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<u>Detecting structural complexity by knot polynomials</u>

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Detecting structural complexity by knot polynomials

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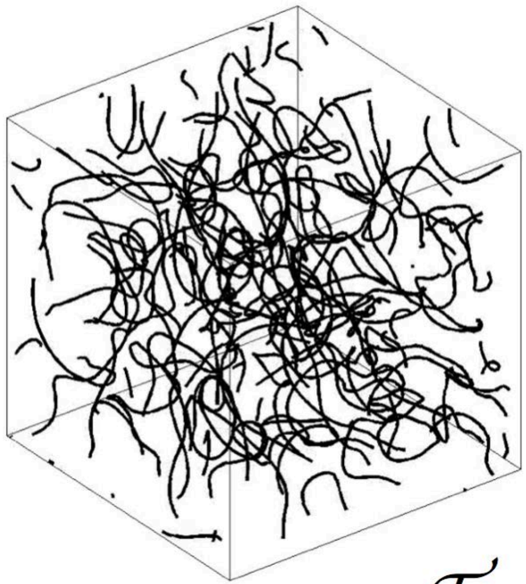
Joint work with **Xin Liu (BJUT, China)**

Objectives

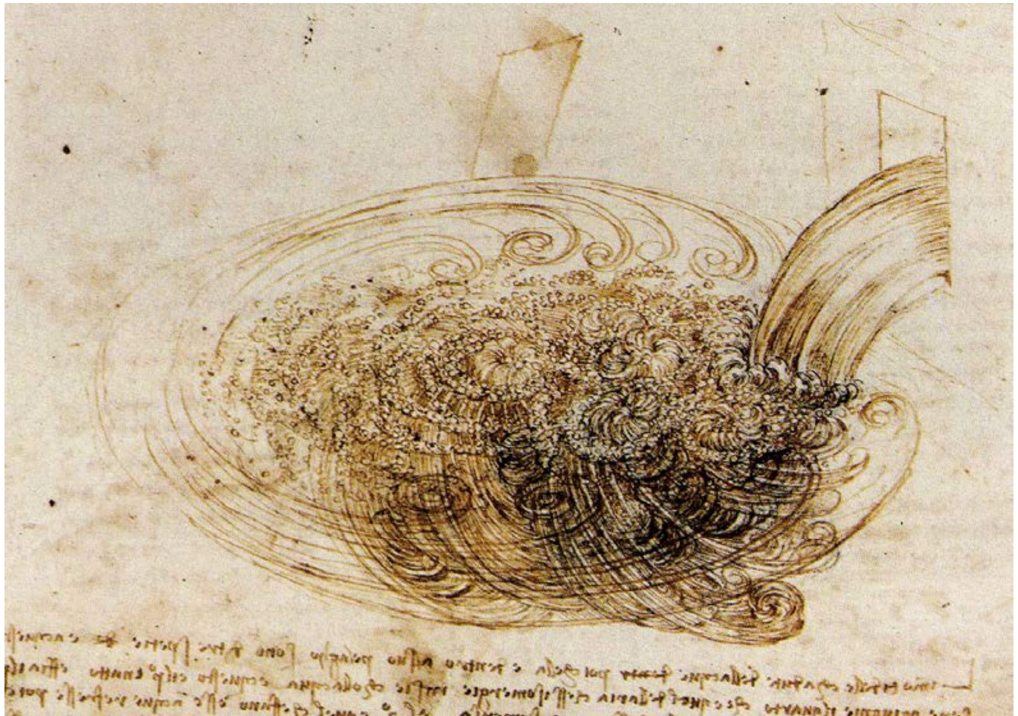
- *Determine relationships between structural complexity of physical knots and energy;*
- *Quantify energy/helicity transfers in dynamical systems.*

↓

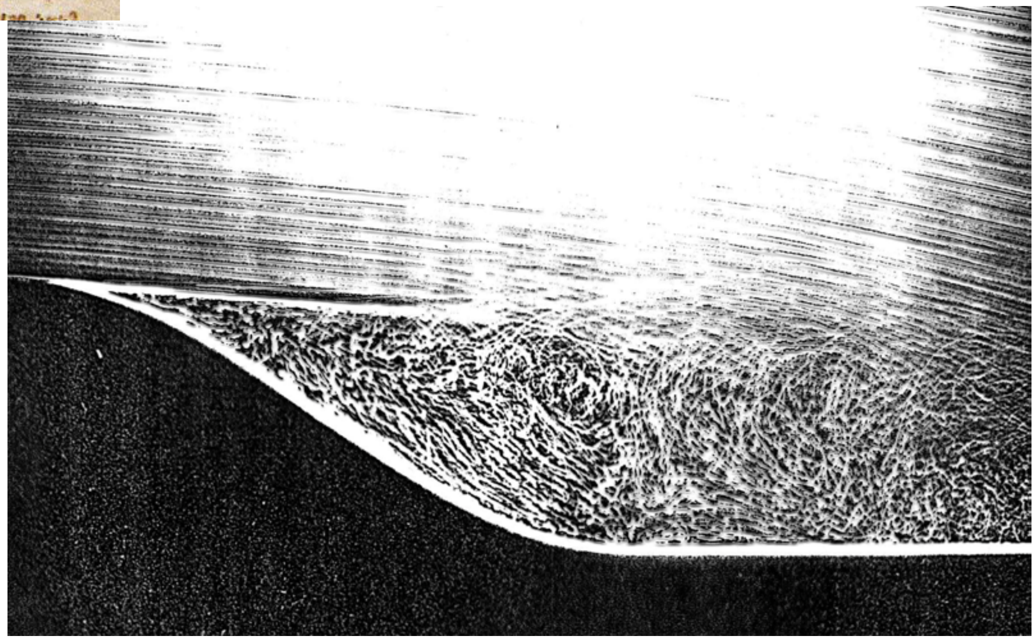
- *Knot polynomials as new physical invariants to quantify topological complexity;*
- *Extend and apply new topological techniques to study complex systems.*



