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# Slide of the Seminar

# **Developments in Large Eddy Simulations of Compressible Magnetohydrodynamic Turbulence in Space Plasma**


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***ERC Advanced Grant (N. 339032) "NewTURB"***
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# Developments in Large Eddy Simulations of Compressible Magnetohydrodynamic Turbulence in Space Plasma

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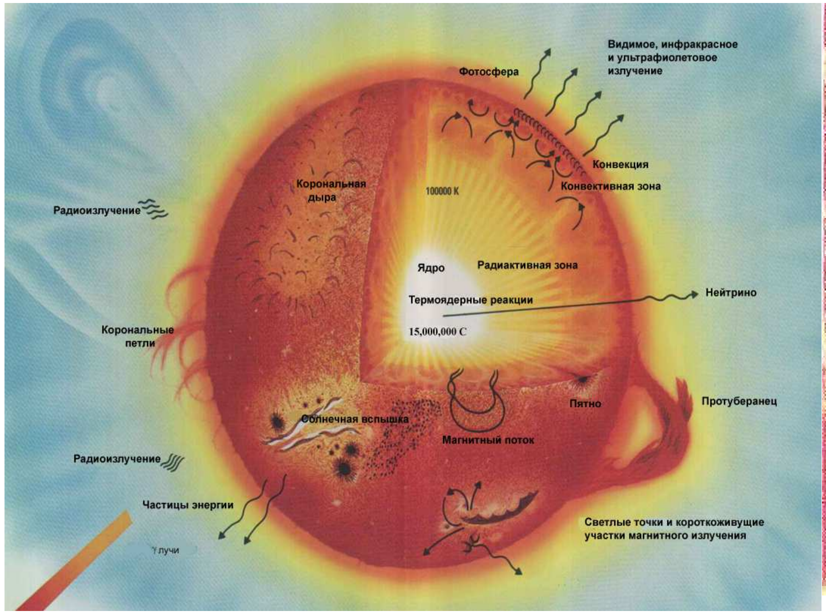
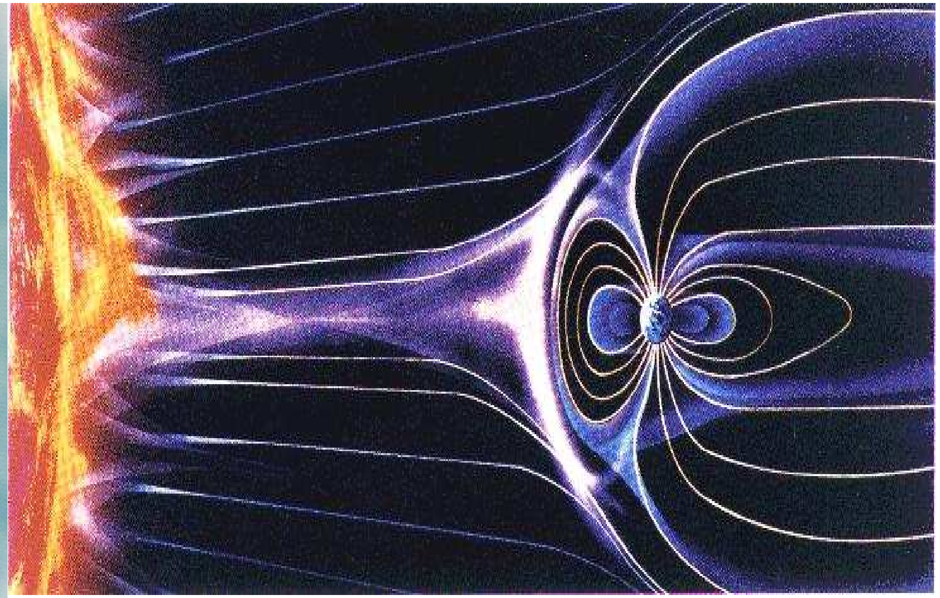
***Rome, 07.11.2014***

# Examples of MHD turbulence

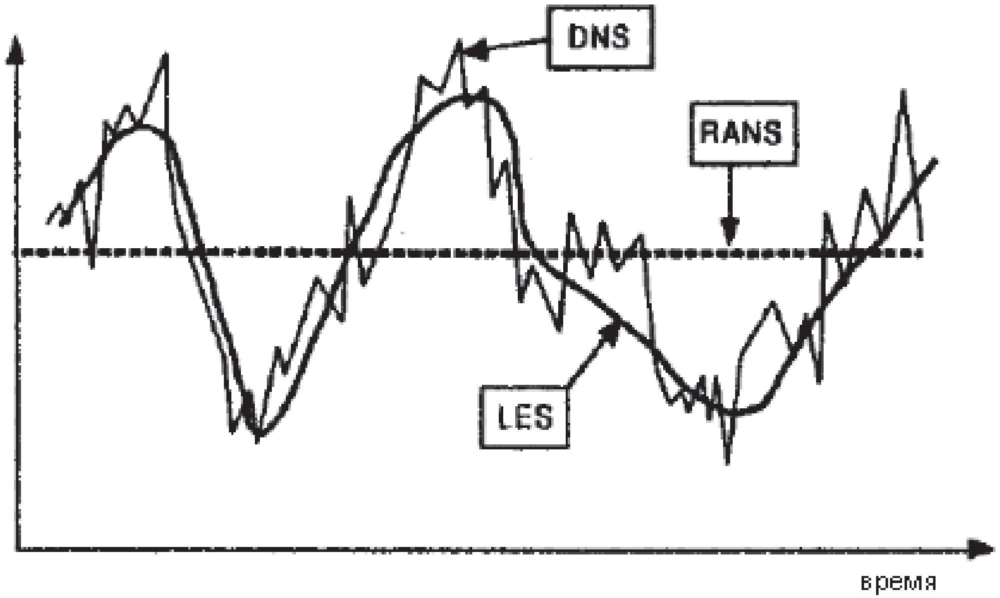


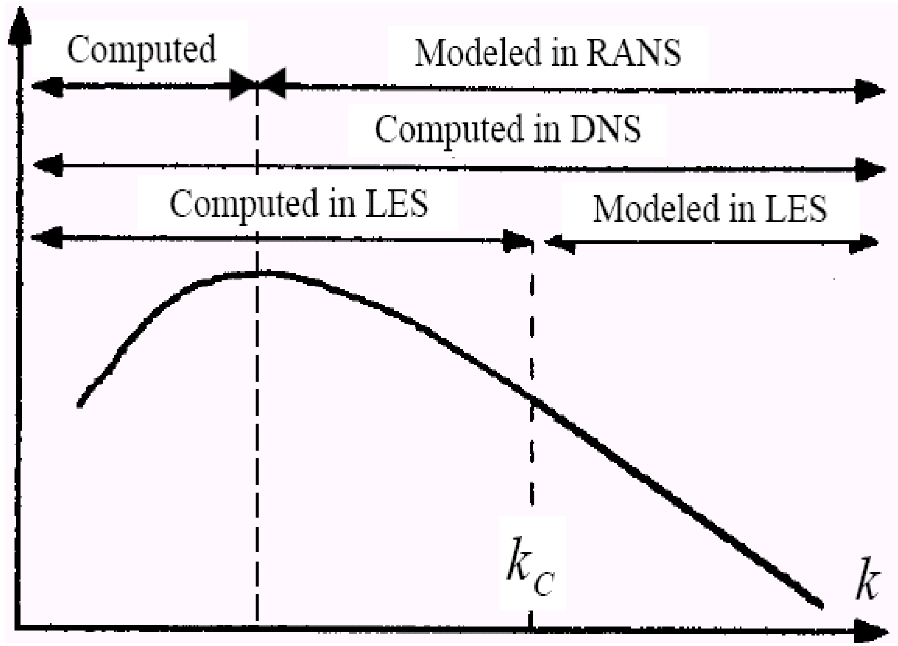
- Solar wind
- Solar tachocline
- Accretion disk

- Solar corona
- Interstellar medium
- Solar convective zone

# RANS, DNS & LES



DNS
RANS
LES
время

Computed
Modeled in RANS
Computed in DNS
Computed in LES
Modeled in LES
$k_C$
$k$

# Plan of presentation

- LES for Perfect gas
- LES for heat-conductive gas
- LES application to space plasma turbulence
- LES for forced MHD turbulence
- Conclusions

# MHD equations

$$
\begin{cases}
\dfrac{\partial \rho}{\partial t} = -\dfrac{\partial \rho u_j}{\partial x_j} \\
\dfrac{\partial \rho u_i}{\partial t} = -\dfrac{\partial}{\partial x_j}(\rho u_i u_j + p\delta_{ij} - \sigma_{ij} - \dfrac{1}{4\pi} B_j B_i + \dfrac{1}{8\pi} B^2) \\
\dfrac{\partial B_i}{\partial t} = -\dfrac{\partial}{\partial x_j}(B_i u_j - B_j u_i) + \eta \nabla^2 B_i \\
\dfrac{\partial B_j}{\partial x_j} = 0
\end{cases}
$$

Polytropic relation: $p = \rho^\gamma$

# Filtering operation

Turbulent flow → Large-scale flow

Turbulent flow → Small-scale flow

$$f = \bar{f} + f'$$

$$\overline{f}(x) = \int_D f(x')G(x, x'; \overline{\Delta})dx'$$

Filter function
(filter $G$ satisfies normalization property)

Filter functions:

- Gaussian filter
- Top-hat filter
- Fourier cutoff filter

# Filtered MHD equations using ordinary filtering

$$\frac{\partial\bar{\rho}\bar{u}_i}{\partial t}+\frac{\partial}{\partial x_j}\left(\bar{\rho}\bar{u}_i\bar{u}_j+\bar{p}\delta_{ij}-\bar{\sigma}_{ij}+\frac{\bar{B}^2}{8\pi}\delta_{ij}-\frac{1}{4\pi}\,\bar{B}_j\bar{B}_i\right)=-\frac{\partial}{\partial x_j}\left(\overline{\rho u_j u_i}-\bar{\rho}\bar{u}_i\bar{u}_j\right)+$$

$$+\frac{1}{4\pi}\frac{\partial}{\partial x_j}\left(\overline{B_j B_i}-\bar{B}_i\bar{B}_j\right)-\frac{\partial}{\partial t}\left(\overline{\rho u_i}-\bar{\rho}\bar{u}_i\right)$$

$$\frac{\partial\bar{\rho}}{\partial t}+\frac{\partial\bar{\rho}\bar{u}_j}{\partial x_j}=-\frac{\partial}{\partial x_j}\left(\overline{\rho u_j}-\bar{\rho}\bar{u}_j\right)$$

additional terms, which are necessary to parametrize!

↓

It requires **additional** numerical resources

# Filtering operation

To simplify equations describing turbulent MHD flow with variable density it is convenient to use Favre filtering (known as mass-weighted filtering) so that to avoid the appearance of extra SGS terms. Therefore, Favre filtering is used in this work.

$$\widetilde{f} = \frac{\overline{\rho f}}{\overline{\rho}}$$

$$f = \widetilde{f} + f''$$

<u>Properties:</u>

$$\overline{\rho u''} \neq 0$$

$$\widetilde{\widetilde{u}} \neq \widetilde{u}$$

$$\widetilde{\widetilde{u}} \neq 0$$

The Favre filtered velocity:

$$\tilde{u}_j = \frac{\overline{\rho u_j}}{\bar{\rho}} = \frac{\int_a^b \rho u_j G(x_j - x'_j, \bar{\Delta}_j) dx'_j}{\int_a^b \rho(x'_j) G(x_j - x'_j, \bar{\Delta}_j) dx'_j}.$$

# Filtered MHD equations

$$\begin{cases} \dfrac{\partial \bar{\rho}}{\partial t} + \dfrac{\partial \bar{\rho}\tilde{u}_j}{\partial x_j} = 0 \\ \dfrac{\partial \bar{\rho}\tilde{u}_i}{\partial t} + \dfrac{\partial}{\partial x_j}\left(\bar{\rho}\tilde{u}_i\tilde{u}_j + \bar{p}\delta_{ij} - \dfrac{1}{\mathrm{Re}}\tilde{\sigma}_{ij} + \dfrac{\bar{B}^2}{2M_A^2} - \dfrac{1}{M_A^2}\bar{B}_i\bar{B}_j\right) = -\dfrac{\partial \tau_{ji}^u}{\partial x_j} \\ \dfrac{\partial \bar{B}_i}{\partial t} + \dfrac{\partial}{\partial x_j}\left(\tilde{u}_j\bar{B}_i - \bar{B}_j\tilde{u}_i\right) - \dfrac{1}{\mathrm{Re}_m}\dfrac{\partial^2 \bar{B}_i}{\partial x_j^2} = -\dfrac{\partial \tau_{ji}^b}{\partial x_j} \end{cases}$$

*Dimensionless form of the equations*

On the right-hand sides of equations the terms designate influence of subgrid terms on the filtered part. To determine these terms, special turbulent parametrizations based on large-scale values describing turbulent MHD flow must be used.

$$\tau_{ij}^u = \bar{\rho}\left((u_j u_i)^{\sim} - \tilde{u}_j\tilde{u}_i\right) - \frac{1}{M_A^2}\left(\overline{B_i B_j} - \bar{B}_j\bar{B}_i\right)$$

$$\tau_{ij}^b = \left(\overline{u_i B_j} - \bar{B}_j\tilde{u}_i\right) - \left(\overline{B_i u_j} - \tilde{u}_j\bar{B}_i\right)$$

Subgrid scale (SGS) or Subfilter-scale (SFS) terms

# Subgrid-scale modeling

Realizability conditions:

$$\tau_{ii} \geqslant 0 \quad \text{for } i \in \{1,2,3\},$$

$$|\tau_{ij}| \leqslant \sqrt{\tau_{ii}\tau_{jj}} \quad \text{for } i,j \in \{1,2,3\},$$

$$\det(\tau_{ij}) \geqslant 0.$$

**Eddy-viscosity model**:

$$\tau_{ij}^{u} - \frac{1}{3}\tau_{kk}^{u}\delta_{ij} = -2\nu_t(\widetilde{S}_{ij} - \frac{1}{3}\widetilde{S}_{kk}\delta_{ij})$$

$$\tau_{ij}^{b} - \frac{1}{3}\tau_{kk}^{b}\delta_{ij} = -2\eta_t\bar{J}_{ij}$$

For obtaining isotropic term from realizability conditions, we have:

$$\tau_{12}^2 + \tau_{13}^2 + \tau_{23}^2 \leq \tau_{11}\tau_{22} + \tau_{11}\tau_{33} + \tau_{22}\tau_{33}$$

Using eddy-viscosity type model:

$$k \geq \frac{1}{2}\sqrt{3}(\nu_t|S^u|) \qquad |\widetilde{S}^u| = (2S_{ij}S_{ij})^{1/2}$$

$$k = \frac{1}{2}(\tau_{11} + \tau_{22} + \tau_{33})$$

$$\widetilde{S}_{ij} = \frac{1}{2}\left(\frac{\partial\widetilde{u}_i}{\partial x_j} + \frac{\partial\widetilde{u}_j}{\partial x_i}\right)$$ - large-scale strain rate tensor

$$\bar{J}_{ij} = \frac{1}{2}\left(\frac{\partial\bar{B}_i}{\partial x_j} - \frac{\partial\bar{B}_j}{\partial x_i}\right)$$ - large-scale magnetic rotation tensor

# Subgrid-scale modeling

Let us define the turbulent viscosity and magnetic diffusion coefficients as follows:

$$\nu_t = C\phi \qquad \eta_t = D\gamma$$

where the arbitrary parameters *C* and *D* are dimensionless functions of space and time while the functional expressions $\gamma$ and $\phi$ may depend on any quantity accessible from within the resolved LES system.

