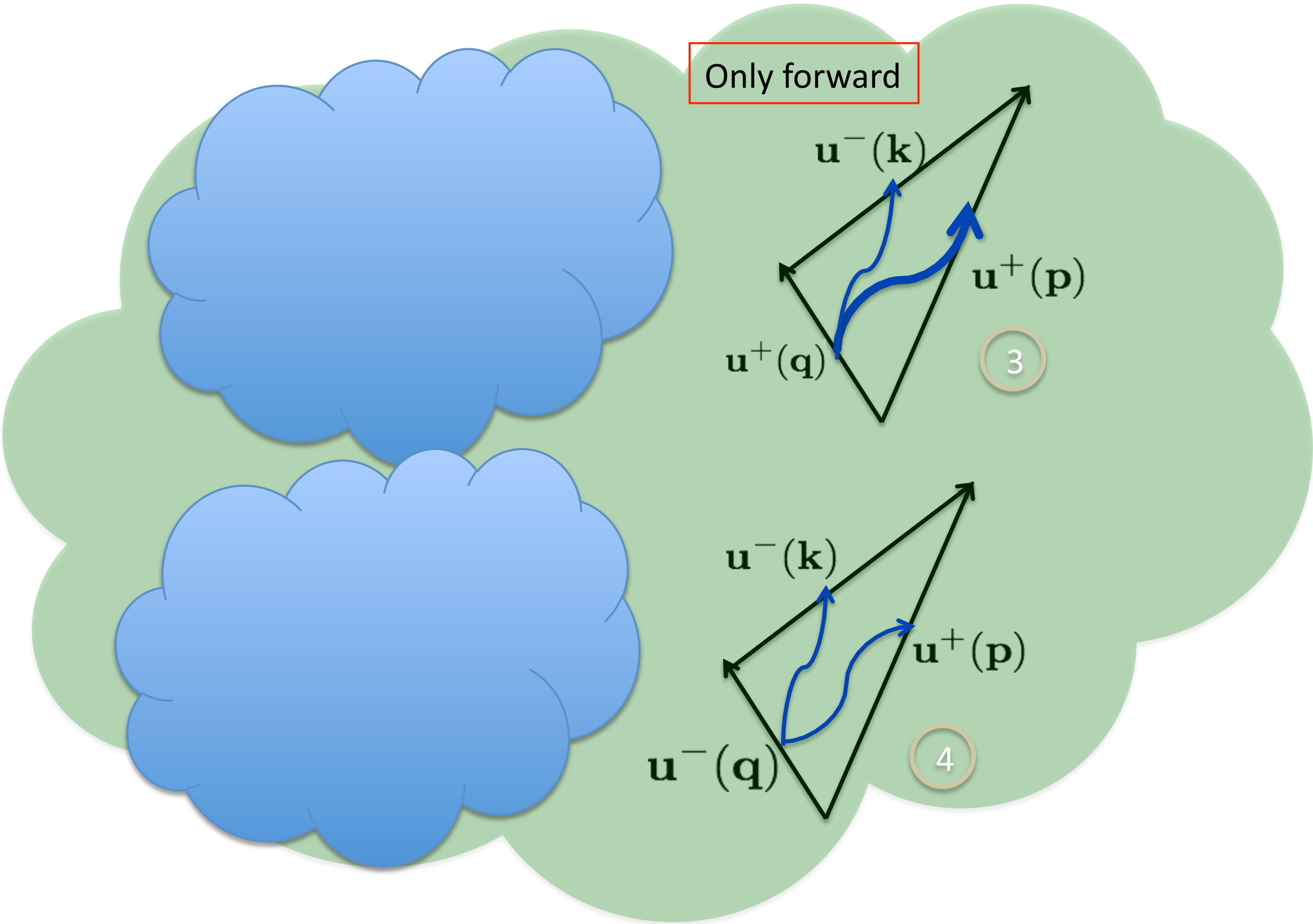
$$\begin{cases} E = \sum_{\boldsymbol{k}} |u^+(\boldsymbol{k})|^2 + |u^-(\boldsymbol{k})|^2; \\ H = \sum_{\boldsymbol{k}} k(|u^+(\boldsymbol{k})|^2 - |u^-(\boldsymbol{k})|^2). \end{cases}$$