Coherent structures

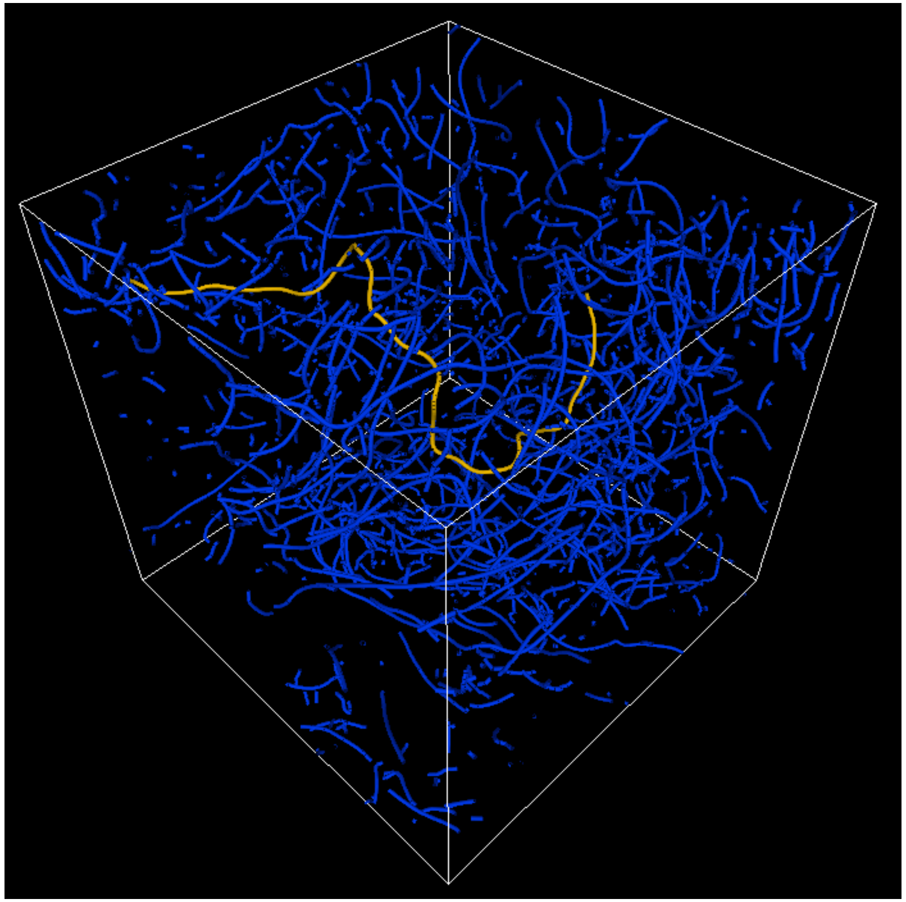


Leonardo da Vinci
(Water Studies 1506)

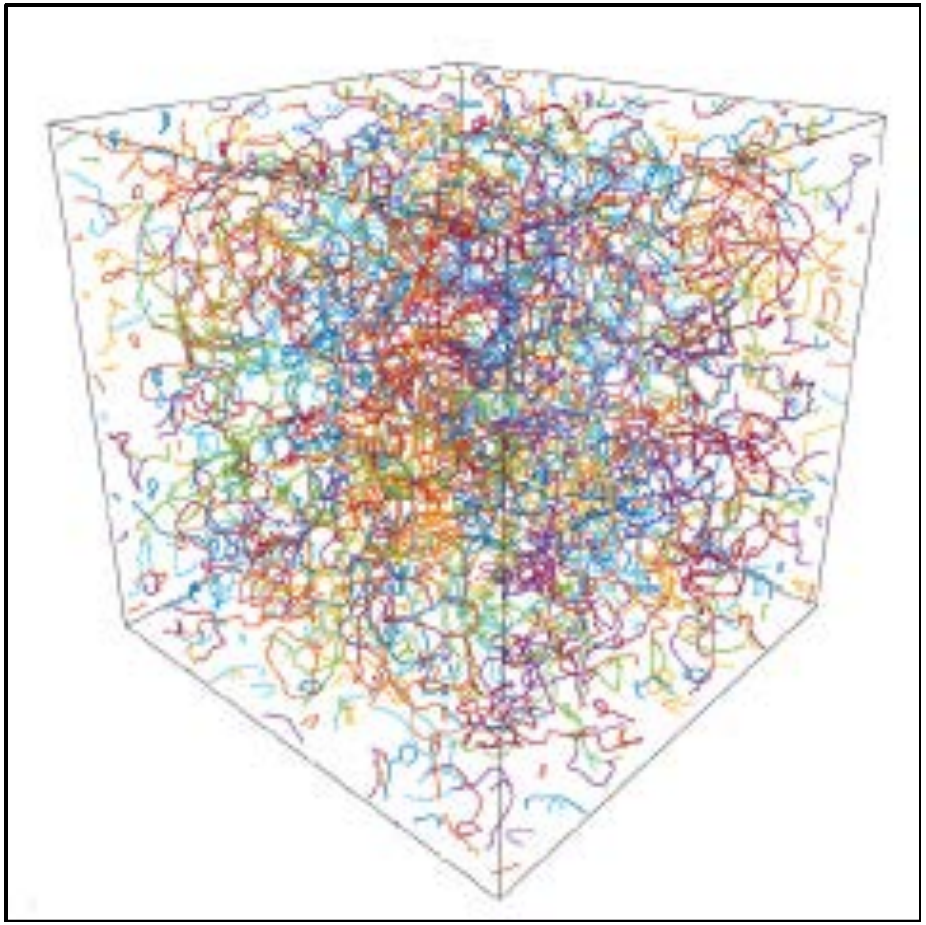


Werlé, ONERA, 1974
(Van Dyke 1982)

Vorticity localization in classical and quantum fluids



Kida et al.
(Toki-Kyoto 2002)



Villois et al.
(PRE 2016)

Modeling vortex tangles by filaments

homogeneous
incompressible
inviscid

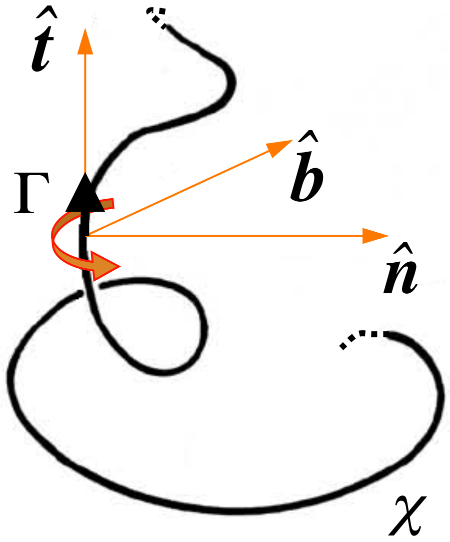
} ***fluid in*** $\mathbb{R}^3$ **:** $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{X}, t)$ $\begin{cases} \nabla \cdot \boldsymbol{u} = 0 & \textit{in } \mathbb{R}^3 \\ \boldsymbol{u} = 0 & \textit{as } X \to \infty \end{cases}$

- ***Vortex line*** χ**:** ***vorticity:*** $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$,

$$\boldsymbol{\omega} = \varpi_0 \hat{\boldsymbol{t}} \ , \quad \varpi_0 = \text{constant} \ ;$$

circulation: $\Gamma = \oint \boldsymbol{\omega} \cdot \mathrm{d}^2 \boldsymbol{X} = \text{constant}$.

Vortex tangle: $\mathcal{T} = \bigcup_i \chi_i \quad i = 1, \ldots, N$.



- ***Kinetic energy:*** $E(\mathcal{T}) = \int_\Omega \boldsymbol{u} \cdot (\boldsymbol{X} \times \boldsymbol{\omega}) \, \mathrm{d}^3 \boldsymbol{X}$

from Lamb (1932), we have: $E(\mathcal{T}) \approx \dfrac{\Gamma^2}{4\pi} \sum_{ij} \iint_{\chi_i \chi_j} \dfrac{\hat{\boldsymbol{t}}_i \cdot \hat{\boldsymbol{t}}_j}{|\boldsymbol{X}_i - \boldsymbol{X}_j|} \, \mathrm{d}s_i \mathrm{d}s_j$.