Following Prandtl's mixing length phenomenology turbulent viscosity and magnetic diffusion can both be approximated as the product of the grid filter width and a characteristic velocity V or magnetic field strength B. Since V and *B* are hard to determine directly in a homogeneously turbulent flow, one can instead extract estimates for them from the respective subgrid energy-dissipation.

$$\phi \sim \overline{l}^{4/3}(\varepsilon^K)^{1/3}$$

$$\gamma \sim \overline{l}^{4/3}(\varepsilon^M)^{1/3}$$

*dimensional considerations*

$\varepsilon^K$ - kinetic subgrid-energy dissipation

$\varepsilon^M$ - magnetic subgrid-energy dissipation

The subgrid models differ in the way they estimate these dissipation functions

# Smagorinsky model for MHD

Approximating the subgrid energy dissipation with the aid of the local resolved dissipation rate, $\varepsilon^K \sim \overline{l^2}(2\overline{\mathbf{S}}^v : \overline{\mathbf{S}}^v)^{3/2}$ and $\varepsilon^M \sim \overline{l^2}|\bar{\mathbf{j}}|^3$ leads to the classical Smagorinsky model and its straightforward MHD extension:

$$\tau^u_{ij} - \frac{1}{3}\tau^u_{kk}\delta_{ij} = -2\nu_t(\tilde{S}_{ij} - \frac{1}{3}\tilde{S}_{kk}\delta_{ij}) \qquad \tau^u_{kk} = 2Y_1\bar{\rho}\overline{\Delta}^2 |\tilde{S}^u|^2 \qquad |\tilde{S}^u| = (2S_{ij}S_{ij})^{1/2}$$

*Turbulent viscosity:* $\nu_t = C_1\bar{\rho}\overline{\Delta}^2 |\tilde{S}^u|$

$\tilde{S}_{ij} = \frac{1}{2}(\frac{\partial \tilde{u}_i}{\partial x_j} + \frac{\partial \tilde{u}_j}{\partial x_i})$ - large-scale strain rate tensor

$\bar{J}_{ij} = \frac{1}{2}(\frac{\partial \bar{B}_i}{\partial x_j} - \frac{\partial \bar{B}_j}{\partial x_i})$ - large-scale magnetic rotation tensor

$$\tau^b_{ij} - \frac{1}{3}\tau^b_{kk}\delta_{ij} = -2\eta_t\bar{J}_{ij}$$

*Turbulent diffusivity:* $\eta_t = D_1\overline{\Delta}^2 |j|$

# Kolmogorov model for MHD case

If the grid filter cutoff lies within the inertial spectral range of the homogeneously turbulent system and the nonlinear exchange between resolved kinetic and magnetic energy is much smaller than the respective dissipation, kinetic subgrid-energy dissipation and magnetic subgrid-energy dissipation can be assumed to depend only on time. Thus except a unit factor carrying the necessary dimensions and an explicit filter scale dependence, both functions $\gamma$ and $\phi$ can be absorbed by the nondimensional parameters yielding the Kolmogorov scaling model

$$\nu_t = C_2 \overline{\rho}\,\overline{\Delta}^{4/3}$$ - *turbulent viscosity*

$$\tau_{kk}^u = 2Y_2 \overline{\rho}\,\overline{\Delta}^{4/3} |\tilde{S}^u|$$ - *isotropic term*

$$\eta_t = D_2 \overline{\Delta}^{4/3}$$ - *turbulent magnetic diffusivity*

# Cross-helicity model

The cross helicity is $H^c = \int_V (u \cdot B) dV$

With regard to the mixing length framework outlined above the functions $\phi$ and $\gamma$ are estimated as the product of subgrid dissipation and an associated length scale. However, instead of the local resolved kinetic and magnetic energy dissipation terms, the corresponding local cross-helicity dissipation expressions

$$\overline{\varepsilon}^{Cv} \sim \overline{\mathbf{S}}^v : \overline{\mathbf{S}}^b \qquad \overline{\varepsilon}^{Cb} \sim \overline{\mathbf{j}} \cdot \overline{\boldsymbol{\omega}}$$

the resolved vorticity $\overline{\boldsymbol{\omega}} = \nabla \times \overline{\mathbf{v}}$

the electric current density $\mathbf{j} = \nabla \times \mathbf{b}$

# Cross-helicity model

The cross-helicity is related to the transfer between kinetic and magnetic energies caused by the Lorentz force. Therefore, the cross helicity allows one to estimate the energy exchange between large and small scales in the LES method:

$$\nu_t = C_3 \overline{\rho} \overline{\Delta}^2 | \widetilde{S}_{ij}^u \widetilde{S}_{ij}^b |^{1/2} \qquad \text{- } \textit{turbulent viscosity}$$

$$\tau_{kk}^u = 2 Y_3 \overline{\rho} \overline{\Delta}^2 | \widetilde{f} || \widetilde{S}^u | \qquad \text{- } \textit{isotropic term} \qquad \bar{f} = | \widetilde{S}_{ij}^u \overline{S}_{ij}^b |^{1/2}$$

$$\eta_t = D_3 \overline{\Delta}^2 \, \mathrm{sgn}(\bar{j} \widetilde{\omega}) | \bar{j} \widetilde{\omega} |^{1/2} \qquad \text{- } \textit{turbulent magnetic diffusivity}$$

Since the energetically most favorable configuration of the local velocity and magnetic field is **V||B**, any decrease of alignment of these two vectors increases locally the magnetic energy. The process works inversely when the local alignment increases whereby the direction of change is given by the sign of the local cross-helicity dissipation. The justification is based on the existence of the inverse magnetic helicity cascade.

# Scale-similarity model for MHD case

The scale-similarity model is not of the eddy-viscosity-type. It is based on the assumption that the component of the SGS most active in the energy transfer from large to small scales can be estimated with sufficient accuracy from the smallest resolved scale, which can be obtained by filtering the subgrid-scale quantities

$$\tau_{ij}^{u} = \overline{\rho}\left(\left(\tilde{u}_j \tilde{u}_i\right)^{\sim} - \tilde{\tilde{u}}_j \tilde{\tilde{u}}_i\right) - \frac{1}{M_A^2}\left(\overline{\overline{B}_i \overline{B}_j} - \overline{\overline{B}}_j \overline{\overline{B}}_i\right)$$

$$\tau_{ij}^{b} = \left(\overline{\tilde{u}_i \overline{B}_j} - \overline{\overline{B}}_j \tilde{\tilde{u}}_i\right) - \left(\overline{\overline{B}_i \tilde{u}_j} - \tilde{\tilde{u}}_j \overline{\overline{B}}_i\right)$$

# Mixed model for compressible MHD turbulence

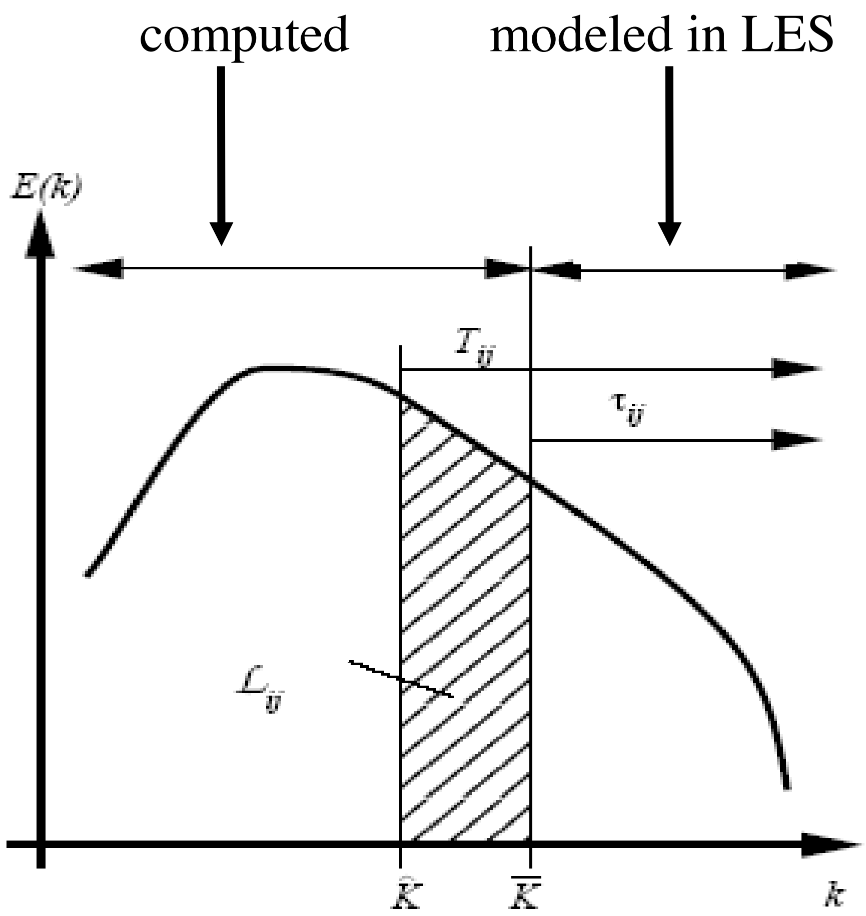
The mixed model is a combination of two subgrid-scale closures: the scale similarity model and the Smagorinsky model fro MHD case.

$$\tau_{ij}^{u} - \frac{1}{3}\tau_{kk}^{u}\delta_{ij} = -2C_5\bar{\rho}\overline{\Delta}^2 \,|\, \tilde{S}^u \,|\, (\tilde{S}_{ij} - \frac{1}{3}\tilde{S}_{kk}\delta_{ij}) + \bar{\rho}((\tilde{u}_j\tilde{u}_i)^{\sim} - \tilde{\tilde{u}}_j\tilde{\tilde{u}}_i) - \frac{1}{M_A^2}(\overline{\overline{B}_i\overline{B}_j} - \overline{\overline{B}}_j\overline{\overline{B}}_i)$$

$$\tau_{kk}^{u} = 2Y_5\bar{\rho}\overline{\Delta}^2 \,|\, \tilde{S}^u \,|^2$$

$$\tau_{ij}^{b} = -2D_5\overline{\Delta}^2 \,|\, \bar{j} \,|\, \bar{J}_{ij} + (\overline{\tilde{u}_i\overline{B}_j} - \overline{\overline{B}}_j\tilde{\tilde{u}}_i) - (\overline{\overline{B}_i\tilde{u}_j} - \tilde{\tilde{u}}_j\overline{\overline{B}}_i)$$

# Dynamic procedure




$$T_{ij}^{b} = \bar{\tau}_{ij}^{b} + L_{ij}^{b} \qquad T_{ij}^{u} = \tilde{\tau}_{ij}^{u} + L_{ij}^{u}$$

Germano assumed that there is an algebraic relation between the stresses at two different filter levels and the resolved stresses. In this model, the model parameter is determined dynamically, and not an *ad hoc* constant.

Applying a second filter to the filtered momentum equations a similar expression is achieved for the new subtest scale stress tensor Tij.

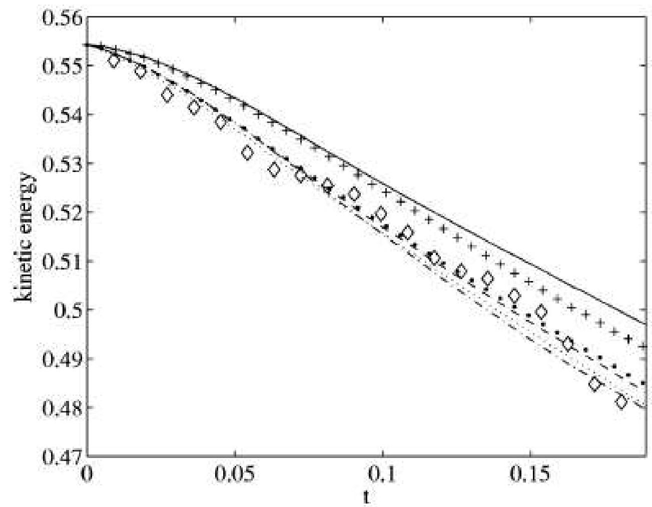
# CASE STUDIES

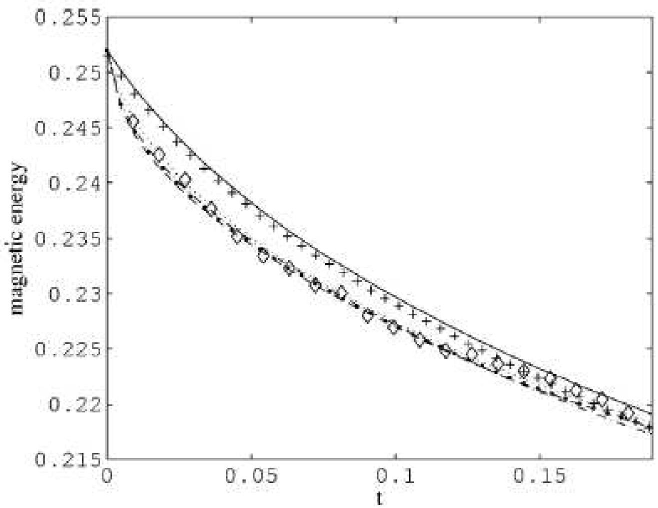
The left boundary for Re is chosen so as to provide a regime of developed turbulence, and the right boundary is the compromise between obtaining adequate DNS results and the necessity of carrying out the comparative analysis with subgrid-scale LES models. The value of the right boundary of the considered interval for magnetic Reynolds number is defined so that we investigate decaying compressible MHD turbulence, and the probability of occurrence of dynamo-processes in three-dimensional charged fluid flow increases with Re_*m*. The left boundary for Re_*m* is determined to express the role of magnetic effects in MHD flow. Mach number is limited by one because in this work approximation of polytropy gas is accepted .The flow with the value Mach number less than 0.2 is not interesting from the point of view of studying compressible flow.

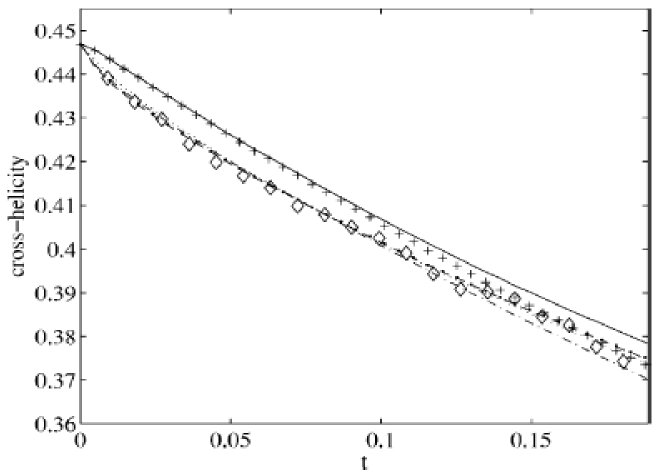
| Case | Re_*l* | Re | Re_*m* | Ms | Ma |
|---|---|---|---|---|---|
| 1 | 50 | 390 | 10 | 0.6 | 0.6 |
| 2 | 25 | 100 | 10 | 0.6 | 0.6 |
| 3 | 100 | 1580 | 10 | 0.6 | 0.6 |
| 4 | 50 | 390 | 2 | 0.6 | 0.6 |
| 5 | 50 | 390 | 20 | 0.6 | 0.6 |
| 6 | 50 | 390 | 10 | 0.2 | 0.6 |
| 7 | 50 | 390 | 10 | 1 | 0.6 |

# CASE #1 - 1

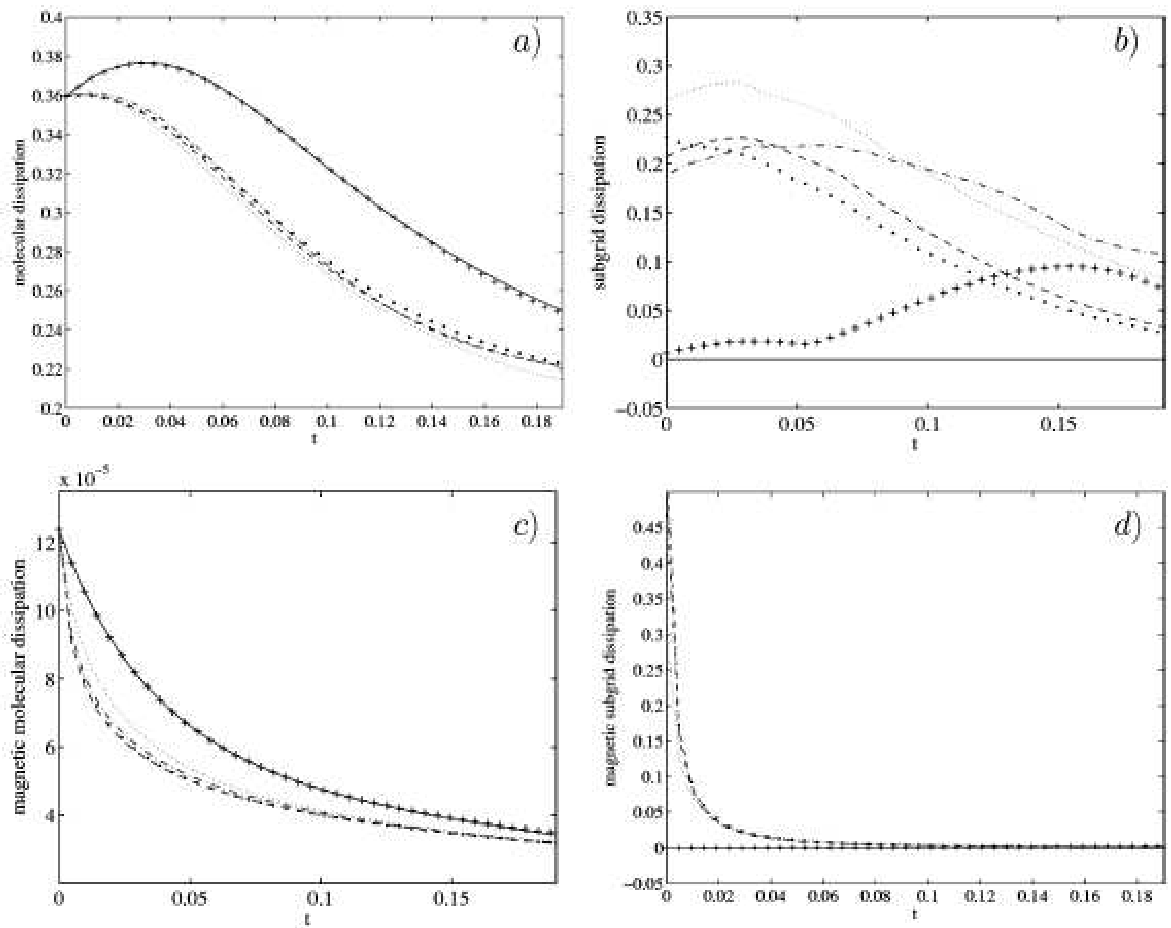
| SGS model | Curve |
|---|---|
| No model | Solid |
| Smagorinsky model | Dashed |
| Kolmogorov model | Dotted |
| Cross-helicity model | Black point |
| Scale-similarity model | Marker + |
| Mixed model | Dashed-dotted |