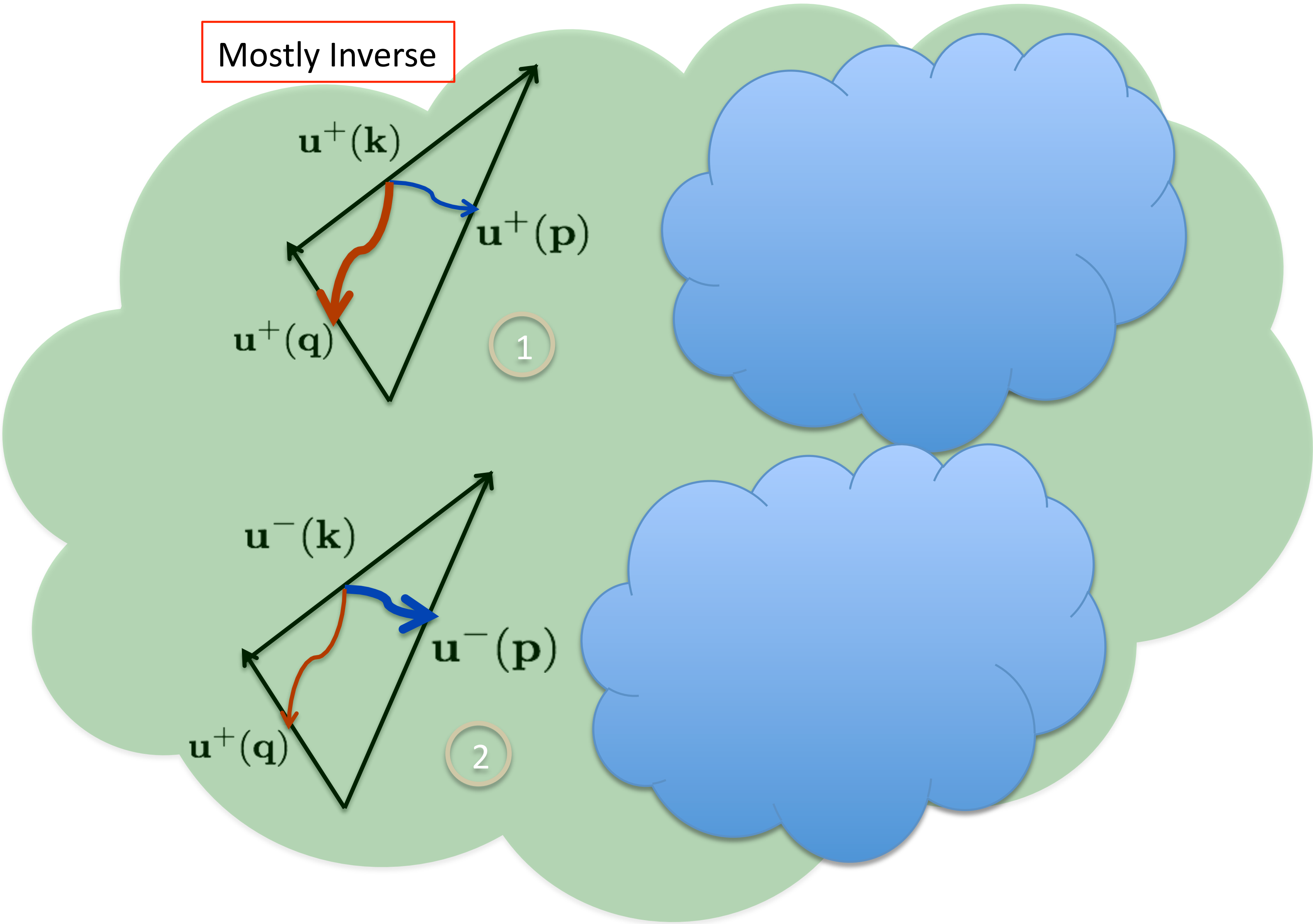
Triadic interactions in the Decimated 3D Navier-Stokes equations