- ***Total length of vortex tangle given by*** $L(\mathcal{T}) = \sum_i \int_{\chi_i} \hat{\boldsymbol{t}}_i \, \mathrm{d}s_i$.

Kinetic helicity and linking numbers

- **Kinetic helicity:**

$$H(\mathcal{T}) = \int_{\mathcal{T}} \boldsymbol{u}\cdot\boldsymbol{\omega}\, \mathrm{d}^3 \boldsymbol{X} = \Gamma \sum_i \int_{\chi_i} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}\,.$$

- **Topological interpretation of kinetic helicity in terms of linking numbers (Moffatt 1969; Ricca & Moffatt 1992):**

$$H(\mathcal{T}) = \Gamma^2 \left(\sum_i SL_i + \sum_{i\neq j} Lk_{ij} \right) \qquad \begin{cases} SL_i = SL(\chi_i) & \text{self-linking number} \\ Lk_{ij} = Lk(\chi_i, \chi_j) & \text{linking number} \end{cases}$$

- **Self-linking number (Călugăreanu-White invariant):**

Consider the ribbon $\Re(\chi_i, \chi_i^*)$ **; then** $SL_i = \lim_{\varepsilon\to 0} Lk(\chi_i, \chi_i^*)$ **,**
where

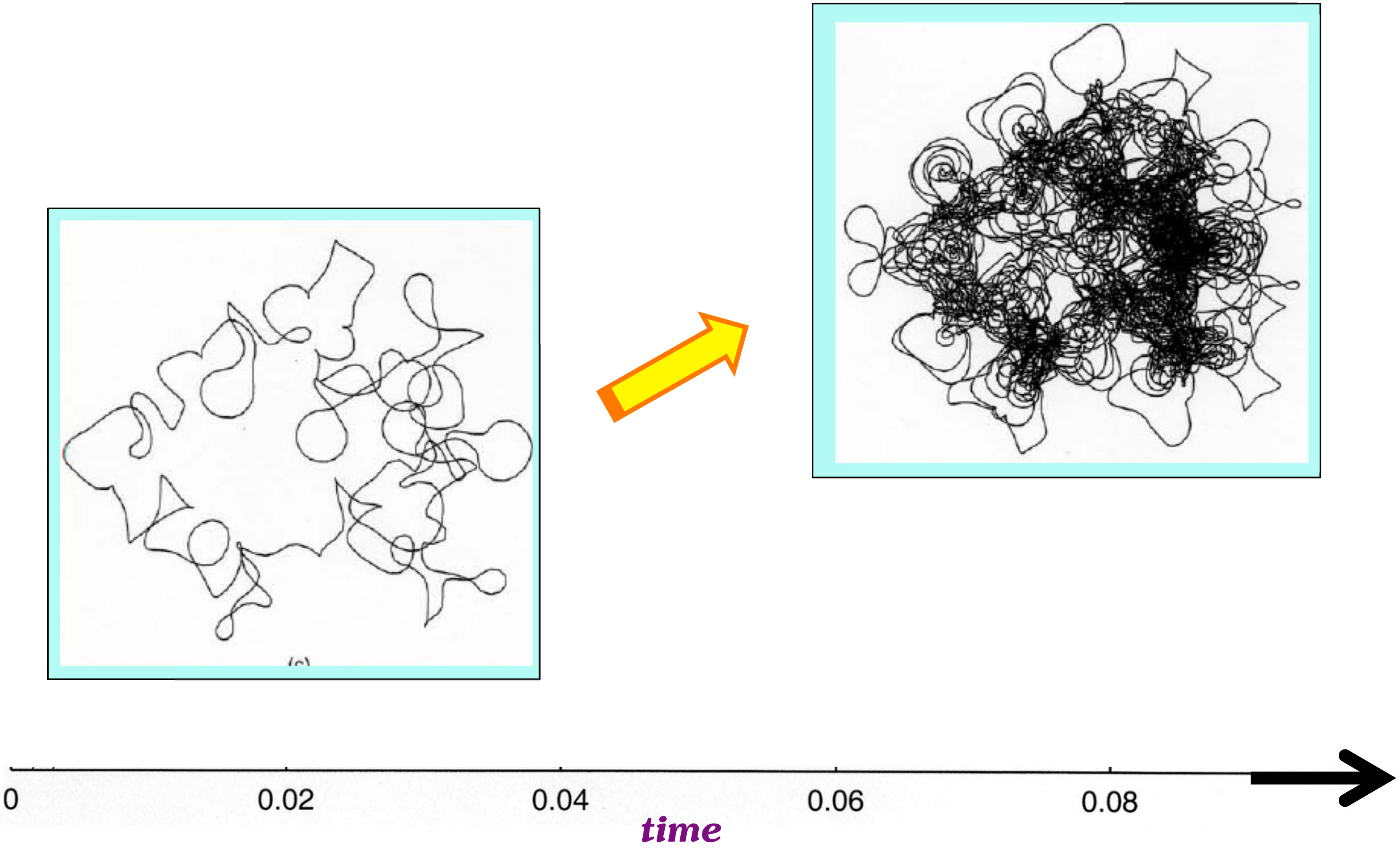
$$SL_i = Wr(\chi_i) + Tw(\chi_i, \chi_i^*)\,,$$