# CASE #1 - 2




$$\varepsilon_{\mu} = 2\mu \widetilde{S}_{ij}\widetilde{S}_{ij}$$

molecular dissipation

$$\chi_{\mu} = -\tau^{u}_{ij}\widetilde{S}_{ij}$$

kinetic energy subgrid-scale dissipation

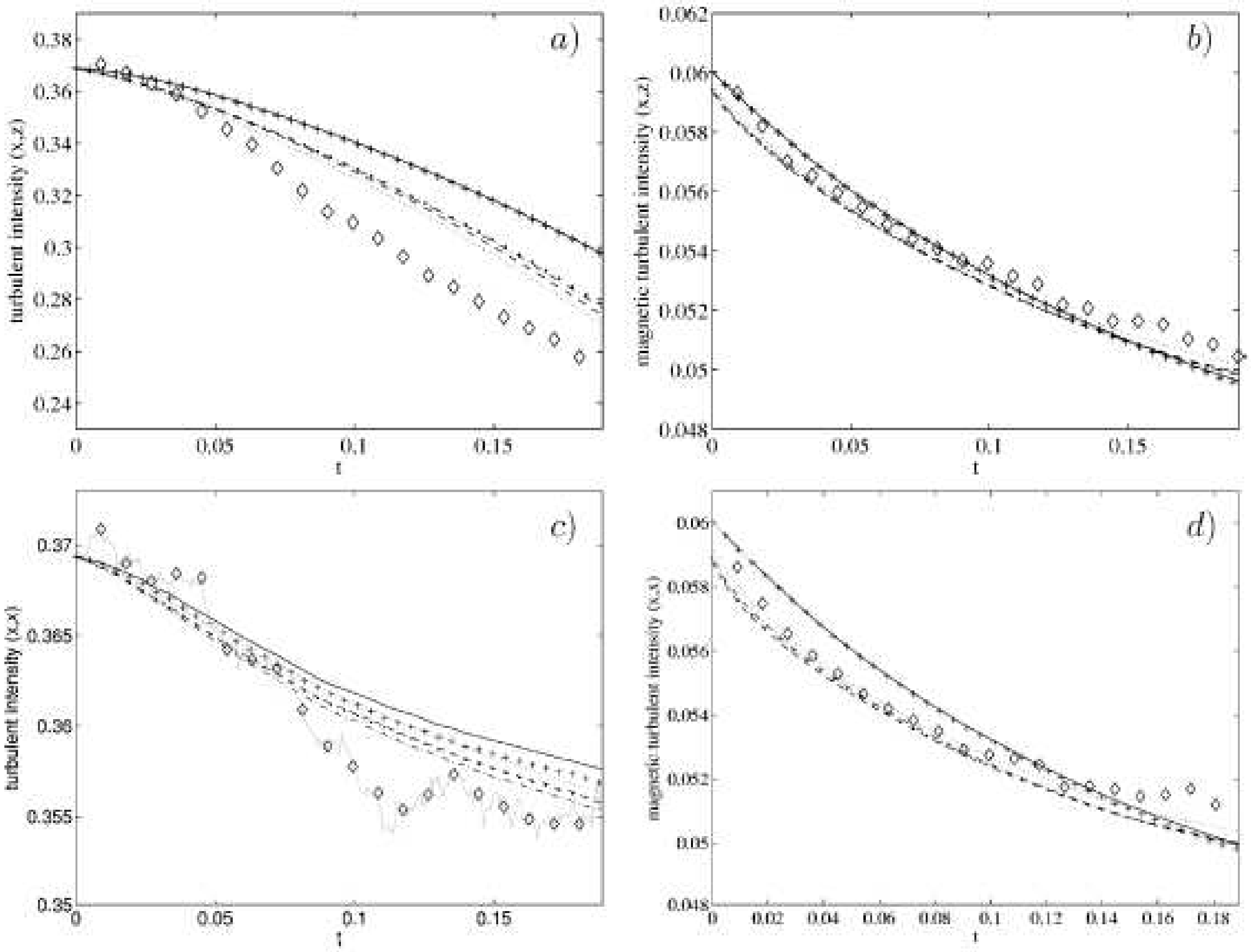
$$\varepsilon_{\eta} = \eta \mid \bar{j} \mid^{2}$$

magnetic molecular dissipation

$$\chi_{b} = -\tau^{b}_{ij}\bar{J}_{ij}$$

magnetic energy subgrid-scale dissipation

# CASE #1 - 3

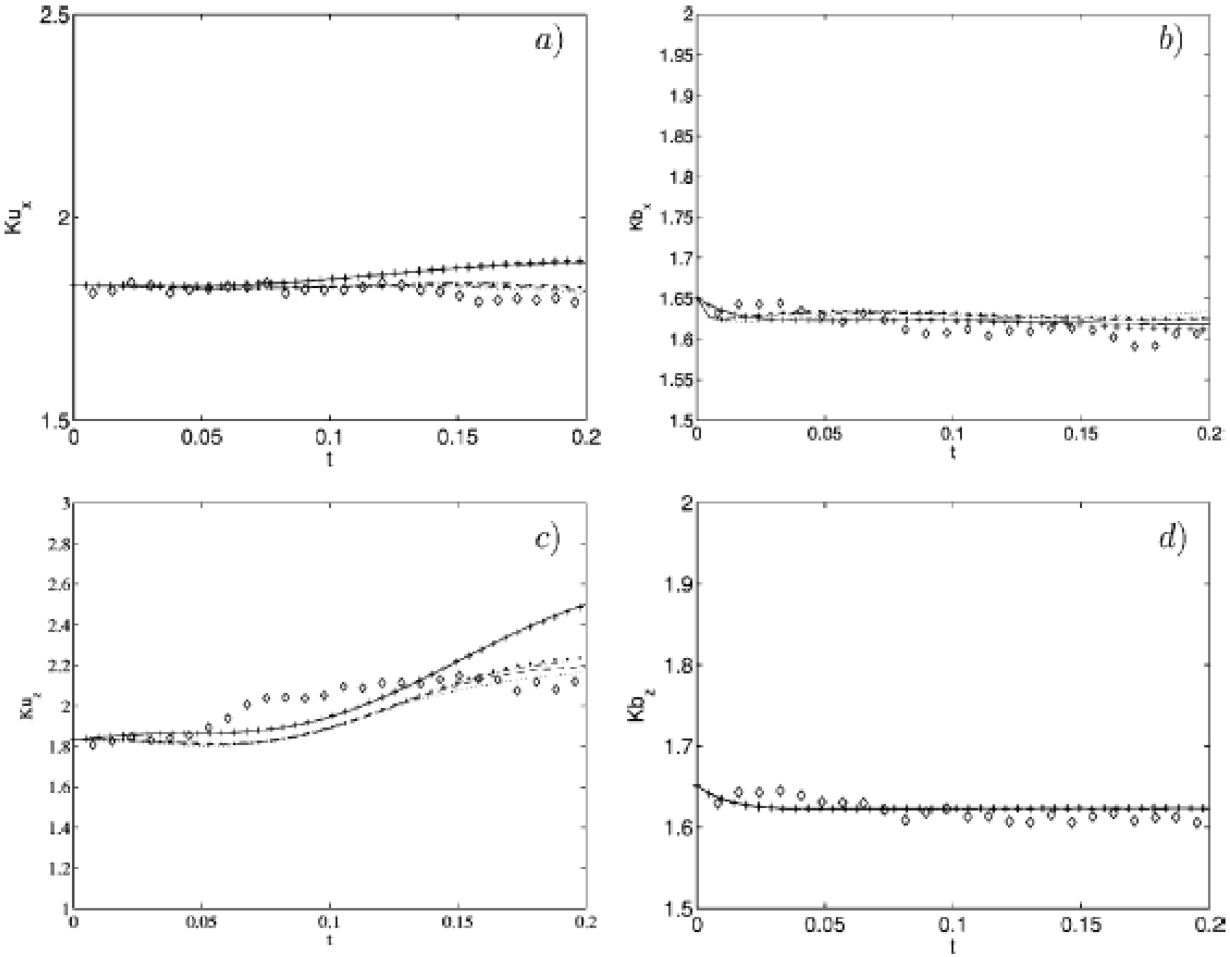



Fluctuating parts:

$$\upsilon_i = \tilde{u} - < \bar{\rho}\tilde{u}_i > / < \bar{\rho} >$$

$$b_i = \bar{B}_i - < \bar{B}_i >$$

# CASE #1 - 4



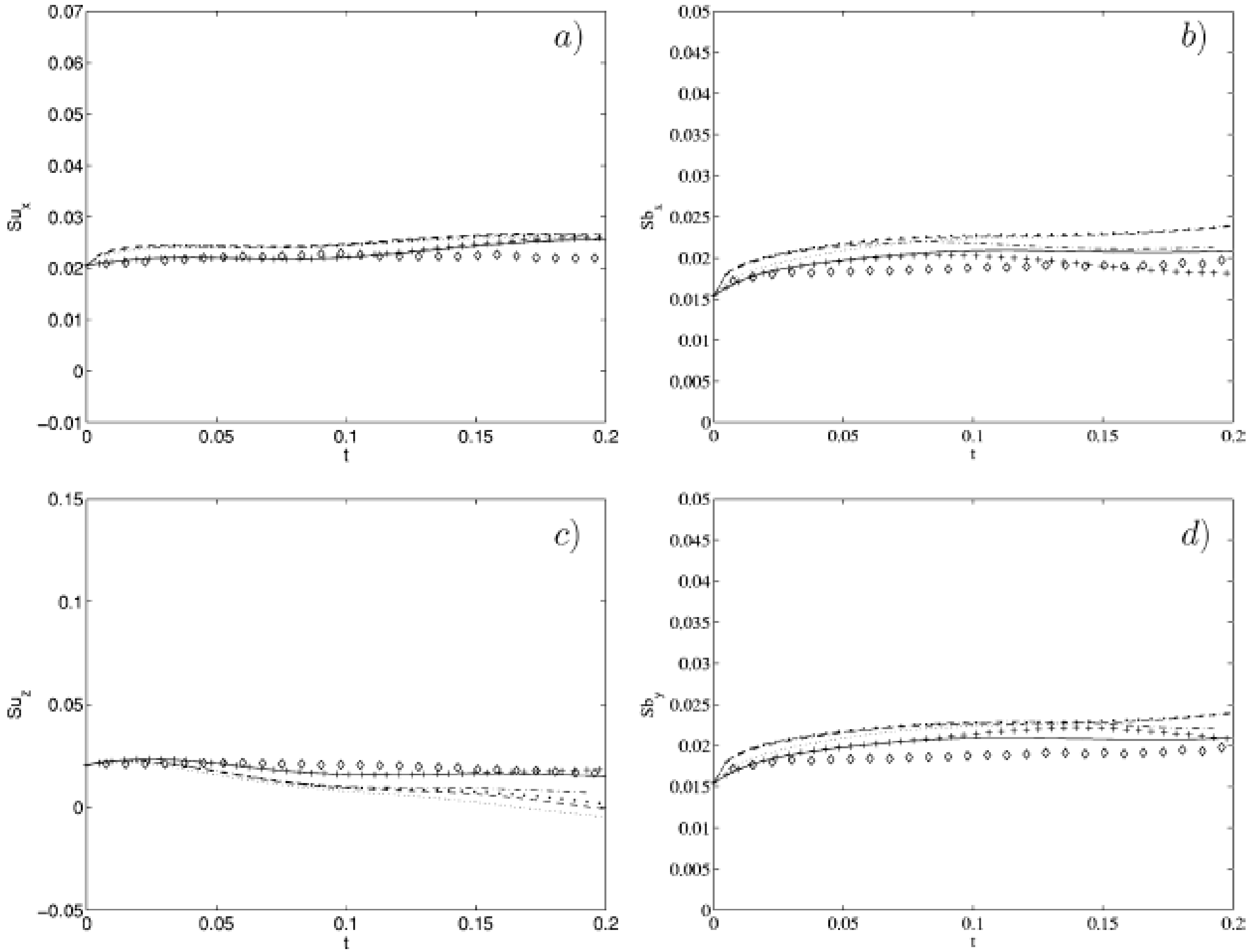

Kurtosis (or flatness) of a velocity component:

$$Ku_j = \frac{\langle u_j^4 \rangle}{(\langle u_j^2 \rangle)^2}$$

Kurtosis (or flatness) of a magnetic field:

$$Kb_j = \frac{\langle B_j^4 \rangle}{(\langle B_j^2 \rangle)^2}$$

# CASE #1 - 5




Skewness of a velocity component:

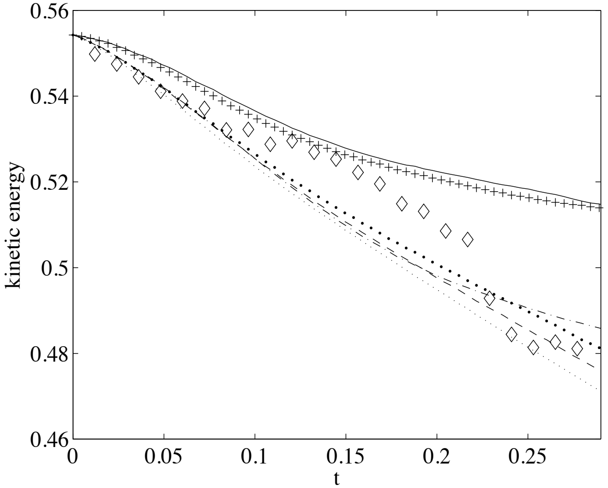
$$Su_j = \frac{\langle u_j^3 \rangle}{(\langle u_j^2 \rangle)^{3/2}}$$

Skewness of a magnetic field:

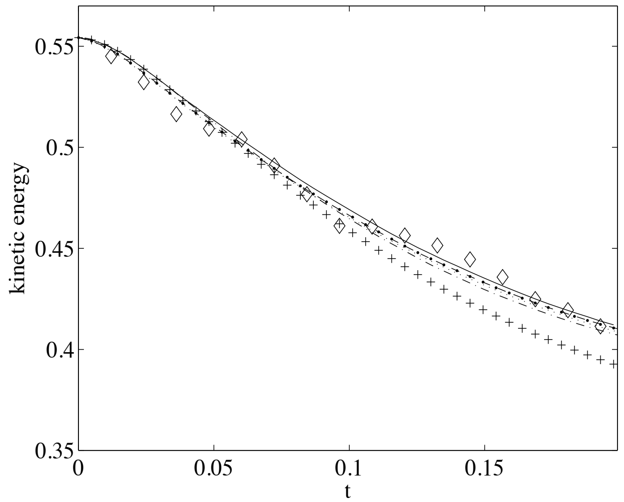
$$Sb_j = \frac{\langle B_j^3 \rangle}{(\langle B_j^2 \rangle)^{3/2}}$$

# Kinetic energy

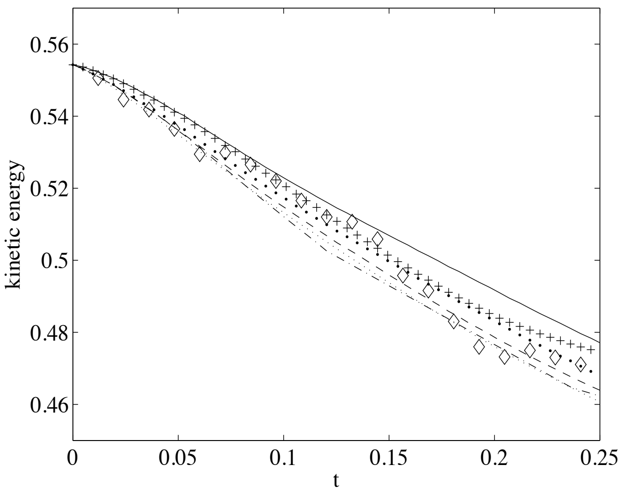
Ms=1



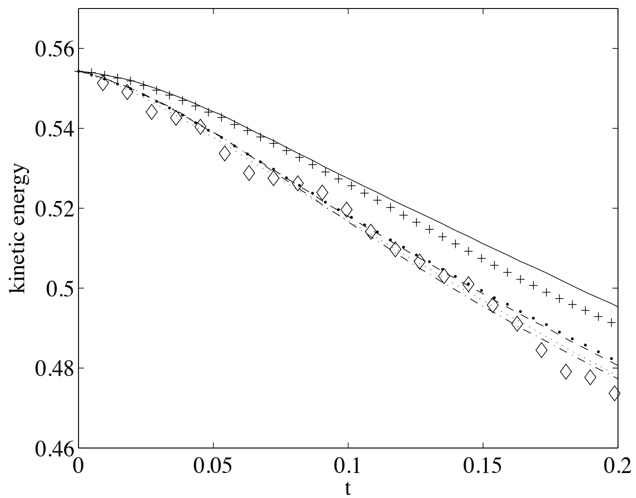

Ms=0.2




Re_m=2




Re_m=20



# Kinetic energy

•For kinetic energy, larger divergence of LES results was observed with a decrease in magnetic Reynolds number using various SGS closures. The scale-similarity model shows the worst results, however, the other SGS closures increase calculation accuracy.