Triadic interactions in the Decimated 3D Navier-Stokes equations



Triadic interactions in the Decimated 3D Navier-Stokes equations

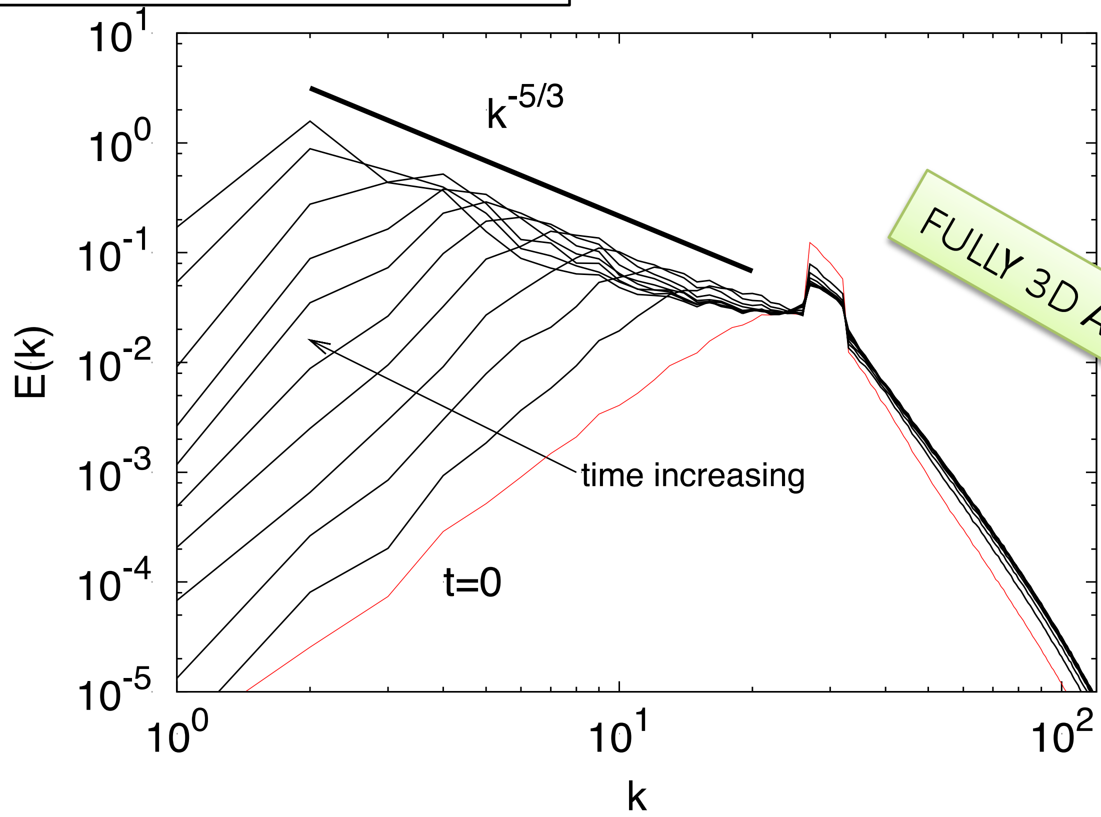


Inverse cascade in DNS of 3D Navier-Stokes equations

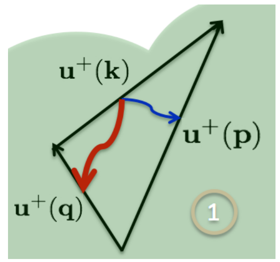
$$\begin{cases} E = \sum_{\boldsymbol{k}} |u^+(\boldsymbol{k})|^2 + |u^-(\boldsymbol{k})|^2; \\ H = \sum_{\boldsymbol{k}} k(|u^+(\boldsymbol{k})|^2 - |u^-(\boldsymbol{k})|^2). \end{cases}$$

Inverse Energy Cascade in Three-Dimensional Isotropic Turbulence

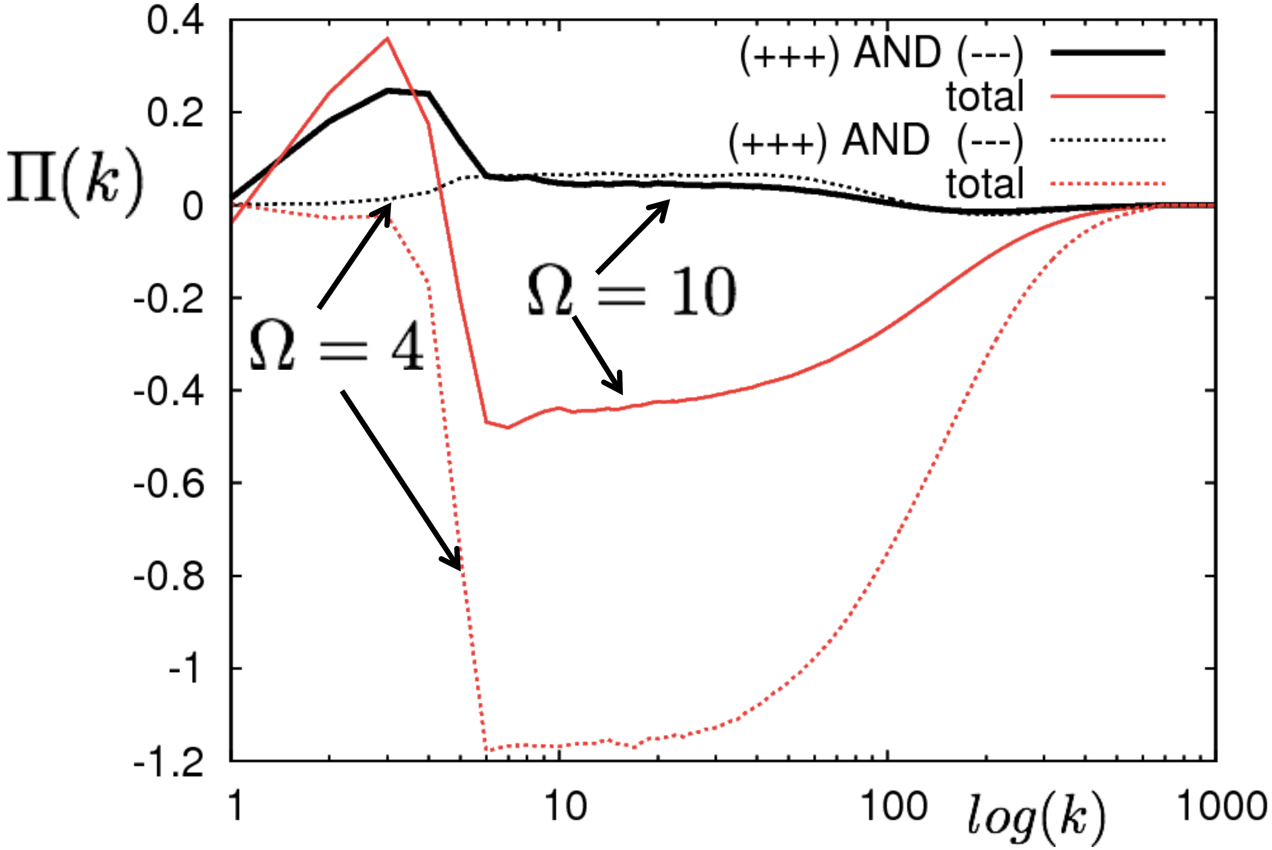
Luca Biferale, Stefano Musacchio, and Federico Toschi
Phys. Rev. Lett. **108**, 164501 – Published 20 April 2012



