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Slide of the Seminar

Entropy Production in Turbulence Paramterisations

Dr. Denny Gohlke

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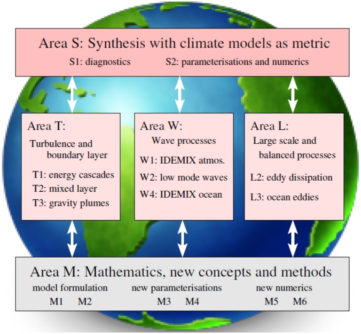
TRR 181 - M4

Entropy Production in Turbulence Parameterisations

by

Denny Gohlke

www.trr-energytransfers.de



- Cooperation: Universities (Hamburg, Bremen) + German research institutes for meteorology and atmospheric physics
- Goal: understanding of interaction of different dynamical regimes; energetically consistent mathematical models

M4 — Entropy Production in Turbulence Parameterisations

Liebniz-Institute of Atmospheric Physics (Kühlungsborn):
Almut Gassmann and Bastian Sommerfeld (PhD student)

University of Hamburg:
Richard Blender, Denny Gohlke (PhD student) and Valerio Lucarini (University of Reading)

Main Idea:
Physically consistent representation of
dissipation and backscatter
by use of the class of Fluctuation Theorems

Keywords: Non-equilibium (entropy production, 2nd law)
Shell models, turbulence models
Inertial range
Subgrid models

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Class of Fluctuation Theorems — Nonequilibrium Systems

Transient Fluctuation Theorem
(Evans & Searles; 1994, 2002):

$$\frac{p\,(\overline{\Omega}_t)}{p\,(-\overline{\Omega}_t)} = e^{\overline{\Omega}_t\;t}$$

Steady State Fluctuation Theorem
(Gallavotti & Cohen; 1995):

$$(t \to \infty)\;\;:\;\frac{p\,(\overline{\Lambda}_t)}{p\,(-\overline{\Lambda}_t)} = e^{\overline{\Lambda}_t\;t}$$

Crooks Fluctuation Theorem (1999):

$$\frac{p_{A\to B}\,(W)}{p_{B\to A}\,(-W)} = e^{\beta\,(W-\Delta F)}$$

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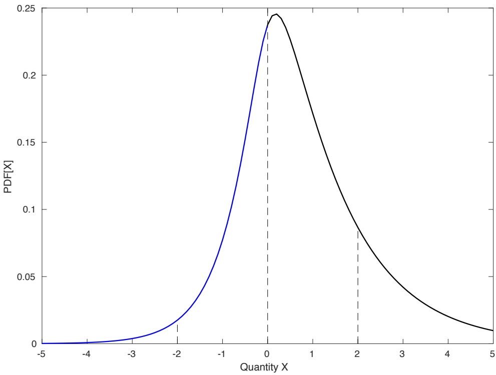

$$\frac{p\left(X\right)}{p\left(-X\right)} = e^{X},\ \langle X\rangle \geq 0$$

Gallavotti - Cohen Steady State Fluctuation Theorem

[Gallavotti G, Cohen EGD. 1995. J. Stat. Phys. 80:93170]

Contraction rate in phase space Λ

$$\frac{d}{dt}x = F(x)\,,\ \Lambda = \nabla \cdot F$$

thermostated:

$$\Lambda \sim EP = {}^{Flux}/_{T}$$

Chaotic hypothesis: $\overline{\Lambda}_t = \frac{1}{t}\int_0^t \Lambda(\tau)d\tau$ (SRB measure)

Large deviation: $p(\overline{\Lambda}_t) \sim e^{t\, I(\overline{\Lambda}_t)}$ $(t \to \infty)$

Time reversibility: $I(\overline{\Lambda}_t) - I(-\overline{\Lambda}_t) = \overline{\Lambda}_t$

$$\lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{p(\overline{\Lambda}_t)}{p(-\overline{\Lambda}_t)}\right) = \overline{\Lambda}_t$$

Universal slope!