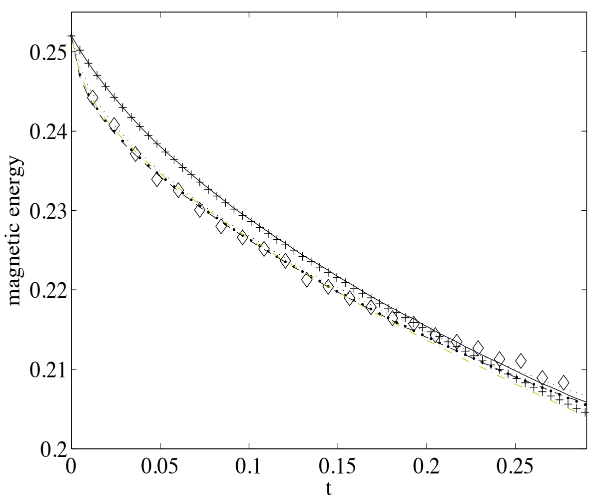
•The changing of Reynolds number produces qualitatively similar results, as the initial conditions of velocity and magnetic fields are the same, and therefore Taylor Reynolds number does not have a significant impact on the choice of subgrid parameterizations in our computations.

•Mach number *Ms* exerts essential influence on results of modeling. The divergence between DNS and LES results for kinetic energy increases with *Ms*.

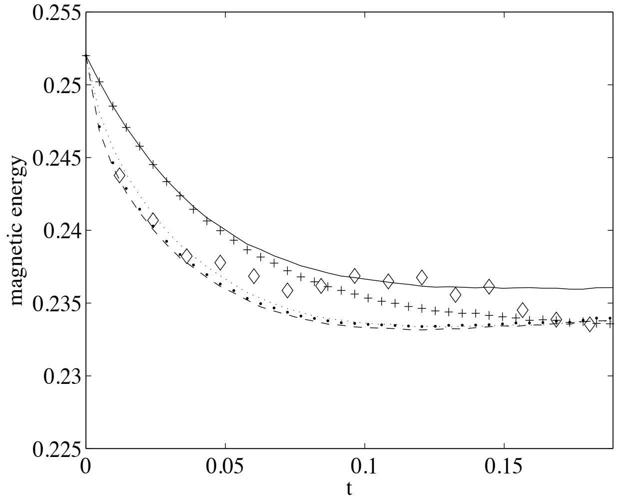
•Generally, the Smagorinsky model and the cross-helicity model yield the best accordance with DNS under various Mach number.

# Magnetic energy

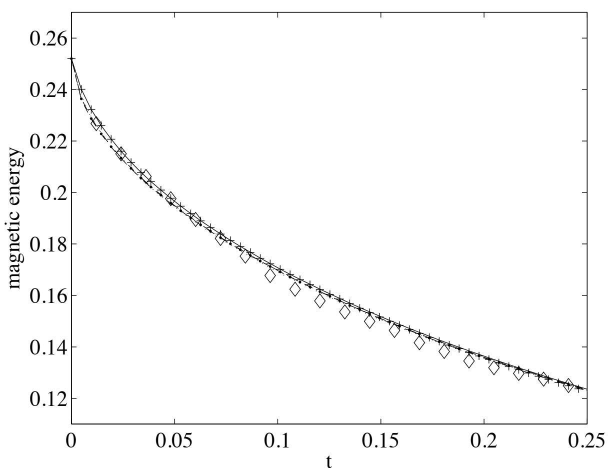
Ms=1



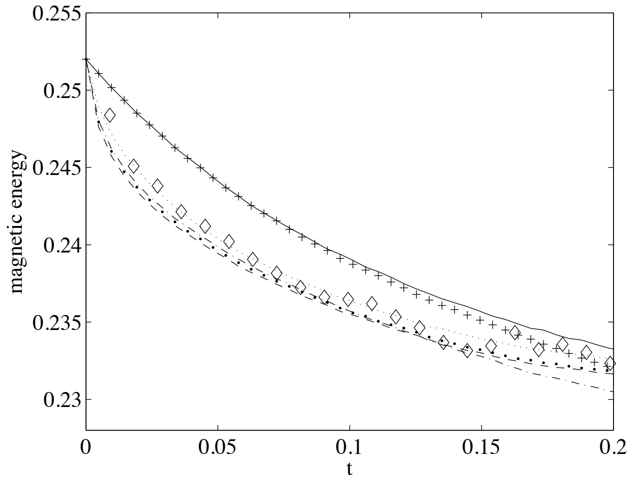

Ms=0.2




Re_m=2




Re_m=20



# Magnetic energy

•The differences between SGS models for magnetic energy are shown to decrease with reducing magnetic Reynolds number and all models above demonstrate good agreement with DNS results at small value of number Re*m*.

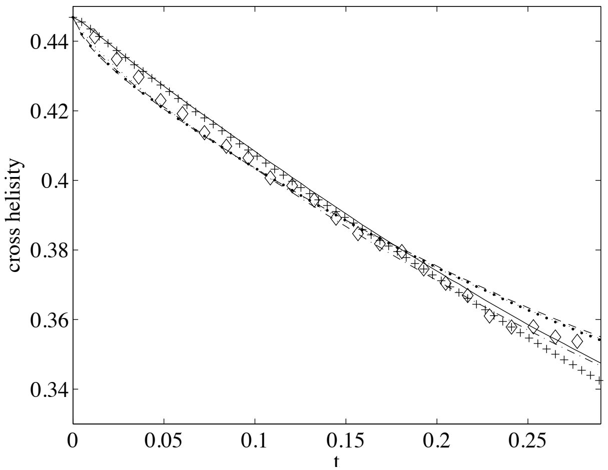
•The effect of subgrid-scale closures increases with magnetic Reynolds number for modeling of compressible MHD turbulence, but the rate of dissipation of the magnetic energy decreases with increasing Re_*m*.

•Generally, the best results are shown for the Smagorinsky, the Kolmogorov, and the cross-helicity models for evolution of the magnetic energy.

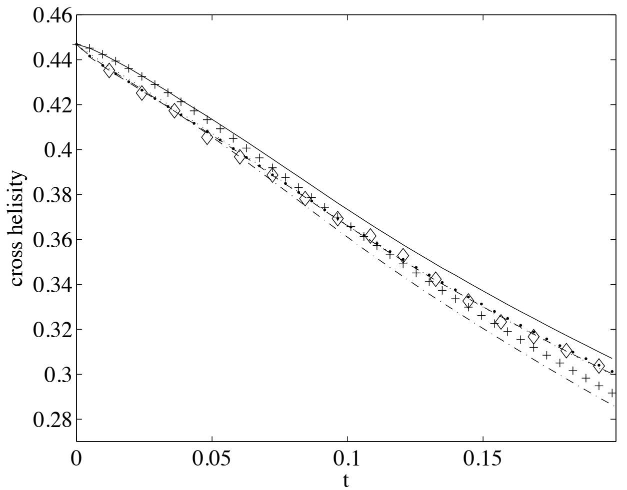
•The deviations in results for magnetic energy decrease with increasing *Ms*. It is necessary to notice, that magnetic energy reaches a stationary level more rapidly with reducing Mach number.

# Cross-helicity

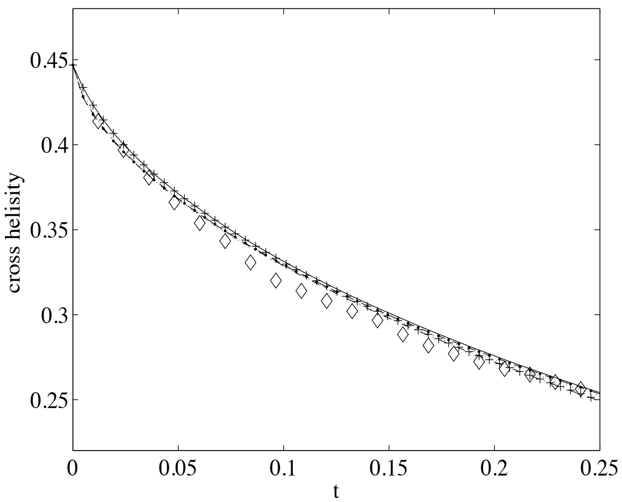
Ms=1



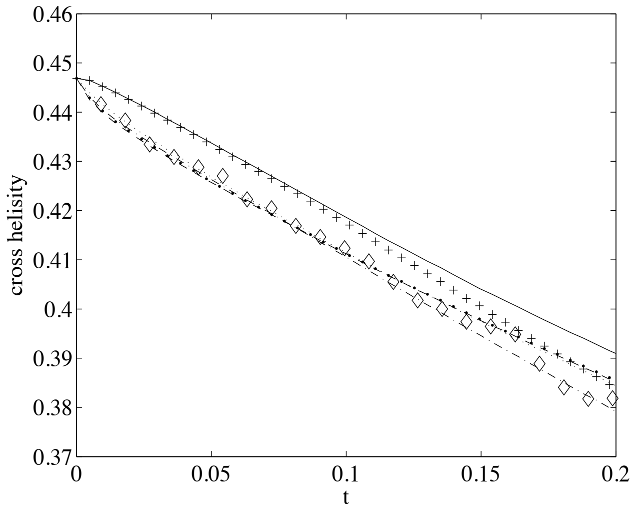

Ms=0.2




Re_m=2




Re_m=20



# Cross-helicity

- For the cross-helicity, the influence of subgrid-scale parametrizations increases with magnetic Reynolds number.

- The scale-similarity model demonstrates the worst results. In the presence of adequate SGS parametrization improves calculation accuracy.

- The Smagorinsky model shows the best results for the cross-helicity both for high and for low Mach numbers.

# Skewness and flatness

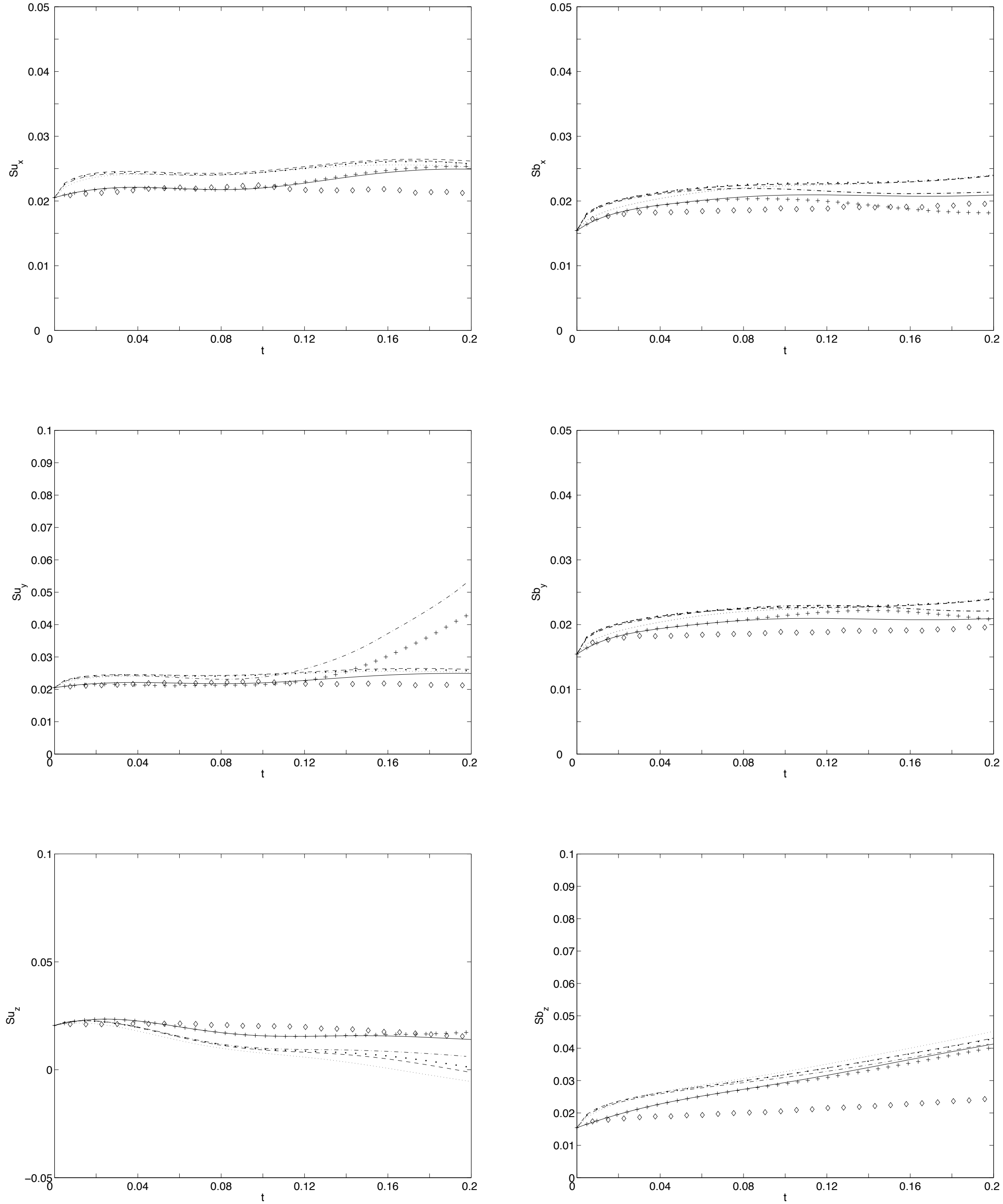
The departure from Gaussianity for fluid turbulence in the laboratory or in numerical simulations is measured in terms of the skewness and flatness factors.