writhing number: $Wr(\chi_i)$

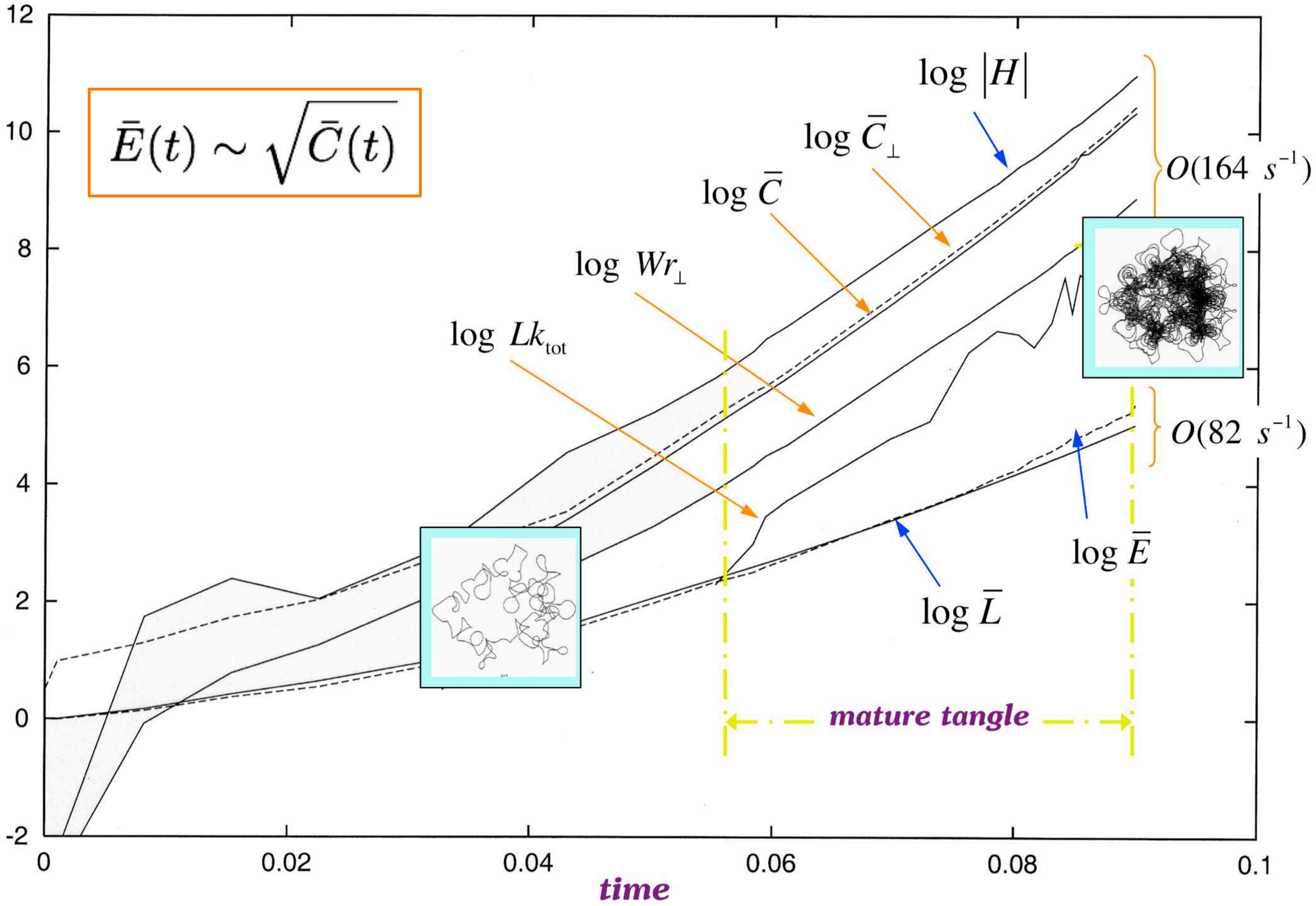
total twist number: $Tw(\chi_i, \chi_i^*)$

Energy-complexity relation: a 15 years-old test case

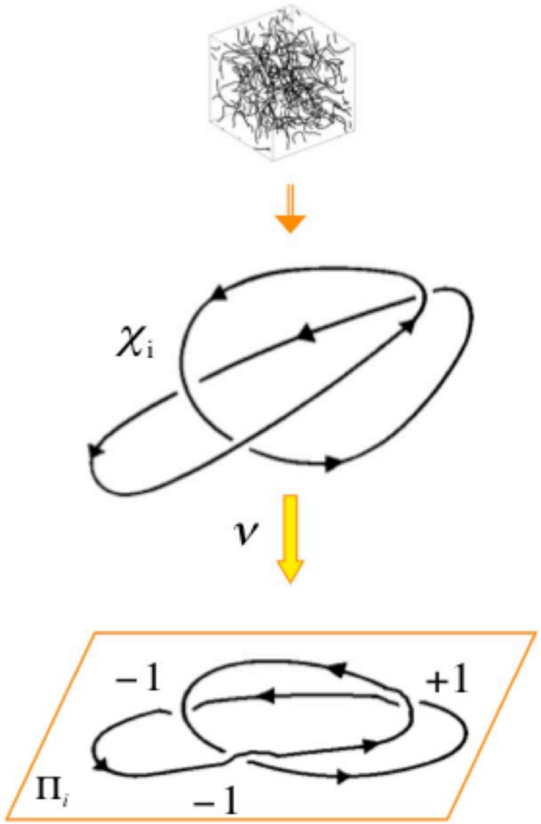
- ***ABC-flow acting on seed vorticity:***



Energy-complexity relation *(Barenghi et al., Physica D 2001)*



Tackling structural complexity by knot polynomials



- ***Helicity and linking number limitations:***

(i) $$H(\mathcal{T}) = f\left(SL_i, Lk_{ij}; \Gamma\right)$$

(ii) $$Lk_{ij} = 0\,, \quad \sum_{i \neq j} Lk_{ij} = 0\,.$$

- ***HOMFLYPT polynomial*** $P(\chi) = P_\chi(a, z)$ **:**

(P.1) $$P(\text{O}) = 1$$

(P.2) aP  $- a^{-1}P$  $= zP$ 

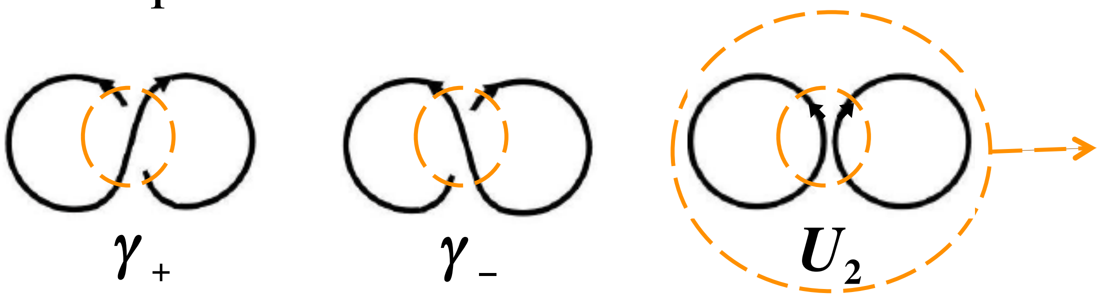
$+$ $\quad -$ $\quad =$

P.1:

U_1 $\quad \gamma_+$ $\quad \gamma_-$

$$P(\text{O}) = P(\gamma_+) = P(\gamma_-) = 1$$

P.2:



γ_+ $\quad \gamma_-$ $\quad U_2$

$$P(U_2) = \frac{a - a^{-1}}{z}$$

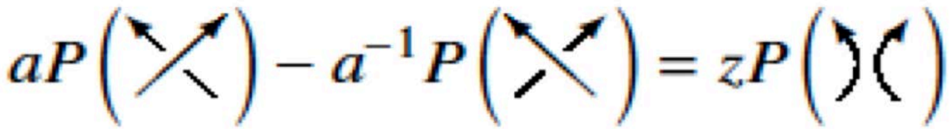
HOMFLYPT polynomial from self-linking

- **Theorem (Liu & Ricca, *JFM* 2015):** *If χ denotes a vortex knot of helicity $H = H(\chi)$, then*

$$e^{SL(\chi)} = e^{\oint_{\chi} \boldsymbol{u}\cdot \mathrm{d}\boldsymbol{l}} ,$$

appropriately rescaled, satisfies (with a plausible statistical hypothesis) the skein relations of the HOMFLYPT polynomial $P(\chi) = P_{\chi}(a,z)$.