FULLY 3D AND ISOTORPIC

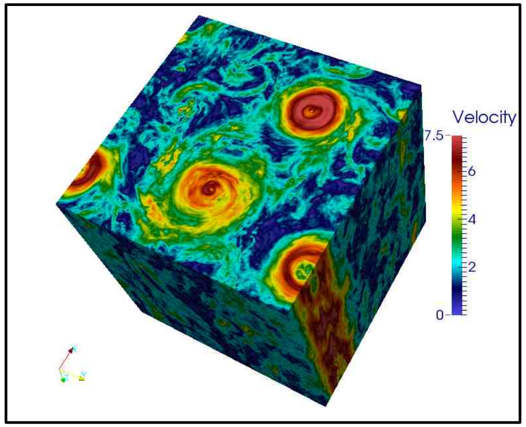


Inverse cascade in DNS of strongly rotating flow



GS and L. Biferale, unpublished

(Rosby = 0.2; 2048^3;



SELF_SIMILAR SOLUTIONS

$$u_n^{\pm} = ik_n^{y-1}U^{\pm}[k_n^y(t-t_c)]\,,$$

$$\Omega = \omega^2 = \sum_{n=1} k_n^2(|u_n^+|^2 + |u_n^-|^2),$$

finite-time blow up

$$\Omega(t) \to \infty \quad \text{as} \quad t \to t_c^-\,.$$

renormalize time to move the blow up at infinite:

$$\frac{d\tau}{dt} = \frac{\omega(t)}{\omega_0}, \qquad w_n^{\pm}(\tau) = -\frac{ik_n u_n^{\pm}(t)}{\omega(t)/\omega_0},$$

finite norm

$$u_n^\pm = i k_n^{y-1} U^\pm[k_n^y (t - t_c)] \,,$$

the form:

$$\left(\frac{d}{d\tau} + A(\tau)\right) w_n^+ = a\lambda^{-2} w_{n+2}^{s_1} w_{n+1}^{s_2 *} + b w_{n+1}^{s_3} w_{n-1}^{s_4 *} + \\ - c\lambda^2 w_{n-1}^{s_5} w_{n-2}^{s_6} \,, \qquad (12)$$

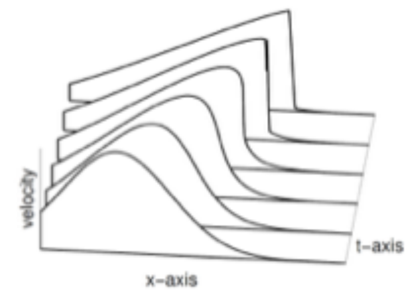
$$\left(\frac{d}{d\tau} + A(\tau)\right) w_n^- = a\lambda^{-2} w_{n+2}^{-s_1} w_{n+1}^{-s_2 *} + b w_{n+1}^{-s_3} w_{n-1}^{-s_4 *} + \\ - c\lambda^2 w_{n-1}^{-s_5} w_{n-2}^{-s_6}, \qquad (13)$$

$$A = \frac{1}{\omega} \frac{d\omega}{d\tau} \,. \qquad w_n^\pm = W^\pm(n - s\tau) \,,$$

$$y = \log_\lambda \frac{\omega(\tau_1)}{\omega_0} \qquad \tau_1 = 1/s, \qquad \omega(\tau) = \omega_0 \exp\left(\int_0^\tau A(\tau') d\tau'\right)$$

Dynamical system view of blowup in inviscid Burgers equation

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad x, t \in \mathbb{R}, \quad \nu \to 0^+ \quad f = u^2/2.$$



Implicit solution: $u = u_0(x_0), \quad x = x_0 + (t - t_0)u$

Blowup (generic solution, simplified by symmetries): $x = ut - u^3 + o(u^3)$

Renormalized (logarithmic) coordinates:

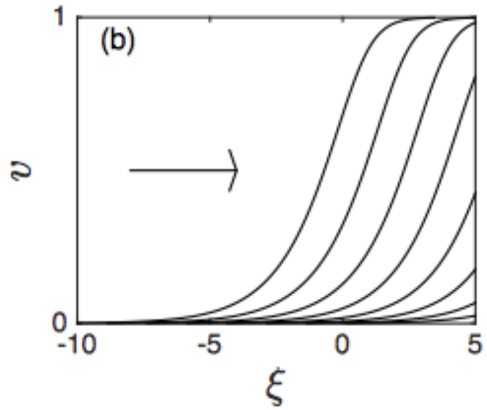
$$t = -e^{-\tau}, \quad x = e^{-\xi}, \quad u = -ve^{\tau - \xi}$$