**The flatness factor** (sometimes also called kurtosis) in turbulent flows is a measure of intermittency. The flatness is an indication of the occurrence of fluctuations far from the mean: it is an indicator of the relative frequency of rare events. Hence the flatness increases with increasing sparseness of the fluctuations:

$$Ku_j = \frac{< u_j^4 >}{(< u_j^2 >)^2} \qquad Kb_j = \frac{< B_j^4 >}{(< B_j^2 >)^2}$$

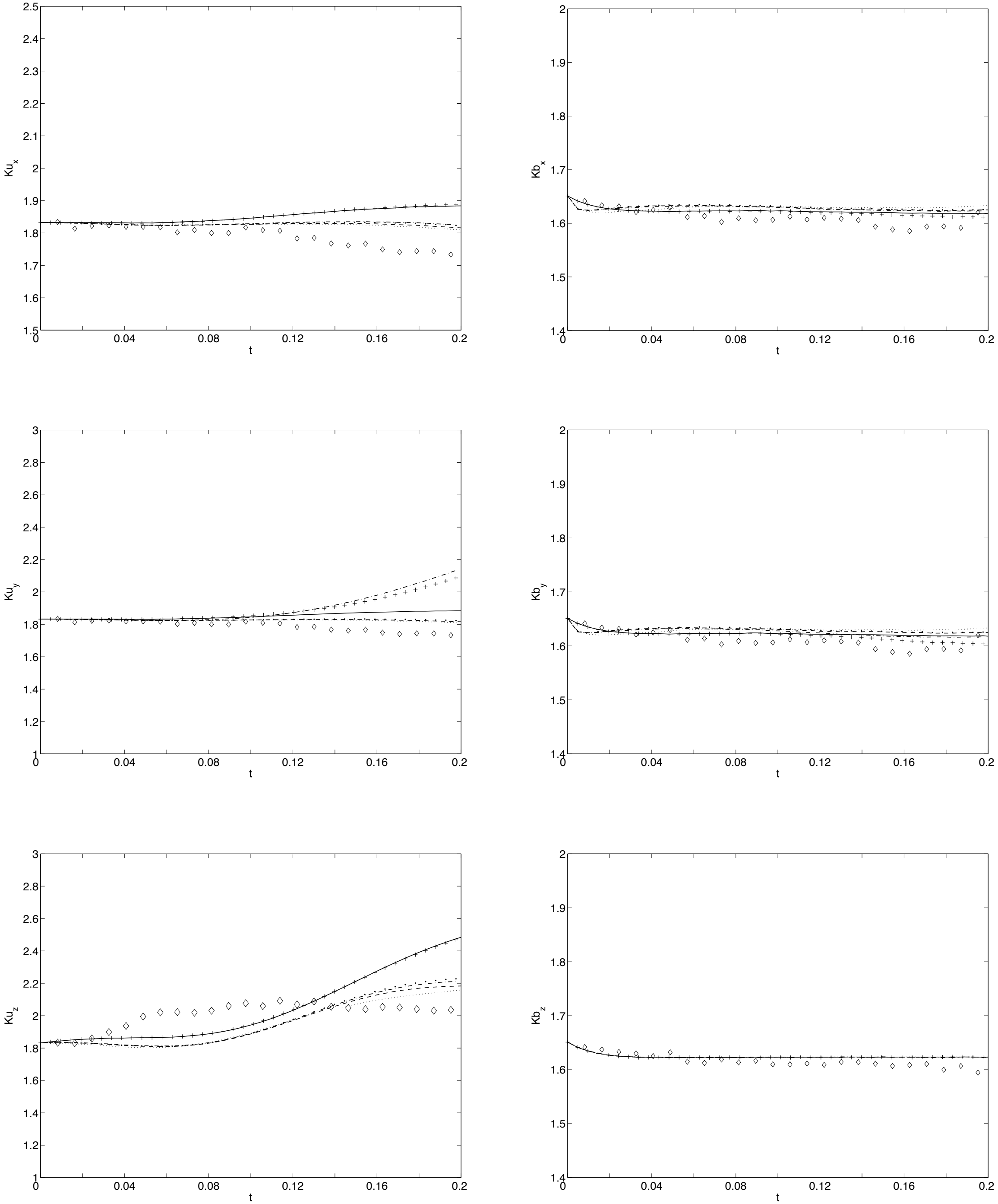
**The skewness** is  related to the asymmetry of the probability density function of the velocity or magnetic filed fluctuations. It is a sensitive indicator of changes in the large scale structure.

$$Su_j = \frac{< u_j^3 >}{(< u_j^2 >)^{3/2}} \qquad Sb_j = \frac{< B_j^3 >}{(< B_j^2 >)^{3/2}}$$




| Results | Curve |
|---|---|
| DNS | Diamond |
| No model | Solid |
| Smagorinsky model | Dashed |
| Kolmogorov model | Dotted |
| Cross-helicity model | Black point |
| Scale-similarity model | Marker+ |
| Mixed model | Dashed-dot |

Time evolution of skewness and flatness of velocity and magnetic field components for the case Re = 100, Re_l = 25, Re_m = 10.0, M_s = 0.6.




Time dynamics of skewness and flatness of velocity and magnetic field components for the case

Re = 100, Re_l = 25,
Re_m = 10.0, M_s = 0.6.

# outcome

- applicability of LES method for studying of non-Gaussian properties of probability density function for turbulent compressible magnetic fluid flow

- potential feasibilities of various subgrid-scale parameterizations by means of comparison with DNS results are explored

- efficiency is demonstrated by various subgrid-scale models depends on similarity numbers of turbulent MHD flow. Lack of dissipation in LES model without any SGS parametrization for kinetic and magnetic energies does not have an effect on determination of the skewness and the flatness, the case without any subgrid modeling sometimes lies even closer to the DNS results. This indicates that the energy pile-up at the small scales, that is characteristic for the case without any SGS closure, does not significantly influence determination of PDF

- among the subgrid models, the best results for studying of the flatness and the skewness of the velocity and the magnetic field components are demonstrated by the Smagorinsky model for MHD turbulence and the model based on cross-helicity for MHD case.

# MHD EQUATIONS FOR HEAT CONDUCTING FLUID

The governing system of compressible electrically conducting fluid is written in the following form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0;$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}\left( \rho u_i u_j + p\delta_{ij} - \sigma_{ij} \right) - \frac{1}{c}\varepsilon_{ijk} j_j B_k = 0;$$

$$\frac{\partial \rho E}{\partial t} + \frac{\partial}{\partial x_j}\left( \left( \rho E + p \right) u_j + q_j - \sigma_{ij} u_i \right) - \Xi j = 0;$$

$$\frac{\partial B_i}{\partial t} = \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} (B_l u_m) + \eta \frac{\partial^2 B_i}{\partial x_j^2};$$

$$\frac{\partial B_i}{\partial x_i} = 0.$$

# MHD EQUATIONS FOR HEAT CONDUCTING FLUID -2

The MHD approximation implies that the energy of the electric field is much less than that of the magnetic field. In this case, the electric field is eliminated from the governing system of equations, and flow characteristics are expressed in terms of magnetic field. Using Maxwell equations for electrodynamic field, we transform MHD equations and reduce to the following dimensionless form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0;$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial}{\partial x_j}\left( \rho u_i u_j + p\delta_{ij} - \frac{1}{Re}\sigma_{ij} + \frac{B^2}{2M_a^2}\delta_{ij} - \frac{1}{M_a^2} B_j B_i \right) = 0;$$

$$\frac{\partial B_i}{\partial t} + \frac{\partial}{\partial x_j}\left( u_j B_i - u_i B_j \right) - \frac{1}{Re_m}\partial^2 B_i \partial x_j^2 = 0;$$

$$\frac{\partial}{\partial t}\left( \rho E + \frac{B^2}{2M_a^2} \right) + \frac{\partial}{\partial x_j}\left( \left( \rho E + P \right)\tilde{u}_j + \frac{1}{\mathrm{Pr}\mathrm{Re}M_s^2(\gamma - 1)} q_j - \frac{1}{\mathrm{Re}}\sigma_{ij}u_i - \frac{1}{M_a^2} B_j B_i u_i \right) -$$

$$- \frac{\partial}{\partial x_j}\left( \frac{\eta}{\mathrm{Re}_m M_a^2} B_i \left( \frac{\partial B_i}{\partial x_j} - \frac{\partial \overline{B}_j}{\partial x_i} \right) \right) = 0.$$

# Filtered MHD equations for heat-conducting plasma

$$
\begin{cases}
\dfrac{\partial \overline{\rho}}{\partial t} + \dfrac{\partial \overline{\rho}\widetilde{u}_j}{\partial x_j} = 0 \\
\dfrac{\partial \overline{\rho}\widetilde{u}_i}{\partial t} + \dfrac{\partial}{\partial x_j}\left(\overline{\rho}\widetilde{u}_i\widetilde{u}_j + \overline{p}\delta_{ij} - \dfrac{1}{\mathrm{Re}}\widetilde{\sigma}_{ij} + \dfrac{\overline{B}^2}{2M_A^2} - \dfrac{1}{M_A^2}\overline{B}_i\overline{B}_j\right) = -\dfrac{\partial \tau_{ji}^u}{\partial x_j} \\
\dfrac{\partial \overline{B}_i}{\partial t} + \dfrac{\partial}{\partial x_j}\left(\widetilde{u}_j\overline{B}_i - \overline{B}_j\widetilde{u}_i\right) - \dfrac{1}{\mathrm{Re}_m}\dfrac{\partial^2 \overline{B}_i}{\partial x_j^2} = -\dfrac{\partial \tau_{ji}^b}{\partial x_j} \\
\dfrac{\partial \overline{\rho}\widetilde{E}}{\partial t} + \dfrac{\partial}{\partial x_j}\left[(\widetilde{E} + \overline{P})\widetilde{u}_j - \dfrac{1}{M_A^2}\overline{B}_i\overline{B}_j\widetilde{u}_i\right] + \dfrac{\partial}{\partial x_j}\left[\dfrac{\widetilde{q}_j}{\mathrm{Pr}\,\mathrm{Re}\,M_s^2(\gamma-1)} - \dfrac{1}{\mathrm{Re}}\widetilde{\sigma}_{ij}\widetilde{u}_i\right] - \\
\dfrac{\partial}{\partial x_j}\left[\dfrac{\eta}{\mathrm{Re}_m M_a^2}\overline{B}_i\left(\dfrac{\partial \overline{B}_i}{\partial x_j} - \dfrac{\partial \overline{B}_j}{\partial x_i}\right)\right] = -\dfrac{\partial}{\partial x_j}\left(\dfrac{1}{\gamma M_s^2}Q_j + \dfrac{1}{2}J_j + \dfrac{1}{2M_a^2}V_j - \dfrac{1}{M_s^2}G_j\right)
\end{cases}
$$

**Equation of state:** $\overline{p} = \dfrac{\widetilde{T}\overline{\rho}}{\gamma M_s^2}$

$$E = \rho e + \frac{1}{2}\rho u_i u_i + \frac{1}{2M_a^2}B_i B_i$$

total energy

$$e = \frac{T\rho}{\gamma(\gamma-1)M_s^2}$$

internal energy

# Subgrid-scale terms of filtered MHD equations for heat-conducting plasma

$$\tau_{ij}^{u} = \overline{\rho}((u_j u_i)^{\sim} - \widetilde{u}_j \widetilde{u}_i) - \frac{1}{M_A^2}(\overline{B_i B_j} - \overline{B}_j \overline{B}_i)$$

- SGS stresses

$$\tau_{ij}^{b} = (\overline{u_i B_j} - \overline{B}_j \widetilde{u}_i) - (\overline{B_i u_j} - \widetilde{u}_j \overline{B}_i)$$

- magnetic SGS stresses

$$Q_j = \overline{\rho}((u_j T)^{\sim} - \widetilde{u}_j \widetilde{T})$$

- SGS heat flux

$$J_j = \overline{\rho}((u_j u_k u_k)^{\sim} - \widetilde{u}_j (u_k u_k)^{\sim})$$

- SGS turbulent diffusion

$$V_j = (\overline{B_k B_k u_j} - \overline{B_j B_i} \widetilde{u}_j)$$

- SGS magnetic energy flux

$$G_j = (\overline{B_j B_k u_k} - \widetilde{u}_k \overline{B}_k \overline{B}_k)$$

- SGS energy of the interaction between the magnetic tension and the velocity

# Models for SGS terms

For SGS stresses we use the Smagorinsky model for the MHD case:

$$\tau_{ij}^{u} = -2C\ \overline{\rho}\overline{\Delta}^{2} \mid \widetilde{S}^{u} \mid (\widetilde{S}_{ij} - \frac{1}{3}\widetilde{S}_{kk}\delta_{ij}) + \frac{2}{3}Y\ \overline{\rho}\overline{\Delta}^{2} \mid \widetilde{S}^{u} \mid^{2} \delta_{ij}$$

$$\tau_{ij}^{b} = -2D\overline{\Delta}^{2} \mid \bar{j} \mid \bar{J}_{ij}$$

The eddy diffusivity model is used for the closure of the subgrid-scale heat flux. This eddy diffusivity model is similar to the molecular heat flux term, but the molecular viscosity and Prandtl number have been replaced by the dynamic eddy viscosity and the turbulent Prandtl number:

$$Q_{j} = -C_{s}\frac{\overline{\Delta}^{2}\overline{\rho} \mid \widetilde{S}^{u} \mid}{\Pr_{T}}\frac{\partial \widetilde{T}}{\partial x_{j}}$$

The model for Jj is based on an analogy to Reynolds-averaged Navier-Stokes equations and on the assumption that $\widetilde{u}_{i} \cong \widetilde{\widetilde{u}}_{i}$

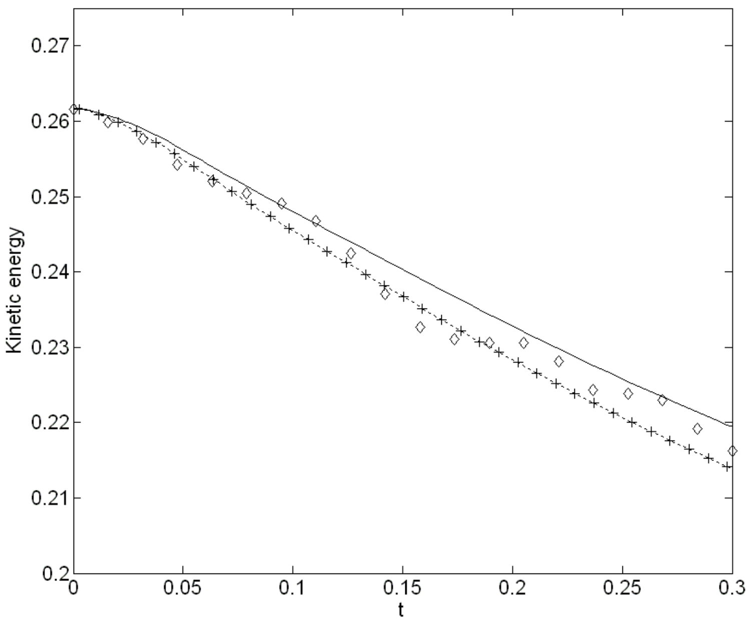
$$J_{j} = \widetilde{u}_{k}\tau_{jk}^{u}$$

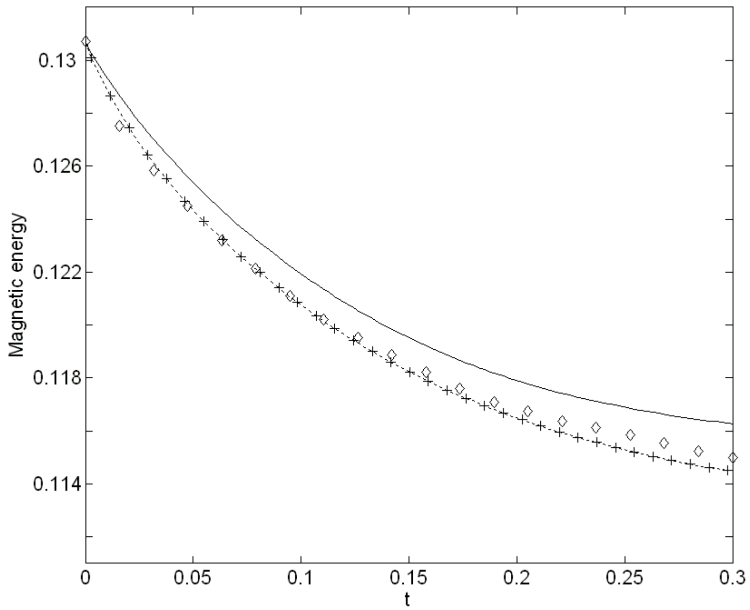
# CASE STUDIES

Since compressibility effects and temporal dynamics of temperature defined from the total energy equation depend nontrivially on the Mach number, in this work we consider three cases:
the Mach number $Ms = 0.38$, that is, the flow is moderately compressible;
the Mach number $Ms = 0.70$ , when compressibility plays an important role in turbulent fluid flow;
the Mach number $Ms = 1.11$ corresponding to appearance of strong discontinuity in essentially compressible flow.

In all three numerical experiments, the following dimensionless parameters for computations are used: the hydrodynamic Reynolds number $Re = 281$, the microscale (Taylor) Reynolds number $Re\_l = 43$, the magnetic Reynolds number $Re\_m = 10$, the magnetic Mach number $Ma = 1.2$, the Prandtl number $Pr = 1.0$ and the ratio of the specific heats $1.5$.