[Gallavotti G, Cohen EGD. 1995. J. Stat. Phys. 80:93170]

Contraction rate in phase space Λ

$$\frac{\mathrm{d}}{\mathrm{dt}}x = F(x)\,,\ \Lambda = \nabla \cdot F$$

thermostated: $\Lambda \sim EP = {}^{Flux}/_{T}$

dissipative: $\Lambda < 0$

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$$\lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{p(X)}{p(-X)}\right) = X\,?$$

Quantity?

Universal slope?

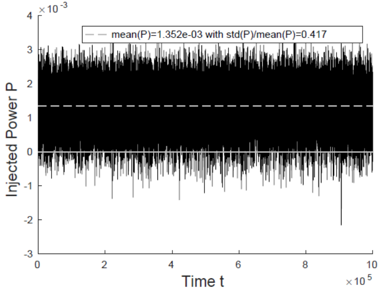
Fluctuation Theorem in dissipative GOY Shell Model?

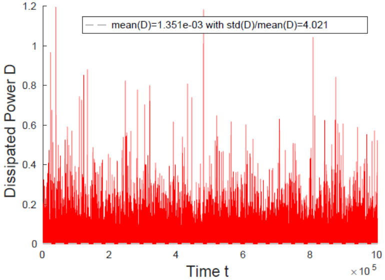
Gallavotti-Cohen-like

$$\lim_{t\to\infty} \frac{1}{t} \ln\left(\frac{p(X)}{p(-X)}\right) = X\ ?$$

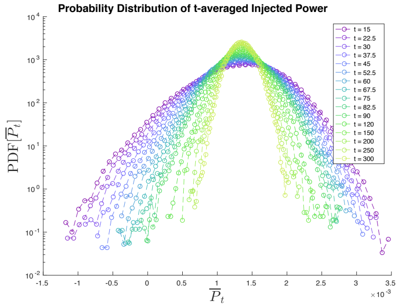
[Aumaitre S, Fauve S, McNamara S, Poggi P. 2001. Eur. Phys. J. B 19: 449-60]
[Gilbert T. 2004. Europhys. Lett. 67 (2): 172-178]

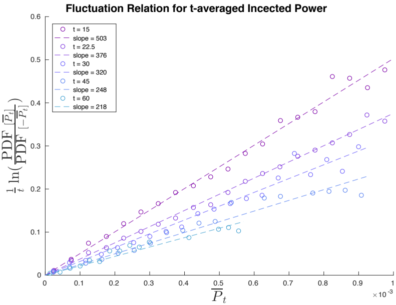
$$\dot{E}(t) = P(t) + D(t), < P >=< D >$$





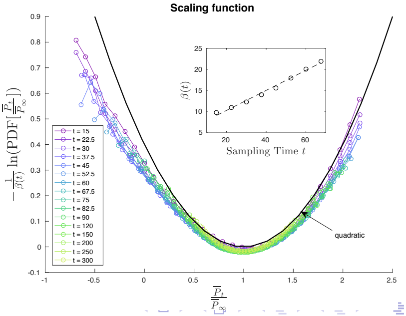
GOY Model — Statistics of t-averaged Injected Power





Distribution of normalised power $Y_t = \frac{\overline{P}_t}{\overline{P}_\infty}$ described by $p(Y_t) \propto e^{-\beta(t)\, I(Y)}$ with scaling function $I(Y)$ and $\beta = A\,t + B$, leading to **non-asymptotic** relation:

$$\ln\left(\frac{p(Y_t)}{p(-Y_t)}\right) = (A\,t + B)\, Y_t$$

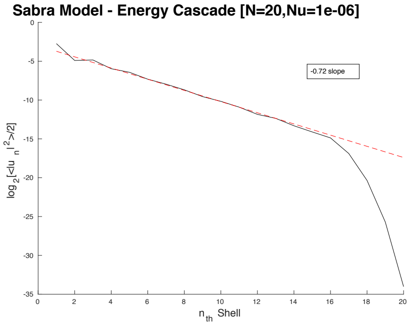


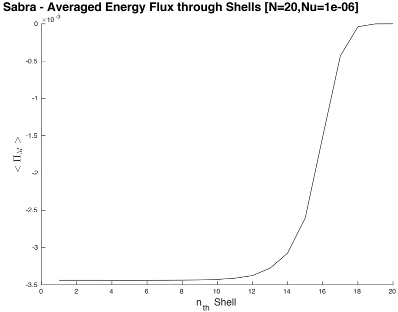
Sabra Model — Statistics of Energy Fluxes Through Shells