Renormalized inviscid equation: $\frac{\partial v}{\partial \tau} = -v + v^2 - v\frac{\partial v}{\partial \xi}$

$$e^{-\xi} = ve^{-\xi} + v^3 e^{3\tau - 3\xi}$$

Renormalized solution: $v = F(\xi - a\tau), \quad a = 3/2,$

$$1 = F + e^{-2\eta}F^3$$



stable steady-state traveling wave

Nonlinearity and nonlocality

Incompressible Navier-Stokes equations:

$$\partial_t v_i + v_j \partial_j v_i = -\partial_i p + \nu \partial_{jj} v_i,$$
$$\partial_i v_i = 0.$$

NS equations resolved w.r.t. pressure:

$$\partial_t v_i + \left(\delta_{i\ell} - \partial_{i\ell}\nabla^{-2}\right)\partial_j\left(v_j v_\ell\right) = \nu\nabla^2 v_i$$

(nonlocal quadratic nonlinearity)

1D models that mimic a nonlocal quadratic nonlinearity:

$$\frac{\partial u}{\partial t} + \frac{\partial g}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f, \quad x, t \in \mathbb{R},$$

$$g(x,t) = \frac{1}{2\pi} \int \int K(y-x, z-x) u(y,t) u(z,t) \mathrm{d}y \mathrm{d}z$$

Extra conditions on the kernel function $K(y,z)$:
energy conservation, Hamiltonian structure. etc.

Example: Constantin–Lax–Majda equation $\omega_t - v_x \omega = 0, \quad v_x = H\omega$

Special cases: Desnyansky-Novikov shell model

$$K(y,z) = -\frac{4}{(y+z)^2} - \frac{4}{(y-2z)^2} - \frac{4}{(z-2y)^2}$$

Solution representation

$$k_n = k_0\lambda^n, \quad \lambda = 2^{3/2}, \quad n \in \mathbb{Z}, \qquad 1 \leqslant k_0 < \lambda.$$

$$u_n(t) = k_n^{1/3}\hat{u}\left(k_n^{2/3}, t\right), \quad \hat{u}(k,t) = \int u(x,t)e^{-ikx}dx$$

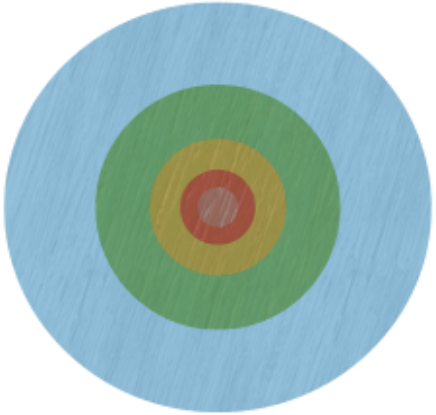
Desnyansky-Novikov shell model equations

$$\frac{\partial u_n}{\partial t} = k_n u_{n-1}^2 - k_{n+1}u_n u_{n+1} - \nu_n u_n + f_n, \quad n \in \mathbb{Z}.$$

Shell speed u_n

Wavenumbers $k_n = k_0\lambda^n$

Viscosity $\nu_n = \nu k_n^2$



Special cases: Sabra shell models

$$K(y,z) = K_\psi(y,z) + K_\psi(z,y), \quad K_\psi(y,z) = \frac{\sigma}{(\sigma y - z)^2} - \frac{(1+c)\sigma^2}{(\sigma^2 y - z)^2} - \frac{c\sigma}{(\sigma y + z)^2}$$

Sabra model equations

$$\frac{\partial u_n}{\partial t} = \mathrm{i}\left[k_{n+1}u_{n+2}u^*_{n+1} - (1+c)k_n u_{n+1}u^*_{n-1} - ck_{n-1}u_{n-1}u_{n-2}\right] - \nu_n u_n + f_n$$

$$\lambda = \sigma^{3/2} = \sqrt{2+\sqrt{5}} \approx 2.058$$

Gledzer-Ohkitani-Yamada (GOY) in 70-80th;
L'vov, Podivilov, Pomyalov, Procaccia, Vandembroucq (Sabra) in 90th

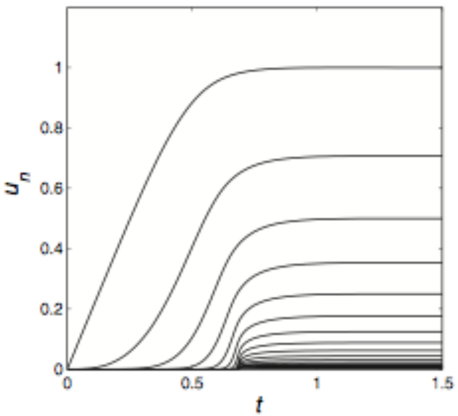
Inviscid invariants: energy, helicity, enstrophy etc. (depending on coefficients)