# M=0.38

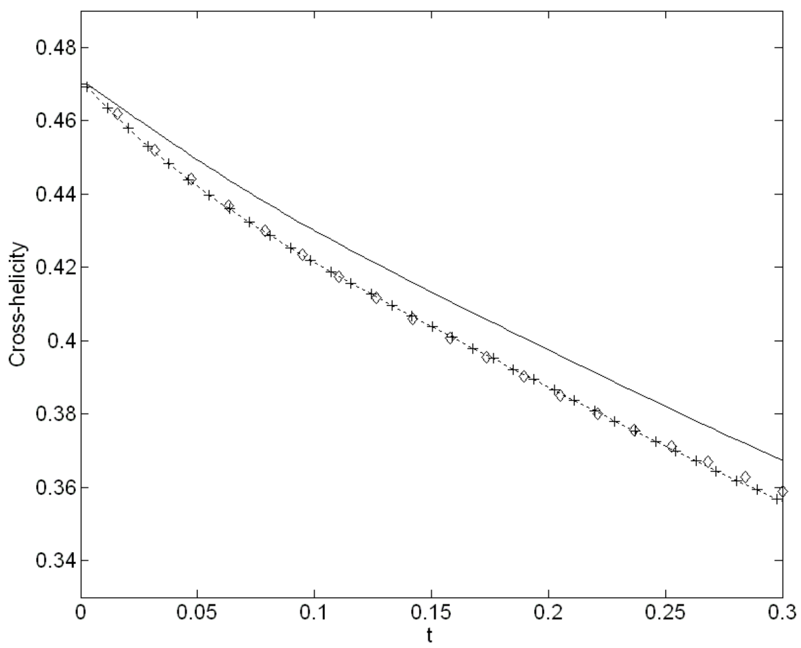


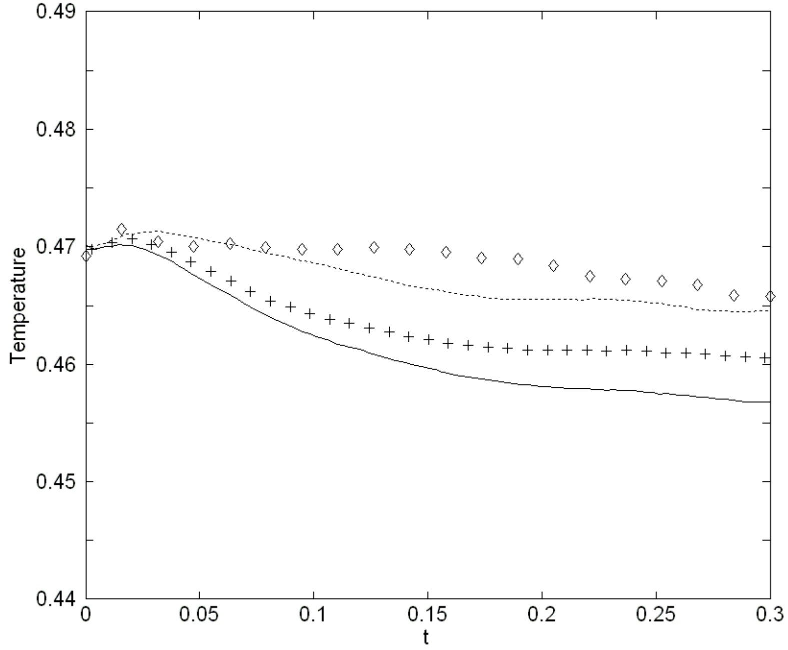




Time dynamics of kinetic and magnetic energy

| Case | Curve |
|---|---|
| DNS | Diamond line |
| LES without any SGS models | Solid line |
| LES | Dotted line |
| LES without energy SGS terms | Marker + |

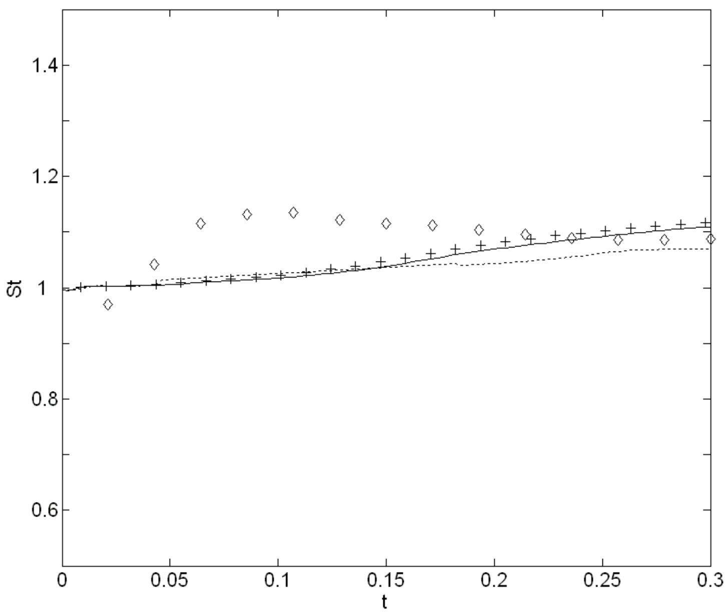
# M=0.38

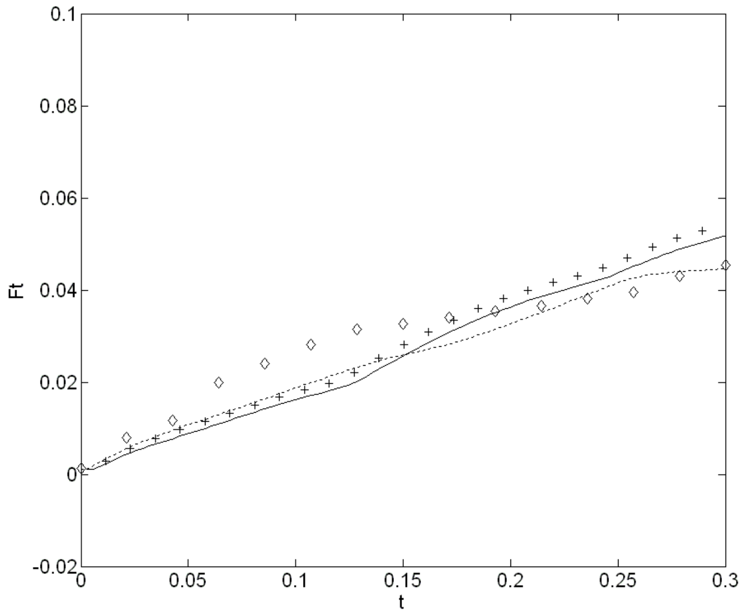



Time evolution of cross-helicity and temperature

# M=0.38



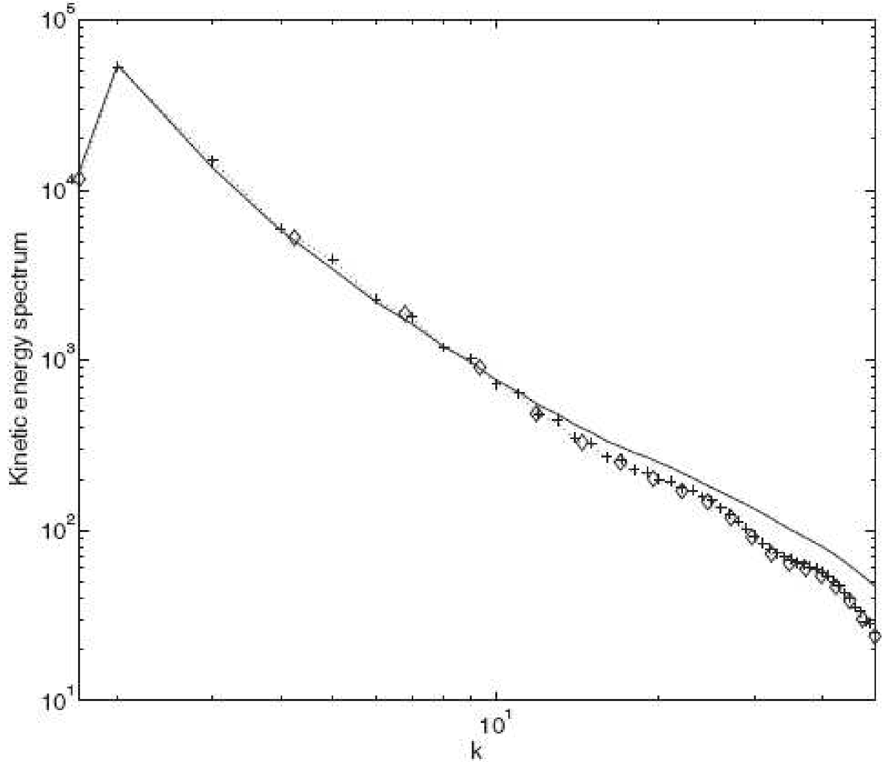



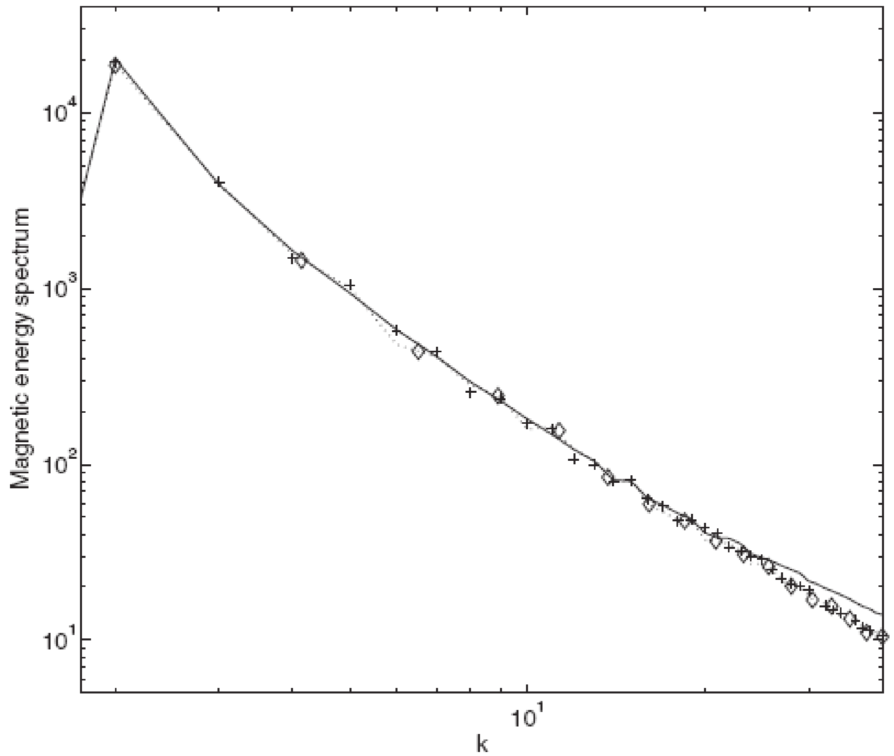

Time dynamics the skewness of the temperature and the parameter Ft

$St = \frac{< T^3 >}{(< T^2 >)^{3/2}}$; - skewness of the temperature

$Ft = (< (T - < T >)^2 >)^{1/2}$ - parameter, describing temperature fluctuations

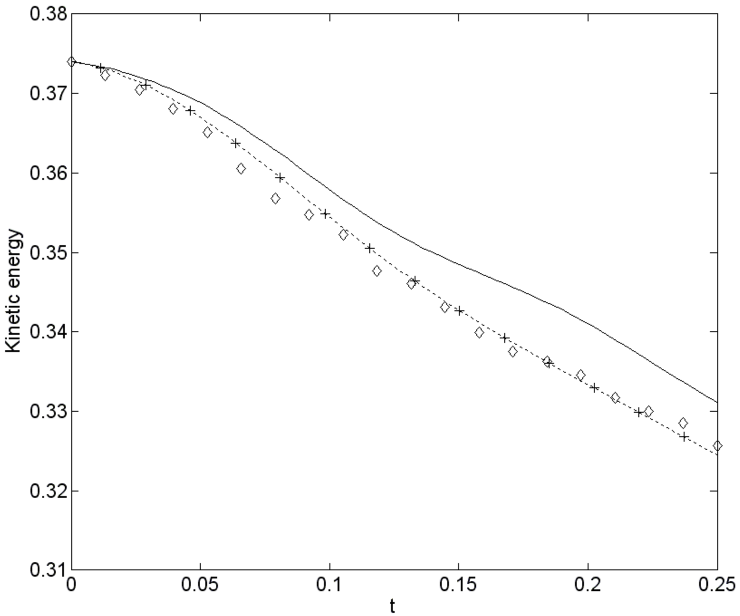
# M=0.38

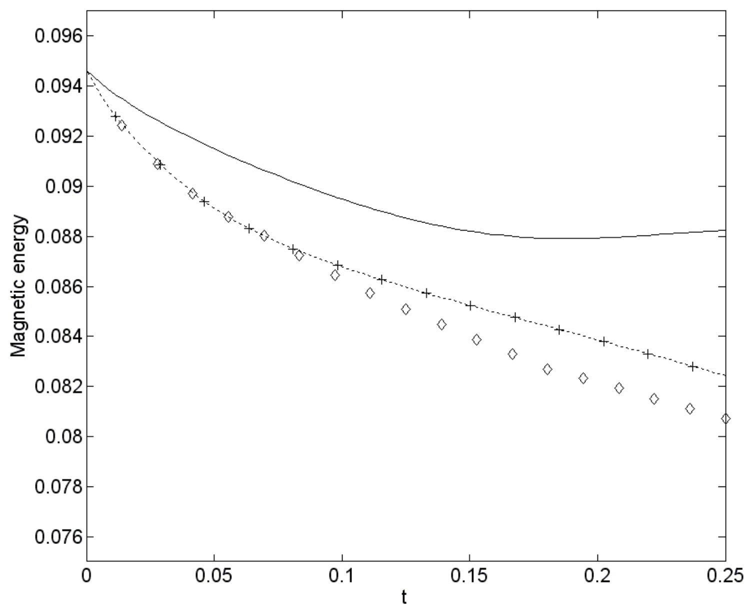



Kinetic and magnetic energy spectra.

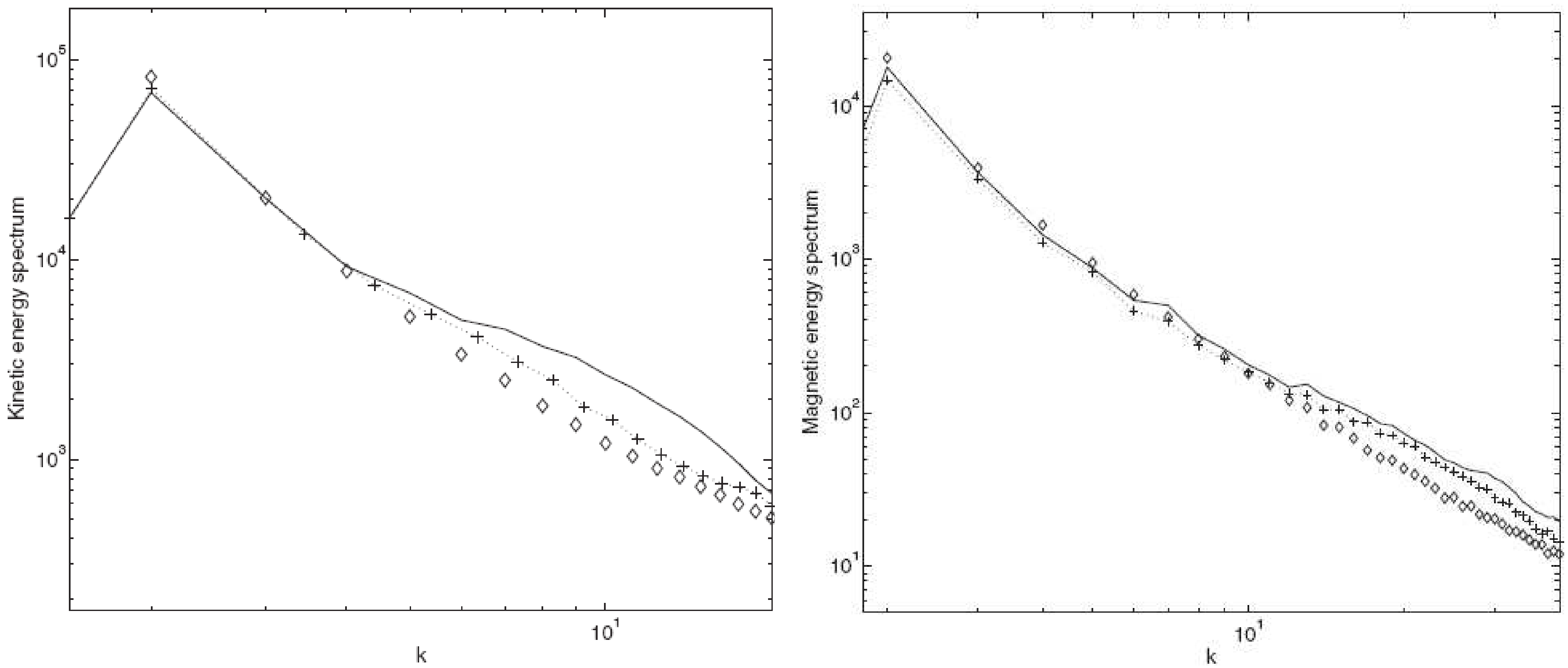
# M=0.70




Time dynamics of kinetic and magnetic energy

# M=0.70




Kinetic and magnetic energy spectra.