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=1}^{M}\frac{1}{2}|u_n|^2 = \underbrace{-k_N\,\Im\left(u_M^* u_{M+1}^* u_{M+2} + \frac{1-\alpha}{2}u_{M-1}^* u_M^* u_{M+1}\right)}_{\Pi_M} + \underbrace{\Re(f\,u_1^*)}_{P} - \nu\underbrace{\sum_{n=1}^{M} k_n^2|u_n|^2}_{D}$$

$$\alpha = 1/2 \text{ (3D-like)}$$

$$<\Pi_M> = -D \text{ (inertial range)}$$





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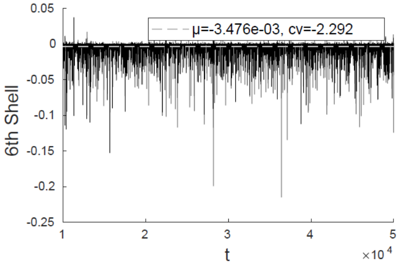

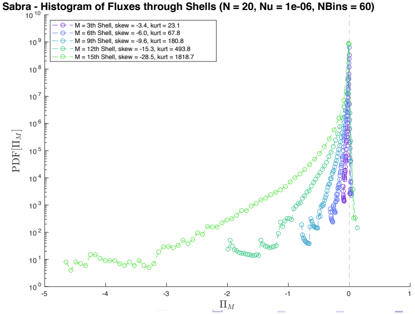

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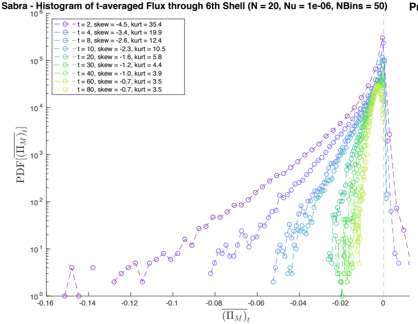
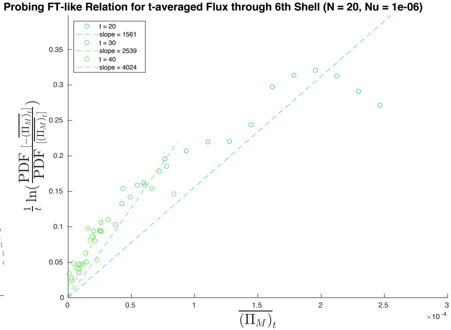
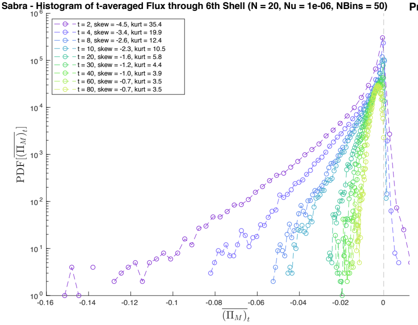
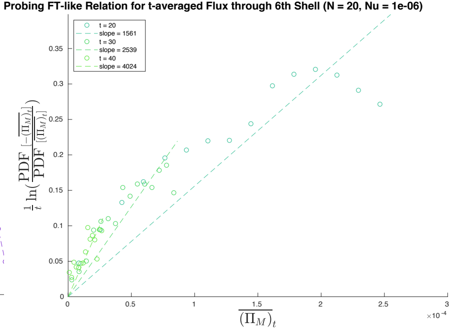




Problems:

- Detection of large deviations
- Definition of entropy production (incl. temperature)