Dynamics in the inviscid Desnyansky-Novikov shell model: blowup to a shock wave

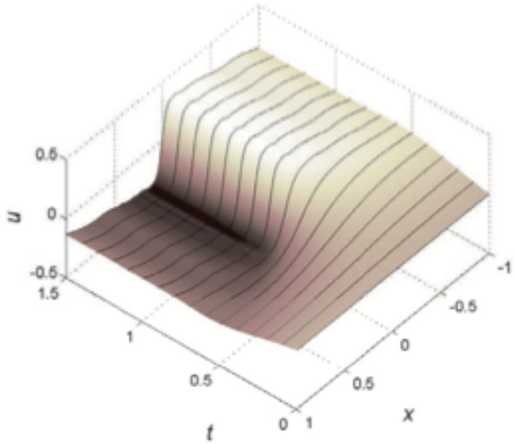
Solution blows up in finite time leading to an asymptotic stationary state with Kolmogorov scaling (for the inviscid limit)

$$u_n = k_n^{-1/3}$$

Dynamics in shell variables



Dynamics in continuous representation



Stationary state: a shock

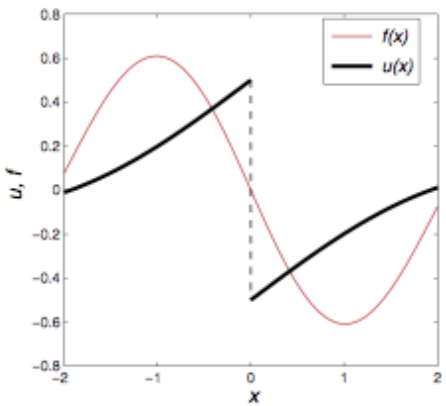
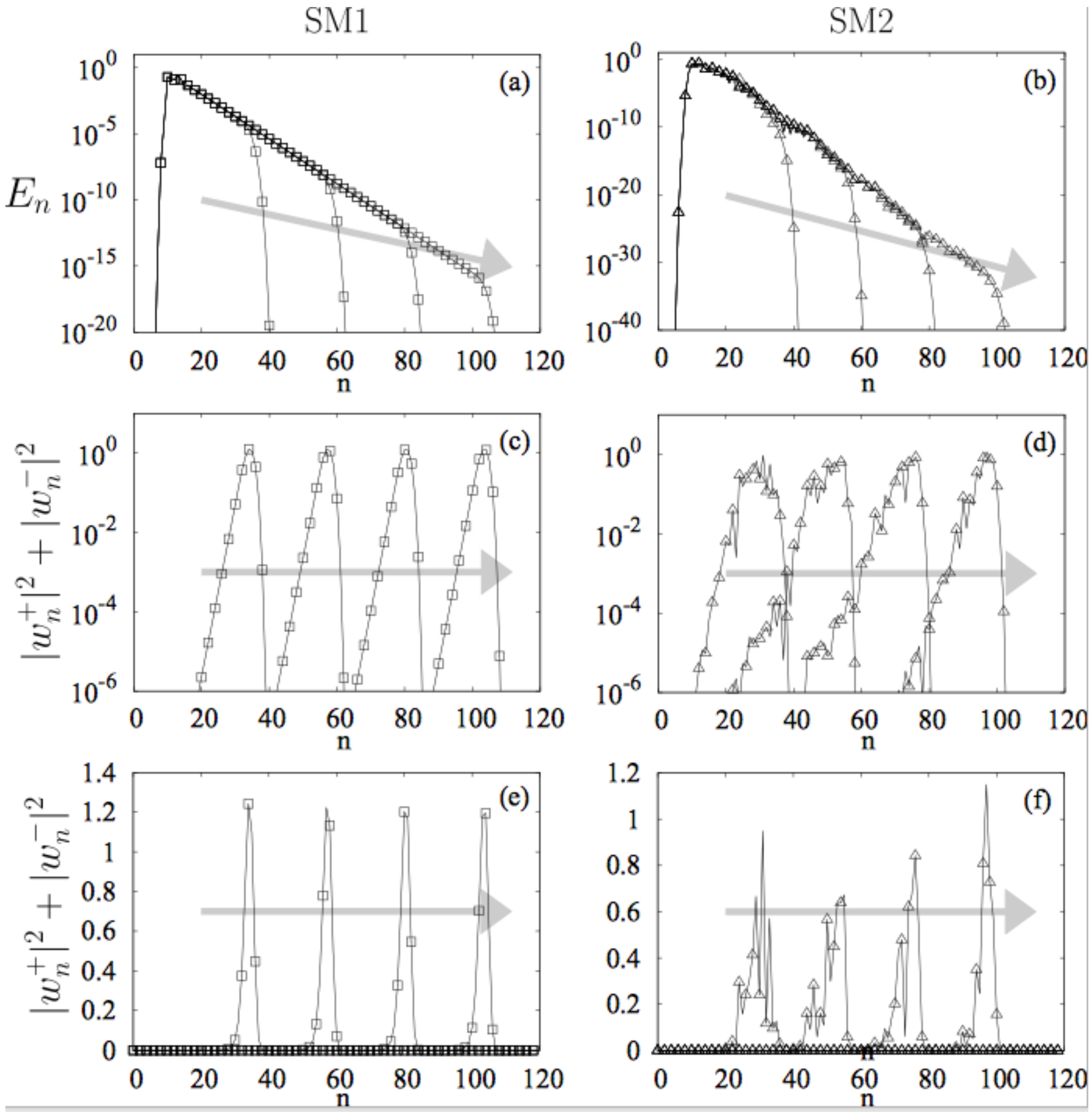


TABLE I. Structure and coefficients of the four helical models (4) and (5). Second column: classes of helical interactions. Without loss of generality, we always choose $a = 1$. These a, b and c coefficients ensure energy and helicity conservation.