- **HOMFLYPT variables in terms of writhe and twist:**



$$\Rightarrow \quad [f(Tw)] = g(Wr) .$$

with $z = k - k^{-1}$ *and*

$$\begin{cases} k = e^{2\omega}, & \omega = \lambda_{\omega}\langle Wr \rangle \\ a = e^{\tau}, & \tau = \lambda_{\tau}\langle Tw \rangle \end{cases} \quad \text{and} \quad \{\lambda_{\omega}, \lambda_{\tau}\} \in (0;1) ,$$

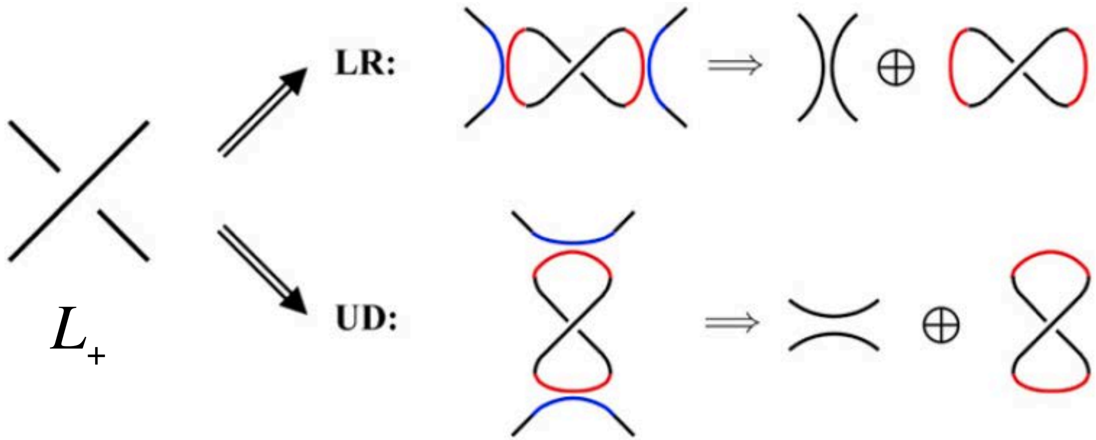
hence $a = f(Tw)$ *and* $z = g(Wr)$.

- **Reduction of HOMFLYPT to Jones:**

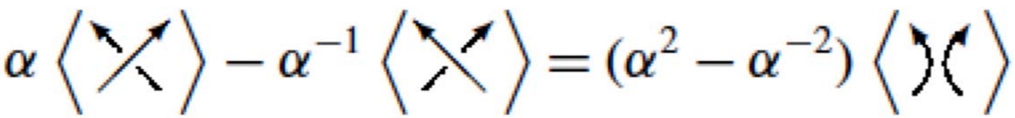
$$ak^2 = e^{\tau}e^{4\omega} = 1 \quad \Rightarrow \quad Wr = -4\lambda\, Tw \qquad (\lambda = \lambda_{\tau}/\lambda_{\omega}) .$$

Sketch of proof

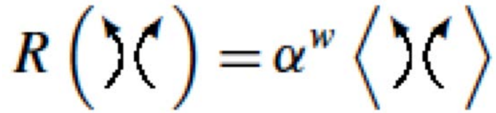
- ***derive the Kauffman bracket $\langle \cdot \rangle$ polynomial for unoriented knot; assume equal probability in state decomposition:***



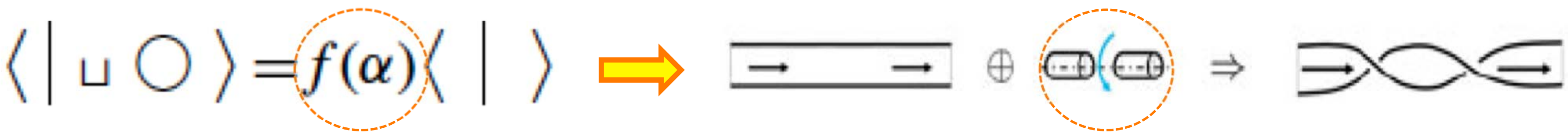
- ***orient knot***
- ***derive skein relation for z in terms of Wr, considering***



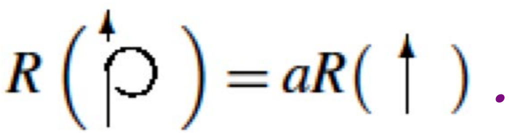
and



- ***note***



- ***derive skein relation for a in terms of twist Tw, considering***



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Quantifying topological complexity

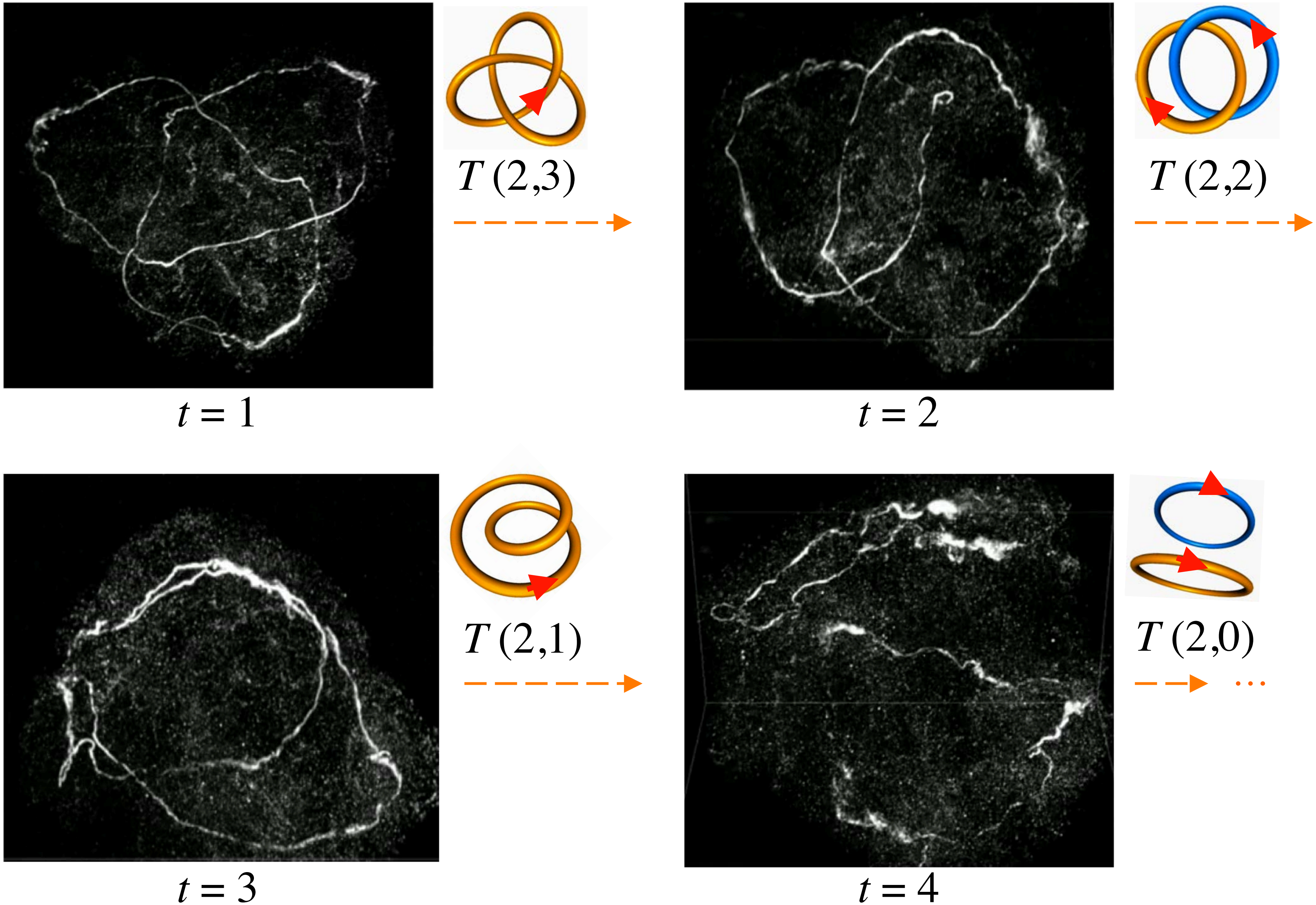
In general we shall have $P(\chi) = f(\chi, \Gamma)$.