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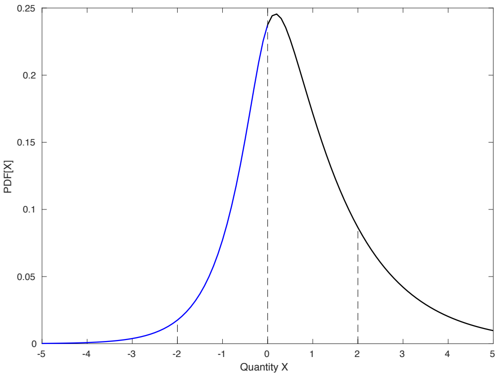

$$\frac{p\left(X\right)}{p\left(-X\right)} = e^{X},\ \langle X\rangle \geq 0$$

[Seifert U. 2008. Eur. Phys. J. B 64]

Coupled Langevin System

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{u} = \vec{F}(\vec{u}) + \vec{\xi}(t)\ ,\ \left\langle \vec{\xi}(t) \otimes \vec{\xi}(t') \right\rangle = 2\,\underline{\underline{D}}\,\delta(t - t')$$

Entropy Change along a Stochastic Trajectory

$$\Delta s_{tot} = \Delta s_{env} + \Delta s_p$$

Concepts from Thermodynamics:

1. From Energy Balance: $\Delta s_{env} = \int_0^\tau \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t}\vec{u}(t)\,\underline{\underline{D}}^{-1}\vec{F}\left(\vec{u}(t)\right)$
2. Entropy of Particle: $\Delta s_p = -\ln \underbrace{p(\vec{u}(t), t)}_{Fokker-Planck} \Big|_0^\tau$

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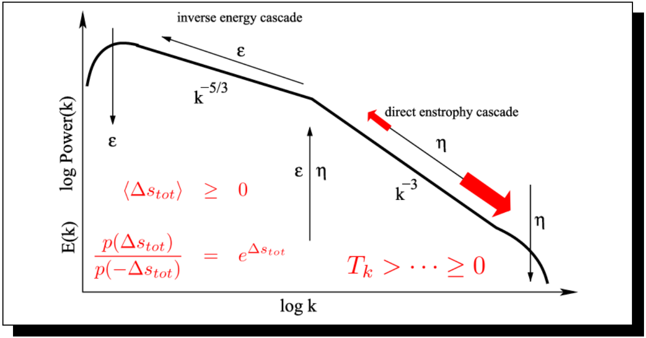
Enstrophy Cascade and Fluctuations in 2D Turbulence



Vorticity Equation

[Baiesi M, Maes C. 2005. Phys. Rev. E 72]

$$d\omega_k = [F_k(\omega) - \nu k^2 \omega_k]\, dt + \sqrt{2\gamma_k}\, dW_k(t)$$

Heat Conduction Problem

$$T_k = \gamma_k / \nu\, k^2$$

$$\Delta s_{tot} = \sum_k J_k / T_k + \ln p(\omega(0)) / p(\omega(\tau))$$

Sabra Shell Model ($k \to k_n$):

$$du_n = \underbrace{[F_n(u)}_{=\, ik_n \left(u^*_{n+1} u_{n+2} - \frac{\alpha}{2} u^*_{n-1} u_{n+1} + \frac{1-\alpha}{4} u_{n-1} u_{n-2}\right)} - \nu k_n^2 u_n]\, dt + \sqrt{2\gamma_n}\, dW_n(t)$$

What about Δs_p?

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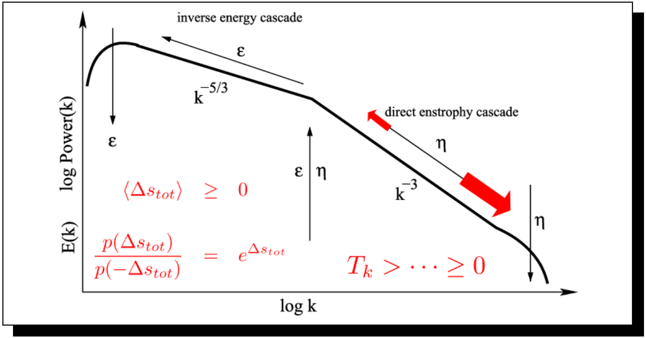