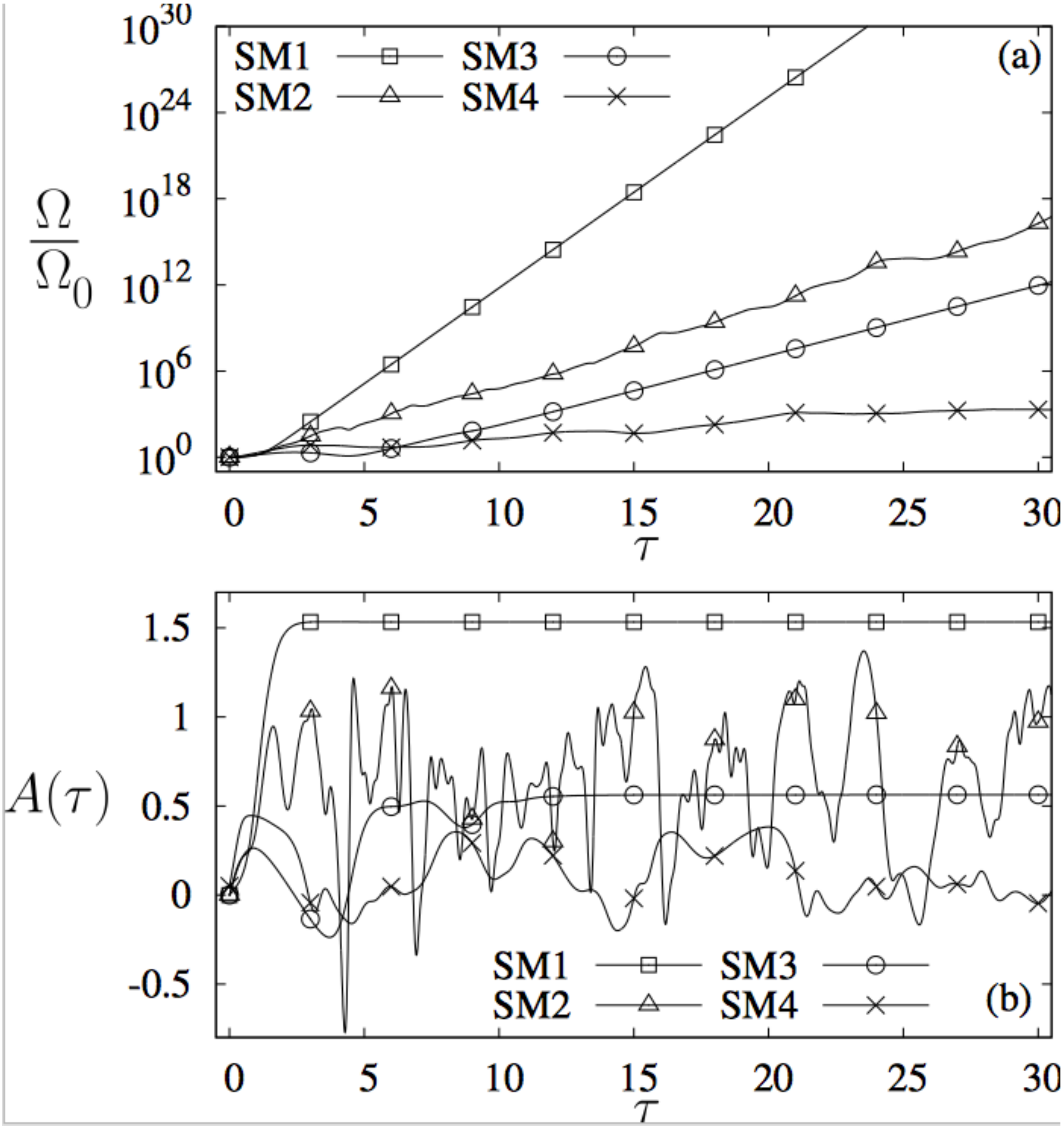
Model	Helical modes coupling	s_1	s_2	s_3	s_4	s_5	s_6	b	c
SM1	$(u_n^+, u_{n+1}^-, u_{n+2}^+)$ or $(u_n^-, u_{n+1}^+, u_{n+2}^-)$	$+$	$-$	$-$	$-$	$-$	$+$	$-1/2$	$1/2$
SM2	$(u_n^+, u_{n+1}^-, u_{n+2}^-)$ or $(u_n^-, u_{n+1}^+, u_{n+2}^+)$	$-$	$-$	$+$	$-$	$+$	$-$	$-5/2$	$-3/2$
SM3	$(u_n^+, u_{n+1}^+, u_{n+2}^-)$ or $(u_n^-, u_{n+1}^-, u_{n+2}^+)$	$-$	$+$	$-$	$+$	$-$	$-$	$-5/6$	$1/6$
SM4	$(u_n^+, u_{n+1}^+, u_{n+2}^+)$ or $(u_n^-, u_{n+1}^-, u_{n+2}^-)$	$+$	$+$	$+$	$+$	$+$	$+$	$-3/2$	$-1/2$

$$\dot{u}_n^+ = i(ak_{n+1}u_{n+2}^{s_1}u_{n+1}^{s_2*} + bk_n u_{n+1}^{s_3}u_{n-1}^{s_4*} + ck_{n-1}u_{n-1}^{s_5}u_{n-2}^{s_6}) + f_n^+ - \nu k_n^2 u_n^+ \,,$$