- *Homogeneous superfluid tangle:* $\Gamma = 1$ *and*

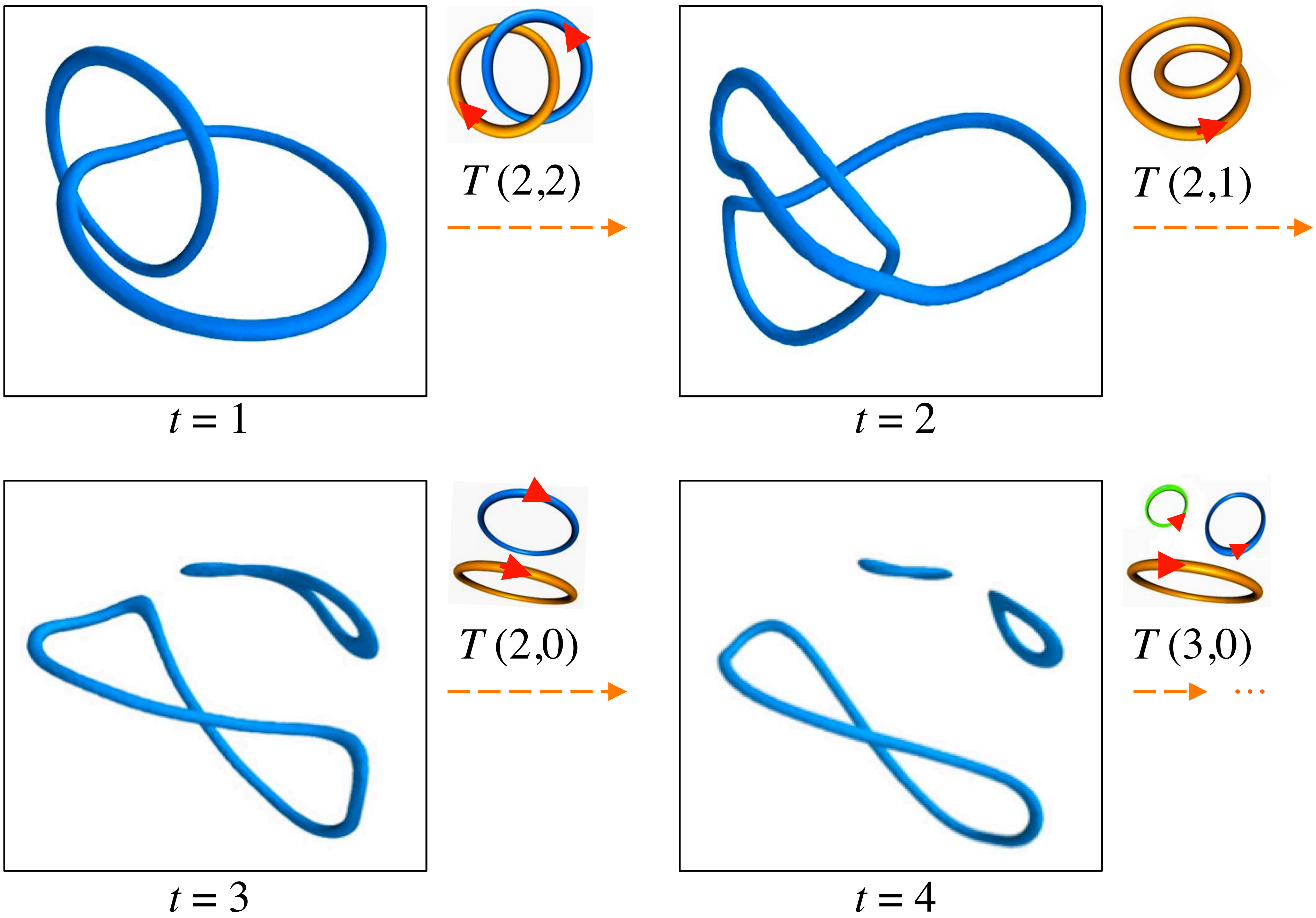
$$\begin{cases} k = e^{2\omega}, & \omega = \lambda_\omega \langle Wr \rangle \\ a = e^{\tau}, & \tau = \lambda_\tau \langle Tw \rangle \end{cases} \quad \textit{with} \quad \begin{matrix} \langle Wr \rangle = \langle Tw \rangle = 1/2 \\ \lambda_\omega = \lambda_\tau = 1/2 \end{matrix} \Rightarrow \begin{cases} z = e^{1/2} - e^{-1/2} \\ a = e^{1/4} \end{cases}$$

Knot type	HOMFLYPT polynomial	Numerical value
U_N	$\delta^{N-1} = [(a - a^{-1})z^{-1}]^{N-1}$	0.48^{N-1}
H_+	$a^{-1}z + (a^{-1} - a^{-3})z^{-1}$	1.10
H_-	$-az - (a - a^3)z^{-1}$	−0.54
T^L	$2a^2 + a^2z^2 - a^4$	2.36
T^R	$2a^{-2} + a^{-2}z^2 - a^{-4}$	1.51
F^8	$a^{-2} - 1 - z^2 + a^2$	0.17
W	$-a^{-1}z^{-1} - a^{-1}z + az^{-1} + 2az + az^3 - a^3z$	1.59
…	…	…

Vortex trefoil cascade process in water (Kleckner & Irvine 2013)



Vortex link cascade in BECs (Zuccher & Ricca, IUTAM 2016)