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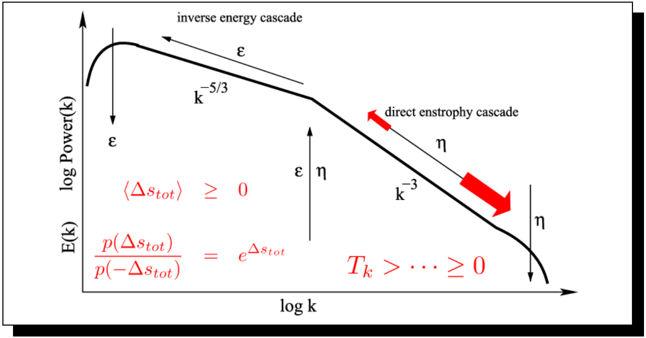

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Reduced Sabra Model — Wouters-Lucarini Parameterisation

$$\frac{\mathrm{d}}{\mathrm{d}t} u_n = \underbrace{ik_n \left(u^*_{n+1}\, u_{n+2} - \frac{\alpha}{2}\, u^*_{n-1}\, u_{n+1} + \frac{1-\alpha}{4}\, u_{n-1}\, u_{n-2}\right)}_{\textit{energy conserving}} + \underbrace{f_n}_{\textit{large scales}} - \underbrace{\nu\, k_n^2\, u_n}_{\textit{small scales}}$$

[Wouters J. Lucarini V. J. Stat. Mech. (2012): P03003]

$$\text{resolved:} \quad \frac{\mathrm{d}}{\mathrm{d}t} X = F_X(X) + \epsilon\, \Psi_X(X, Y)$$

$$\text{unresolved:} \quad \frac{\mathrm{d}}{\mathrm{d}t} Y = F_Y(Y) + \epsilon\, \Psi_Y(X, Y)$$

Weak coupling + Chaotic dynamics

$$\rho_{XY,\epsilon}(A(X)) = \rho_{X_{Reduced},\epsilon}(A(X)) + \mathcal{O}(\epsilon^3) \quad \text{(SRB measures)}$$

$$\text{Reduced:} \quad \frac{\mathrm{d}}{\mathrm{d}t} X = F_X(X) + \epsilon \underbrace{D(X)}_{=<\Psi_X>_{\rho_{Y,0}}} + \epsilon^2 \underbrace{S(X,t)}_{corr(\Psi'_X;\, \rho_{Y,0})} + \epsilon^2 \underbrace{M(X,t)}_{=\int_0^\infty h(X(t-s),s)\, \mathrm{d}s}$$

$$h(X, s) = < \Psi_Y(X, Y)^T \cdot \partial_Y \Psi_X(f^s_X(X), f^s_Y(Y)) >_{\rho_{Y,0}}$$

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Reduced Sabra Model — Wouters-Lucarini Parameterisation

$$M(X,t) = \int_0^\infty \sum_{k,p,q} C^{(1)}_{kpq}\, f_X^s(X_k(t-s))\, X_p(t-s)\, X_q(t-s)\, \mathrm{d}s$$

$$+ \int_0^\infty \sum_{p,q} C^{(2)}_{pq}\, X_p(t-s)\, X_q(t-s)\, \mathrm{d}s$$

$$+ \int_0^\infty \sum_{k,r} C^{(3)}_{kr}\, f_X^s(X_k(t-s))\, X_r(t-s)\, \mathrm{d}s$$

$$+ \int_0^\infty \sum_{r} C^{(4)}_{r}\, X_r(t-s)\, \mathrm{d}s$$

Weighting terms determined by:

$$< \partial_Y \otimes Y^s >_{\rho_{Y,0}} \qquad < Y^s \otimes \partial_Y \otimes Y^s >_{\rho_{Y,0}} \qquad < Y \otimes Y^s \otimes \partial_Y \otimes Y^s >_{\rho_{Y,0}}$$

Reduced Sabra Model — Wouters-Lucarini Parameterisation