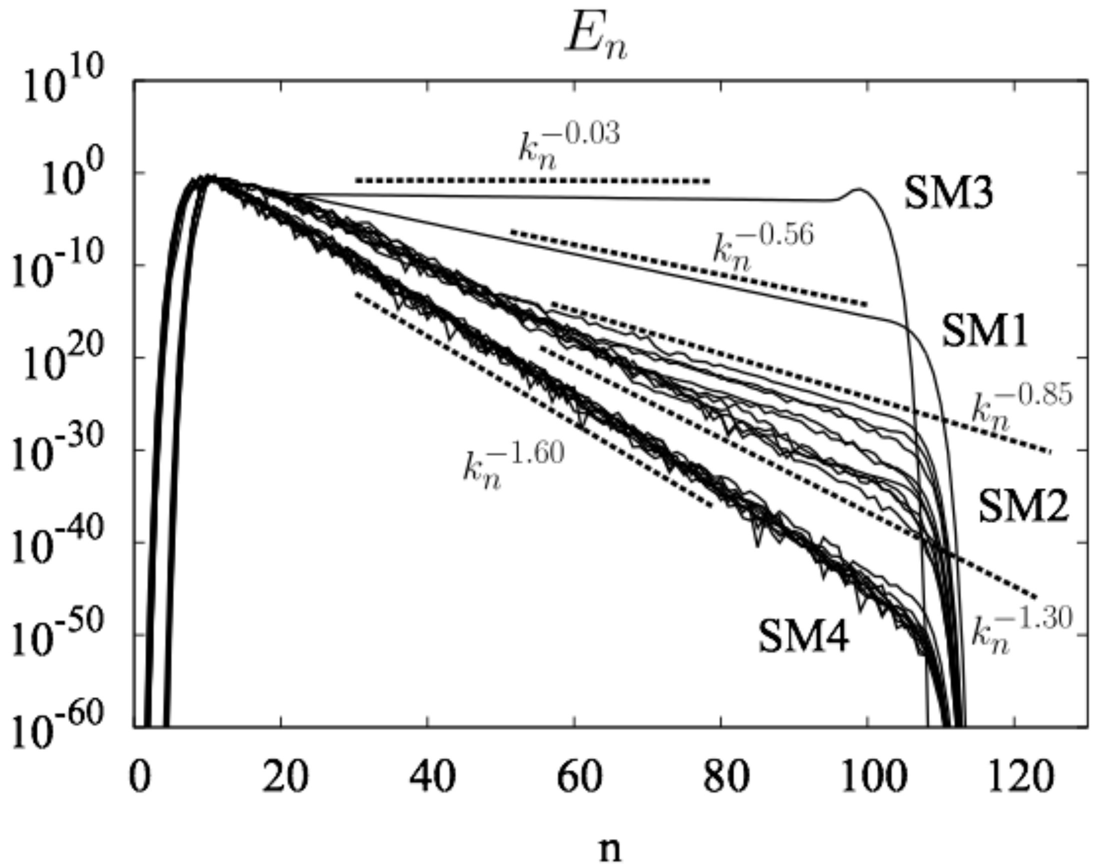
$$\dot{u}_n^- = i(ak_{n+1}u_{n+2}^{-s_1}u_{n+1}^{-s_2*} + bk_n u_{n+1}^{-s_3}u_{n-1}^{-s_4*} + ck_{n-1}u_{n-1}^{-s_5}u_{n-2}^{-s_6}) + f_n^- - \nu k_n^2 u_n^- \,,$$



SM1
SM2
(a)
(b)
(c)
(d)
(e)
(f)
E_n
$|w_n^+|^2 + |w_n^-|^2$
$|w_n^+|^2 + |w_n^-|^2$
n



SM1
SM2
SM3
SM4
(a)
$\frac{\Omega}{\Omega_0}$
10^{30}
10^{24}
10^{18}
10^{12}
10^{6}
10^{0}
0 5 10 15 20 25 30
τ
$A(\tau)$
1.5
1
0.5
0
-0.5
(b)


E_n
$k_n^{-0.03}$
SM3
$k_n^{-0.56}$
SM1
$k_n^{-0.85}$
$k_n^{-1.60}$
SM2
$k_n^{-1.30}$
SM4
10^10
10^0
10^-10
10^20
10^-30
10^-40
10^-50
10^-60
0
20
40
60
80
100
120
n

1

TABLE II. Summary of the dynamical properties of helical shell models (4)–(5) and (26)–(27) in both the blowup and stationary regimes.

	Instanton dynamics		Stationary dynamics	
Model	Type	Energy transfer	Intermittency	Energy dynamics
SM1	Smooth	Forward	yes	Forward cascade
SM2	Chaotic	Forward	no	Forward cascade
SM2E	Chaotic	Backward	no	Backward cascade
SM3	Smooth	Forward	yes	Forward cascade
SM4	Chaotic	Backward	no	Backward flux + quasi-equilibrium