# outcome

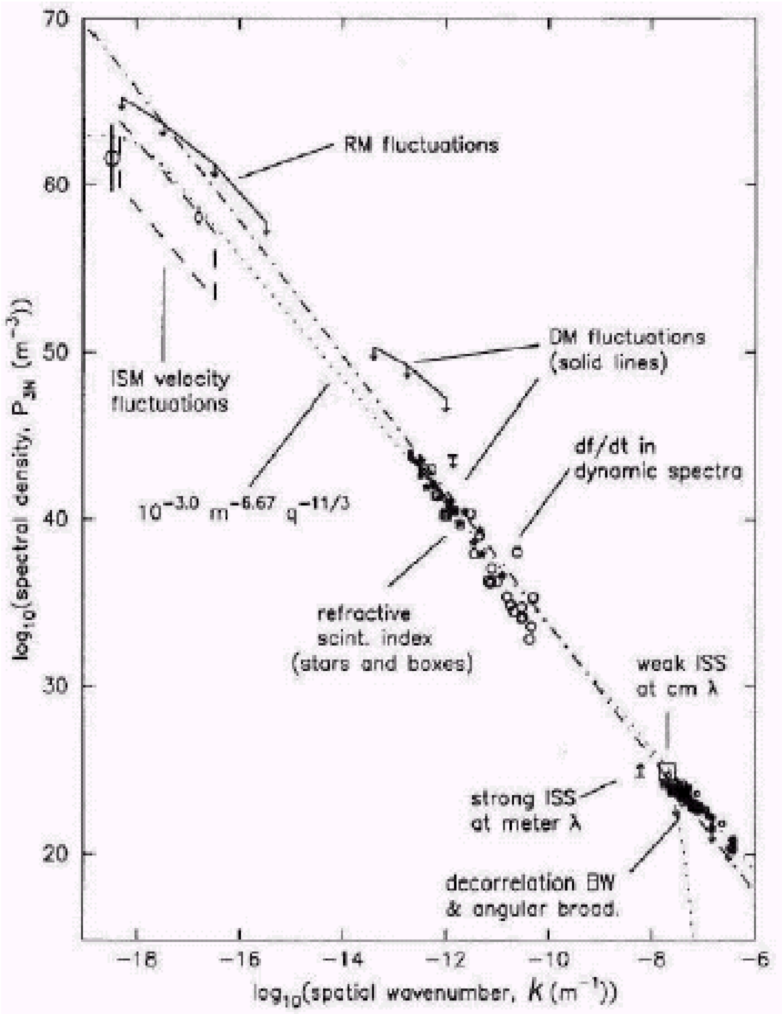
• The system of the filtered MHD equations with the total energy equation using the mass-weighted filtering procedure has been obtained. Novel subgrid-scale terms arise in total energy equation due to the presence of energy equation.

• New subgrid-scale models for the SGS terms, appearing after filtering procedure in the total energy equation in the presence of magnetic field, are suggested.

• Consideration of the SGS terms in the energy equation scarcely affects the kinetic and the magnetic energy even at high Mach numbers, while for the temperature (same as for the internal energy) the presence of SGS models in the energy equation is an important condition for improvement of calculation accuracy of thermodynamic quantities.

• Generally, LES method using explicit mass-weighted filtering demonstrates good results for modeling of electrically and heat conducting fluid in MHD turbulence when the medium is weakly or moderately compressible.

# Local Interstellar Medium




*Armstrong et al., ApJ (1995), 443:209-221*

There is growing interest in observations and explanation of the spectrum of the density fluctuations in the interstellar medium. These fluctuations are responsible for radio wave scattering in the interstellar medium and cause interstellar scintillation fluctuations in the amplitude and phase of radio waves. Kolmogorov-like $k^{-5/3}$ spectrum of density fluctuations have been observed in wide range of scales in the local interstellar medium (from an outer scale of a few parsecs to scales of about 200 km).

# Parameters of numerical study of local interstellar medium

For study of compressible MHD turbulence in interstellar, medium we use large eddy simulation (LES) method. Smagorinsky model for compressible MHD case for subgrid-scale parameterization is applied. The Smagorinsky model for compressible MHD turbulence showed accurate results under various range of similarity numbers.

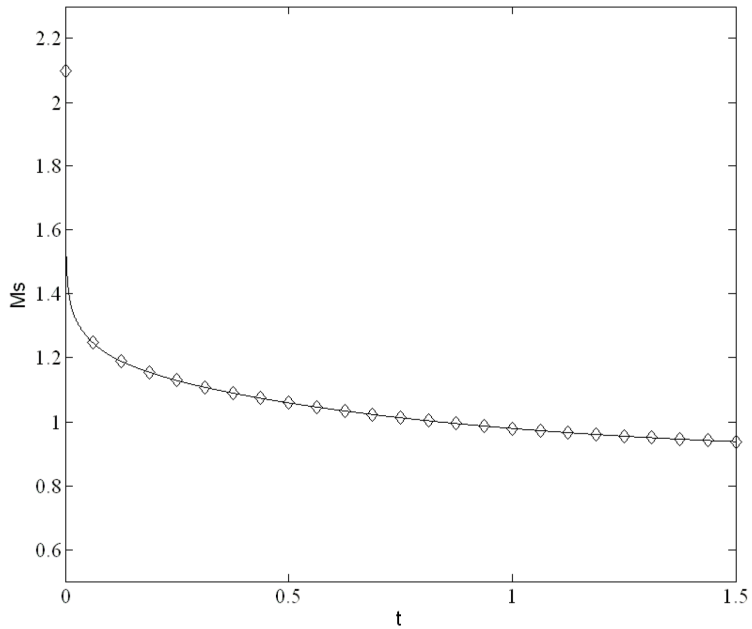
Initial parameters: $\mathrm{Re} \approx 2000$ $\quad M_s \approx M_A \approx 2.2$

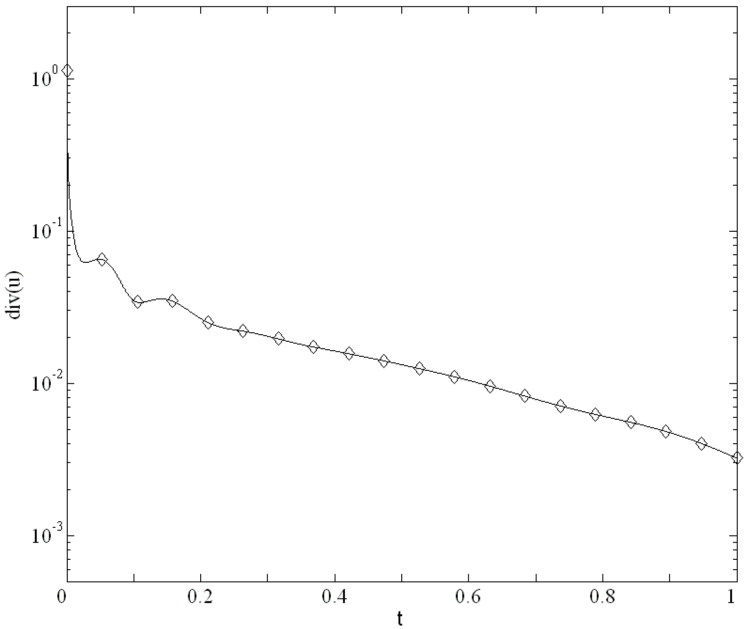
$\mathrm{Re}_m \approx 200$ (ambipolar diffusion)

The initial isotropic turbulent spectrum was chosen for kinetic and magnetic energies in Fourier space to be close to $k^{-2}$ with random amplitudes and phases in all three directions. The choice of such spectrum as initial conditions is due to velocity perturbations with an initial power spectrum in Fourier space similar to that of developed turbulence.

The simulation domain is a cube with dimensions of $\pi^3$

# Compressibility properties

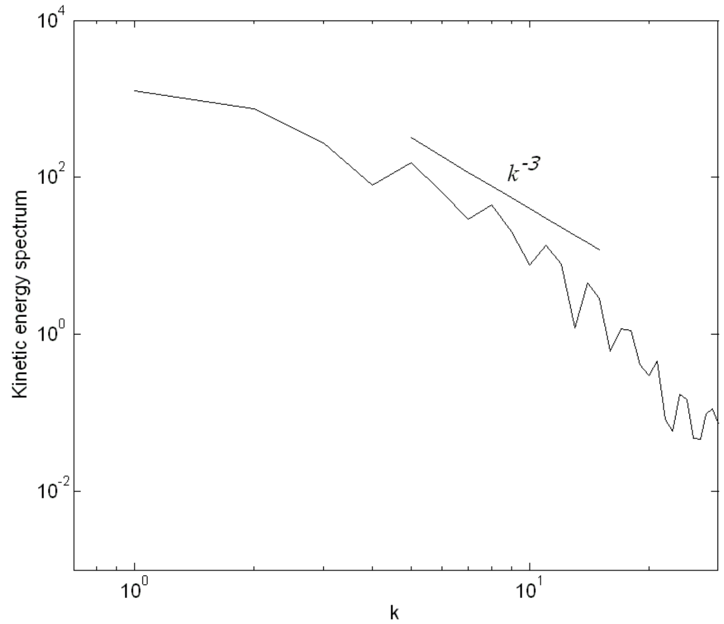


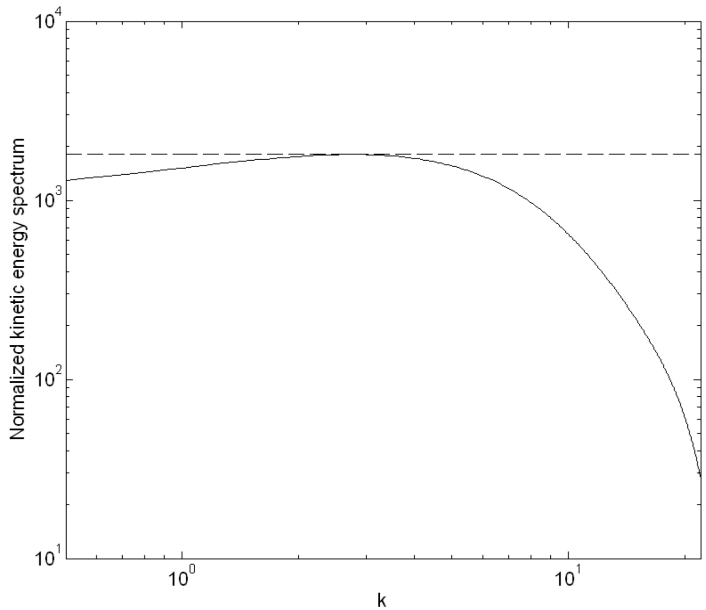




Decay of turbulent small-scale Mach number with time. A transition from a supersonic to a subsonic regime can be observed.

Time dynamics of the velocity divergence. The velocity divergence describing medium compressibility attenuates and the flow becomes weakly compressible with time.

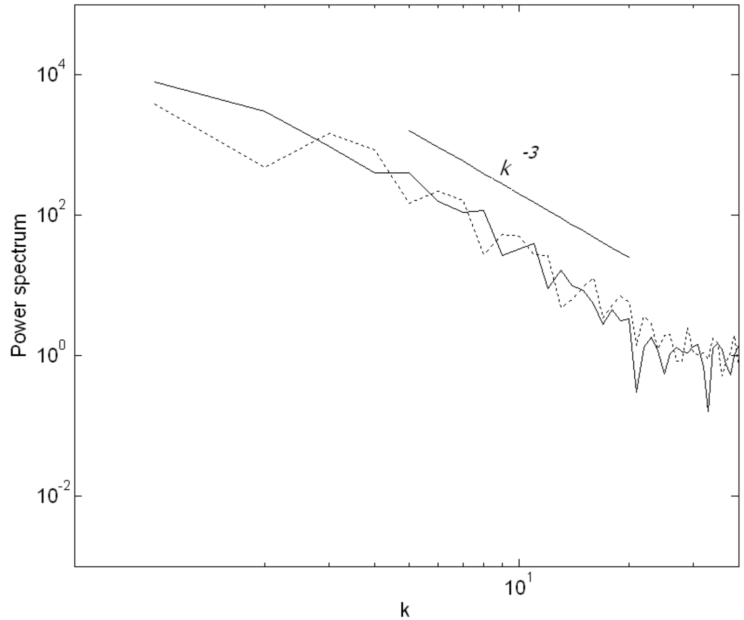
# Turbulent spectra in the local interstellar medium

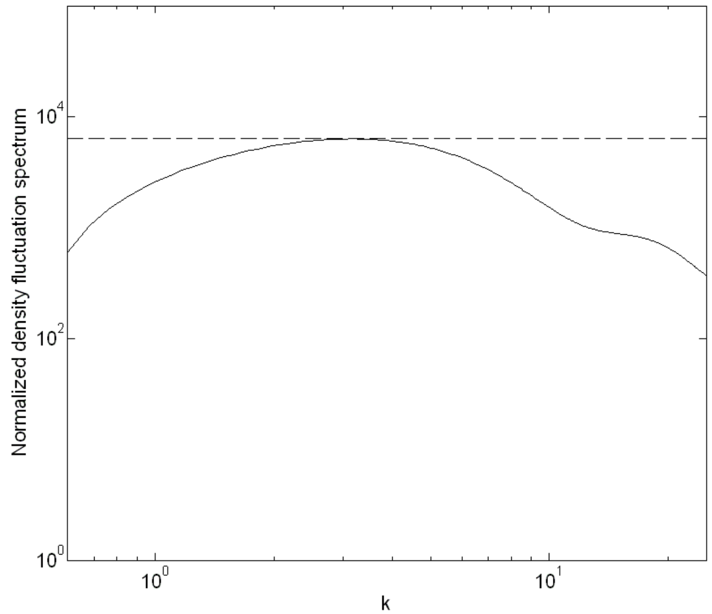



*The kinetic energy spectrum (**left**). Normalized and smoothed spectrum of kinetic energy, multiplied by $k^{5/3}$ (**right**). Notice that the spectrum is close to $k^{-3}$ in a forward cascade regime of decaying turbulence. However, there is well-defined inertial Kolmogorov-like range of $k^{-5/3}$*

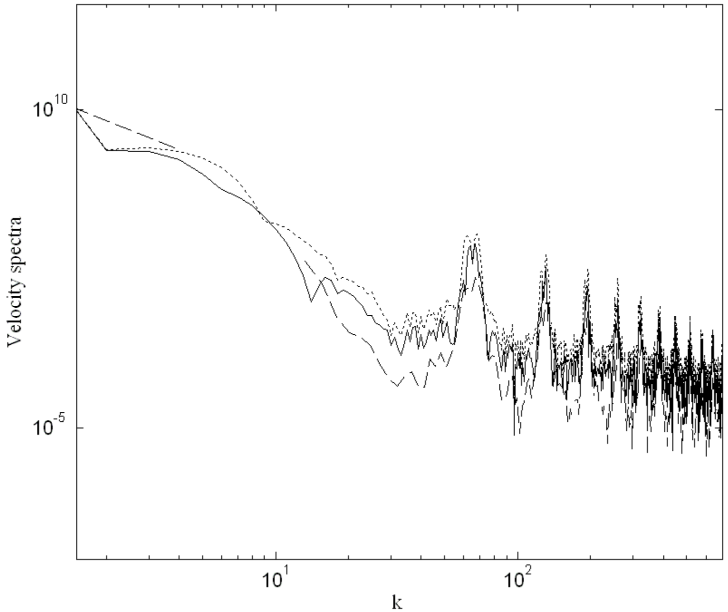
# Turbulent spectra in the local interstellar medium

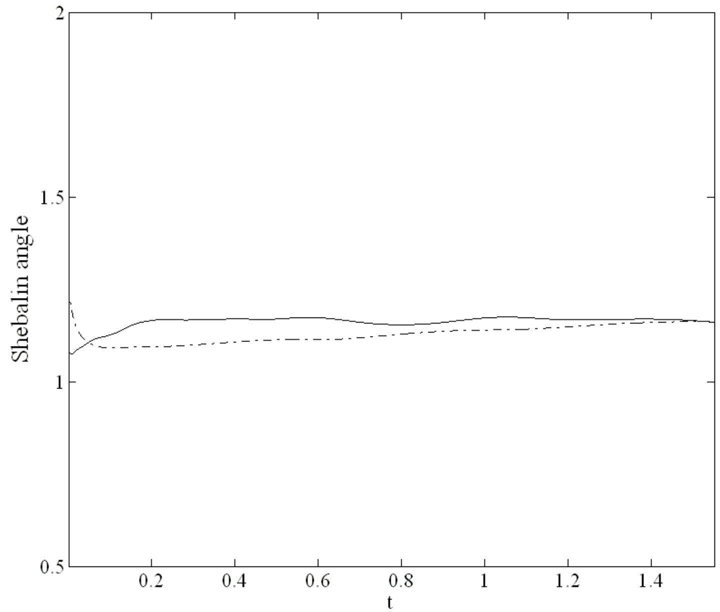



The density spectrum is the solid line and the density fluctuations spectrum is the dot line (**left**). Normalized and smoothed spectrum of density fluctuations, multiplied by $k^{5/3}$ (**right**). Both graphs (in the left figure) have spectral index close to $k^{-3}$ . Moreover, there is well-defined inertial Kolmogorov-like range of $k^{-5/3}$ that confirms observation data.