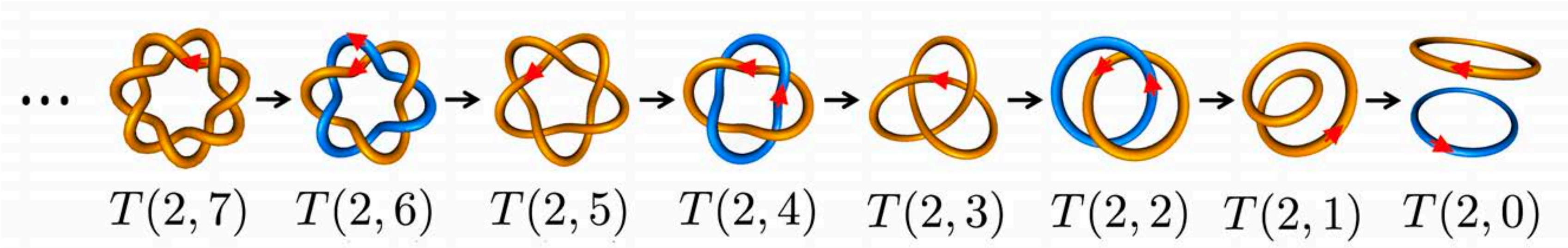


Ideal torus knots & links cascade

Consider the cascade process:

(i)

(ii)



$$\{T(2,n)\}: \quad \ldots \rightarrow T(2,2n+1) \rightarrow T(2,n) \rightarrow \ldots \rightarrow T(2,0)\ .$$

Assumptions:

- ***all torus knots*** $T(2,2n+1)$ ***and links*** $T(2,2n)$ ***are standardly embedded on a mathematical torus in closed braid form;***
- ***all torus knots and links form an ordered set*** $\{T(2,n)\}$ ***of elements listed according to their decreasing value of topological complexity given by*** $c_{\min} = n$ ***;***
- ***any topological transition between two contiguous elements of*** $\{T(2,n)\}$ ***is determined by a single, orientation-preserving reconnection event.***

Torus knots cascade detected by HOMFLYPT

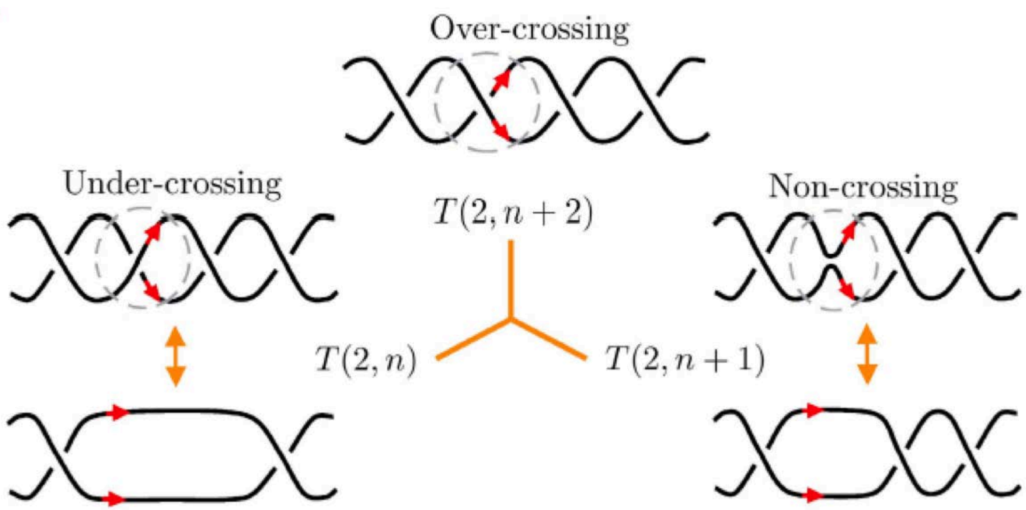
- **Theorem** (Liu & Ricca, *Sci Rep* 2016): *HOMFLYPT computation of $P_{T(2,n)}$ generates, for decreasing n, a monotonically decreasing sequence of numerical values given by*

$$P_{T(2,3+q)} = A_q(\tau, \omega) P_{T(2,3)} + B_q(\tau, \omega) P_{T(2,2)} \quad (q \in \mathbb{N}) \, ,$$

where $A_q(\tau,\omega)$ and $B_q(\tau,\omega)$ are known functions of τ and ω, with initial conditions $P_{T(2,3)}$ and $P_{T(2,2)}$.

Sketch of proof.

Apply (P.2) to



to obtain

$$P_{T(2,n+2)} = a^{-1} z P_{T(2,n+1)} + a^{-2} P_{T(2,n)} \, .$$

Recursively, we have

$$P_{T(2,n+2)} - \alpha P_{T(2,n+1)} = \beta^{n-1}\left[P_{T(2,3)} - \alpha P_{T(2,2)}\right], \quad n \geq 2,$$

and after some algebra

$$P_{T(2,n)} = \left(\frac{\beta^{n-2} - \alpha^{n-2}}{\beta - \alpha}\right) P_{T(2,3)} - \left(\alpha\beta\frac{\beta^{n-3} - \alpha^{n-3}}{\beta - \alpha}\right) P_{T(2,2)}, \quad n \geq 4\,.$$

Hence, by setting $k = e^{2\omega}$ ***and*** $a = e^{\tau}$***, we have:***

$$P_{T(2,3+q)} = A_q(\tau,\omega) P_{T(2,3)} + B_q(\tau,\omega) P_{T(2,2)} \quad (q \in \mathbb{N}),$$

with

$$A_q(\tau,\omega) = \frac{e^{2(1+q)\omega} - (-1)^{1+q} e^{-2(1+q)\omega}}{e^{q\tau}(e^{2\omega} + e^{-2\omega})}\,, \quad B_q(\tau,\omega) = \frac{e^{2q\omega} - (-1)^q e^{-2q\omega}}{e^{(1+q)\tau}(e^{2\omega} + e^{-2\omega})}\,.$$

and

$$P_{T(2,3)} = 2a^{-2} + a^{-2}z^2 - a^{-4}, \qquad P_{T(2,2)} = a^{-1}z + (a^{-1} - a^{-3})z^{-1}.$$