# Anisotropic turbulence




Anisotropy and symmetry breakdown are caused first of all by the magnetic field at low value of the plasma beta when the role of the magnetic field is substantial. Anisotropic cascades are observed to be due to propagating compressible acoustic modes that hinder spectral transfer in the local Fourier space at high value of plasma beta when the role of the magnetic field is little. These modes in compressible MHD turbulence could be excited either by a large-scale or ambient velocity component of the background hydrodynamic turbulence.

$$tan^2\theta_u = 2\frac{G^u_{xx} + G^u_{xy} + G^u_{xz}}{G^u_{yx} + G^u_{yy} + G^u_{yz}}$$

$$tan^2\theta_b = 2\frac{G^b_{xx} + G^b_{xy} + G^b_{xz}}{G^b_{yx} + G^b_{yy} + G^b_{yz}}$$

- the Shebalin angles (or anisotropy angles)

# outcome

- It is shown that density fluctuations are a passive scalar in a velocity field in weakly compressible magnetohydrodynamic turbulence and demonstrate Kolmogorov-like spectrum

- The decrease of energy-containing large eddies and inertial range with time, and the increase of dissipative scale are also represented

- It is shown, that the turbulent sonic Mach number decreases significantly from a supersonic turbulent regime, where the medium is strongly compressible, to a subsonic value of Mach number describing weakly compressible flow

- In local interstellar medium, the transition of MHD turbulent flow from a strongly compressible to a weakly compressible state not only transforms the characteristic supersonic motion into subsonic motion, but also attenuates plasma magnetization, which is shown in this work because plasma beta increases with time, thus, role of magnetic energy decreases in comparison with plasma pressure.

- The anisotropy of turbulent flow is considered and it is demonstrated that large-scale flow shows anisotropic properties while small-scale structures are isotropic.

# Compressible MHD equations

The system of equations of compressible magnetohydrodynamic turbulence in the presence of external force is written in the following form:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial \rho u_j}{\partial x_j}$$

$$\frac{\partial \rho u_i}{\partial t} = -\frac{\partial}{\partial x_j}\left(\rho u_i u_j + p\delta_{ij} - \sigma_{ij} - \frac{1}{4\pi} B_j B_i + \frac{1}{8\pi} B^2\right) + F_i^u$$

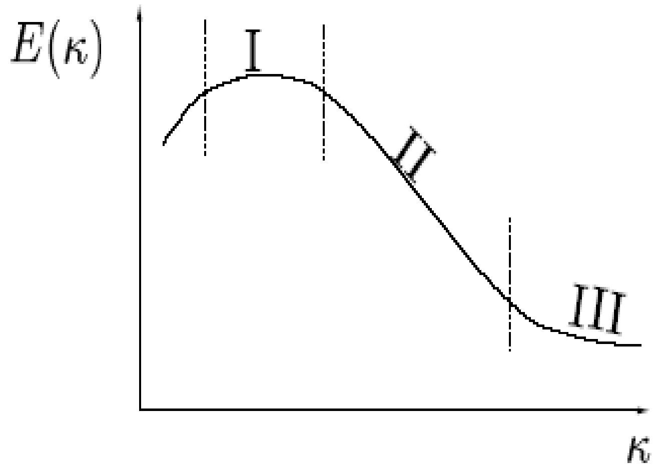
$$\frac{\partial B_i}{\partial t} = -\frac{\partial}{\partial x_j}\left(B_i u_j - B_j u_i\right) + \eta \nabla^2 B_i + F_i^b$$

$$\frac{\partial B_j}{\partial x_j} = 0$$

Driving forces

Polytropic relation: $p = \rho^\gamma$

# Turbulence




$I.\quad Integral\ Scale \qquad l$

$II.\quad Inertial\ Subrange \qquad \eta$

$III.\quad Viscous\ Subrange \qquad \lambda$

Kolmogorov-Obukhov spectrum $E(k) \sim k^{-5/3}$

Iroshnikov-Kraichnan spectrum $E(k) \sim k^{-3/2}$

# Linear forcing

Idea essentially consists in adding a force proportional to the fluctuating velocity. Linear forcing resembles a turbulence when forced with a mean velocity gradient, that is, a shear. This force appears as a term in the equation for fluctuating velocity that corresponds to a production term in the equation of turbulent kinetic energy.

The equation for the fluctuating part of the velocity in a compressible MHD turbulent flow are written as

$$\rho[\frac{\partial \acute{u}_i}{\partial t} + U_j \frac{\partial \acute{u}_i}{\partial x_j}] = -\frac{\partial \acute{p}}{\partial x_j} + \frac{\partial \acute{\sigma}_{ij}}{\partial x_j} - \boxed{\rho \acute{u}_j \frac{\partial U_i}{\partial x_j}} - [\rho \acute{u}_j \frac{\partial \acute{u}_i}{\partial x_j} - \rho \langle \acute{u}_j \frac{\partial \acute{u}_i}{\partial x_j} \rangle] - \frac{\partial}{\partial x_j} \frac{\acute{B}^2}{8\pi} + \frac{1}{4\pi}[\text{ß}_j \frac{\partial \acute{B}_i}{\partial x_j} + \acute{B}_j \frac{\partial \text{ß}_i}{\partial x_j}] - \frac{1}{4\pi}[\acute{B}_j \frac{\partial \acute{B}_i}{\partial x_j} - \langle \acute{B}_j \frac{\partial \acute{B}_i}{\partial x_j} \rangle]$$

*Here following decomposition referred to as the Reynolds decomposition is used:*

$$u_i = U_i + \acute{u}_i,\ B_i = \text{ß}_i + \acute{B}_i,\ B_i = \text{ß}_i + \acute{B}_i,\ p = P + \acute{p},\ \sigma_{ij} = \Sigma_{ij} + \acute{\sigma}_{ij}$$

# Linear forcing

In symbolic terms, derivation of turbulent kinetic energy equation can be written as $\langle u \cdot NS\ eq. \rangle - U \langle NS\ eq. \rangle$ which yields:

$$\frac{\partial}{\partial t}\langle \frac{1}{2}\rho u'^2 \rangle + \frac{\partial}{\partial x_j}(\langle \frac{1}{2}\rho u'^2 \rangle U_j + \langle \frac{1}{2}\rho u'^2 u'_j \rangle - \langle \beta_{ij} u'_i \rangle) = -\langle u'_i \frac{\partial p'}{\partial x_i} \rangle + \langle u'_i \frac{\partial \sigma'_{ij}}{\partial x_j} \rangle - \langle \rho u'_i u'_j \frac{\partial U_i}{\partial x_j} \rangle - \langle \beta_{ij} \frac{\partial u'_i}{\partial x_j} \rangle$$

where $\beta_{ij} = \frac{B'_i B'_j}{4\pi} - \frac{B'^2}{8\pi}\delta_{ij}$ - turbulent magnetic tensor

production of turbulent energy per unit volume per unit time resulting from the interaction between the Reynolds stress and the mean shear.

# Linear forcing

$F_i^u = \Theta \rho u_i$ - driving term proportional to the velocity

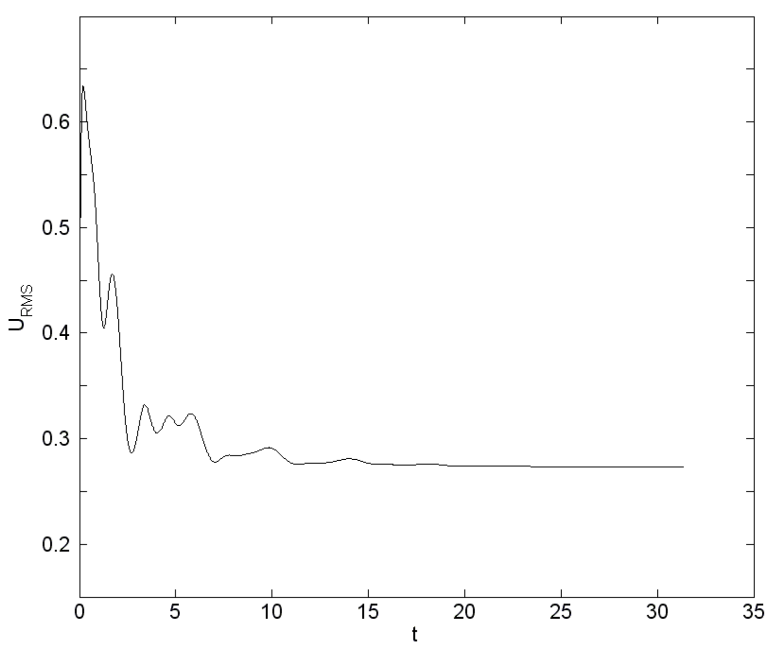
$\Theta$ ↓ coefficient which is determined from a balance of kinetic energy for a statistically stationary state:

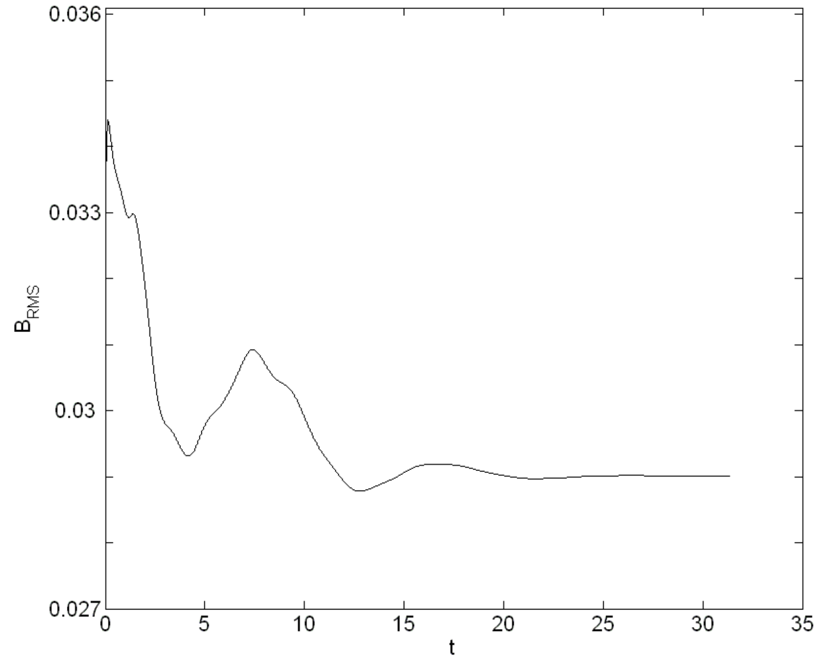
$$\Theta = \frac{1}{3\langle\rho\rangle u_{rms}^2}[\langle u_j \frac{\partial}{\partial x_j} p\delta_{ij}\rangle + \varepsilon + \frac{1}{8\pi}\langle u_j \frac{\partial}{\partial x_j} B^2 \delta_{ij}\rangle]$$

$\varepsilon \;=\; -\langle u_j \;\; \partial\sigma/\partial x_j\rangle$ - mean dissipation rate of turbulent energy into heat

$$1/(\langle\rho u^2\rangle) \;=\; 1/(3\langle\rho\rangle u_{rms}^2)$$

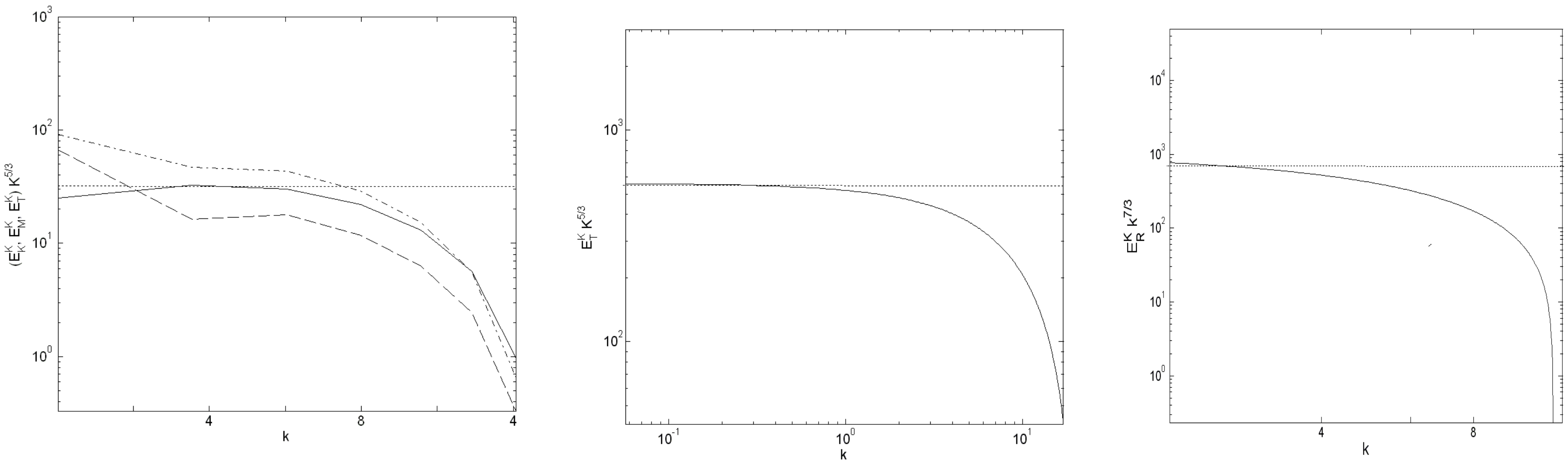
# Polytropic plasma -1



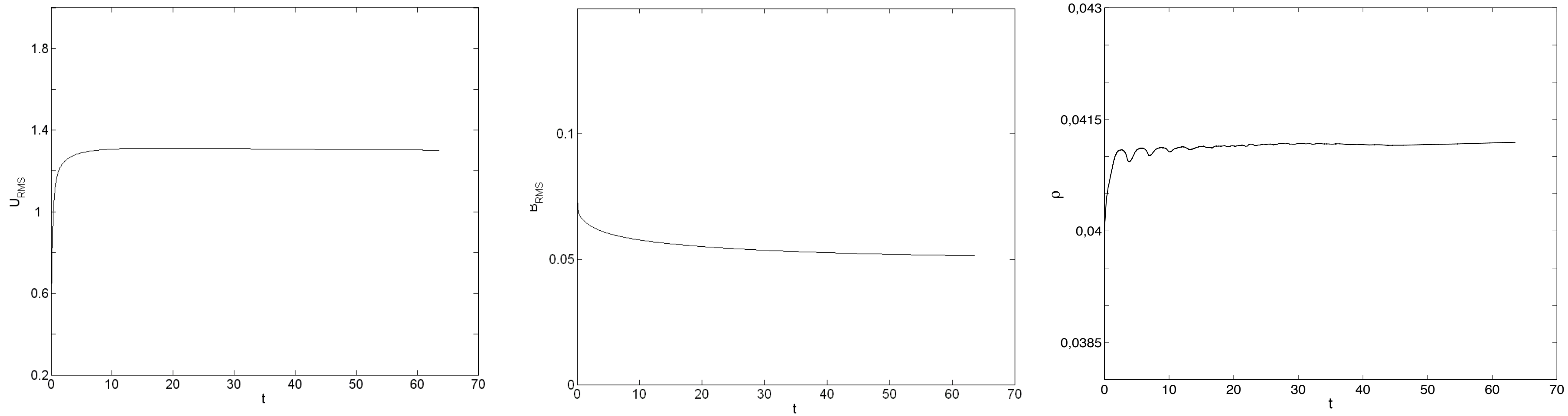

Time evolution of U_rms and B_rms

# Polytropic plasma

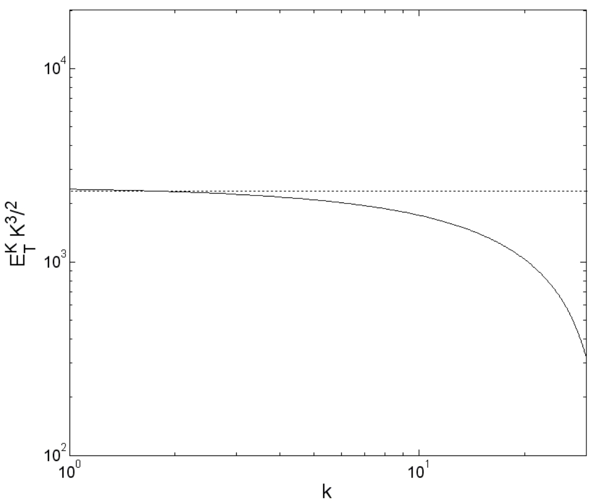



Spectra of MHD turbulence

# Polytropic plasma




Time dynamics of rms velocity, rms magnetic field and mean density.




Spectrum of total energy

# outcome

The theory of linear forcing is developed for study of compressible MHD turbulence  in coordinate space. The expressions of external force which provide obtaining a statistically stationary regime of turbulence are derived. The formulae used for the formulation of large eddy simulation approach are obtained. The potential possibilities of LES method to reproduce physics of flow under investigation in a stationary regime both for polytropyc and for heat-conducting plasmas are studied.

Spectra of MHD turbulence is obtained and studied. Type of obtained spectra is determined. Kolmogorov and Iroshnikov-Kraichnan spectra of total energy are obtained and conditions of their occurrence are showed.

Efficiency of LES method for studying of scale-invariant properties of compressible MHD turbulence is demonstrated.