Since for mirror knot $P(\chi) \to P(\tilde{\chi})$ ***by changing***

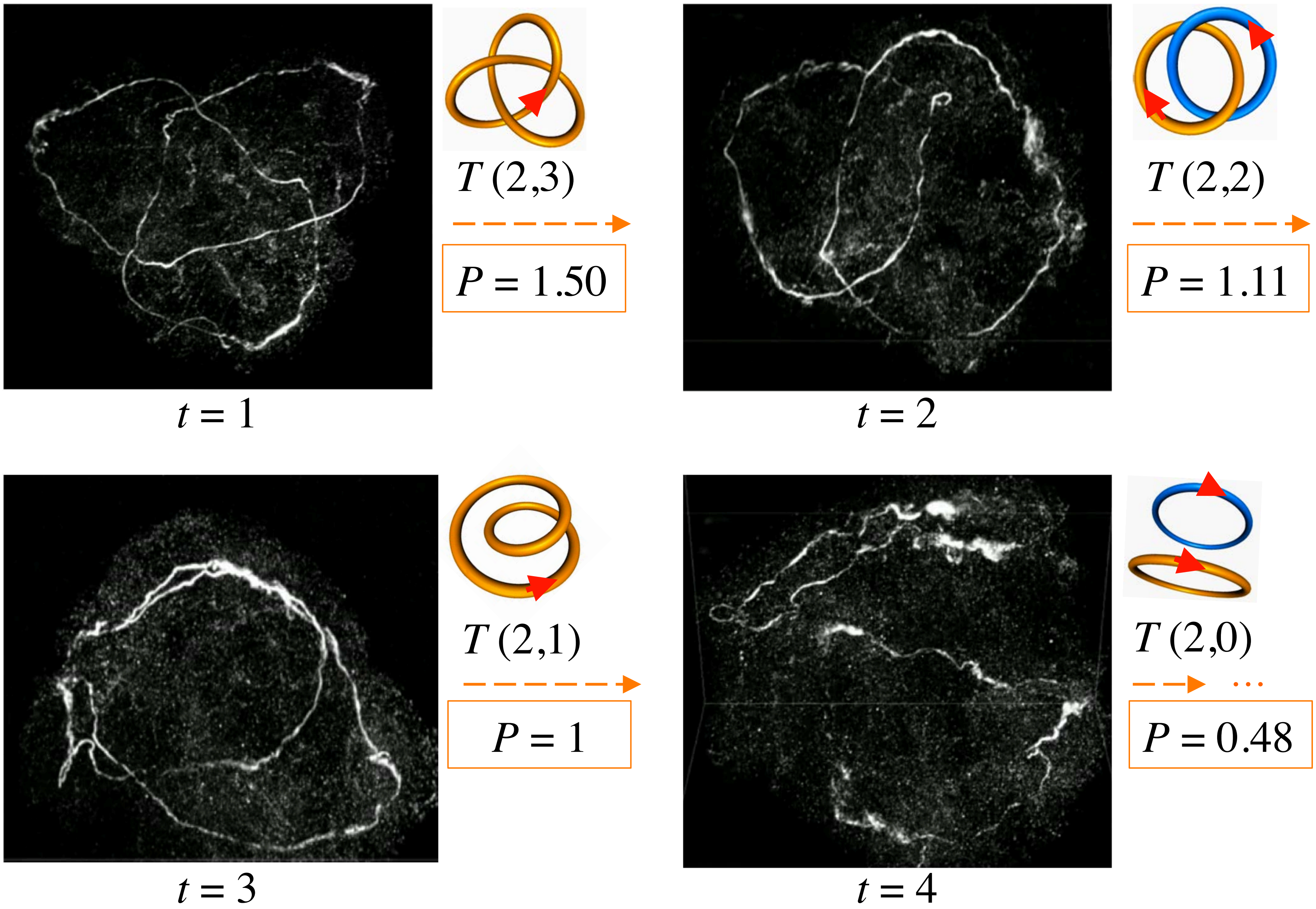
$$a \to a^{-1} \ (\tau \to -\tau), \qquad z \to -z \ (\omega \to -\omega),$$

then

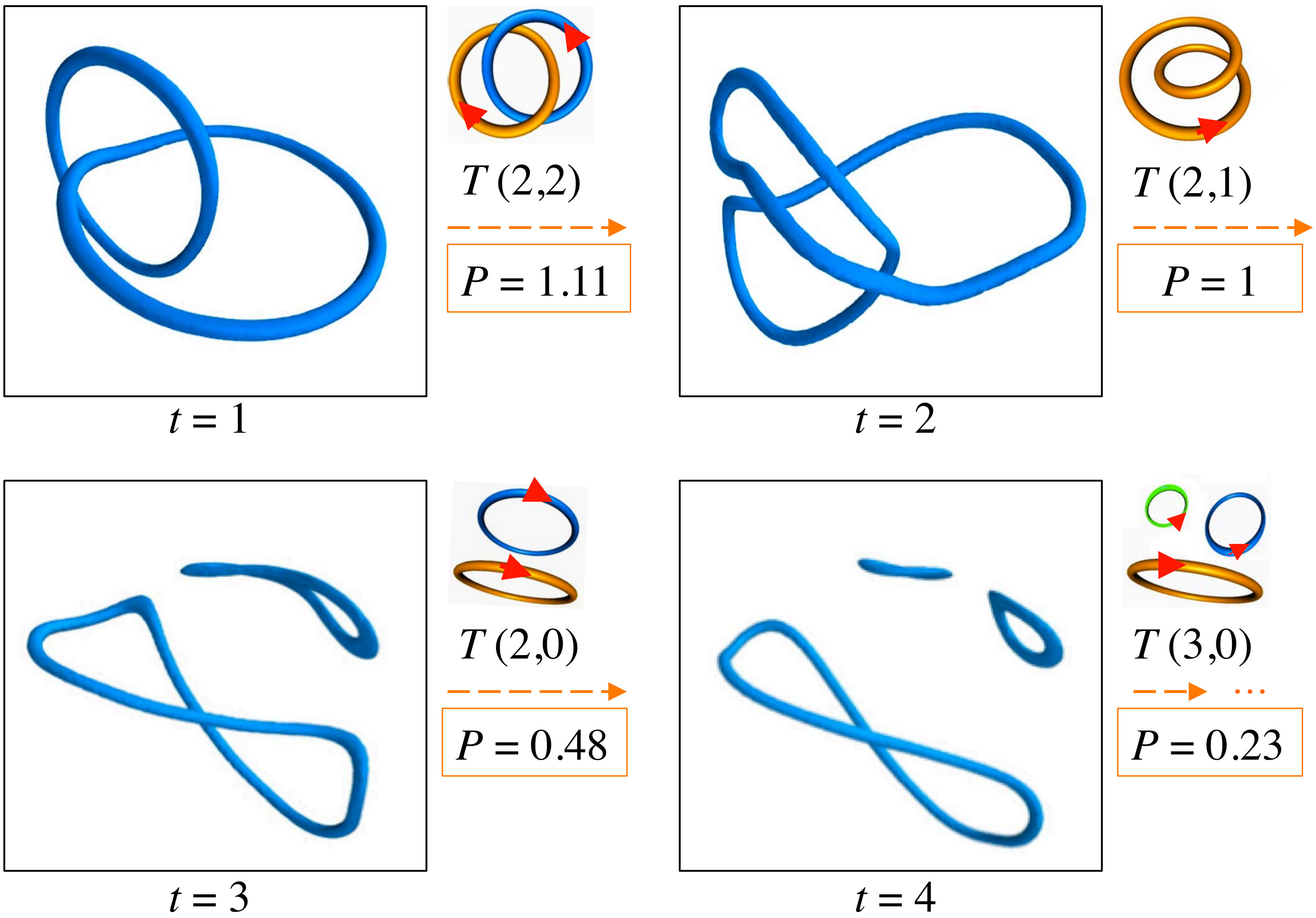
$$P\{T(2,n)\}_+ = P\{T(2,n)\}_- = P_{T(2,n)}\,.$$

■

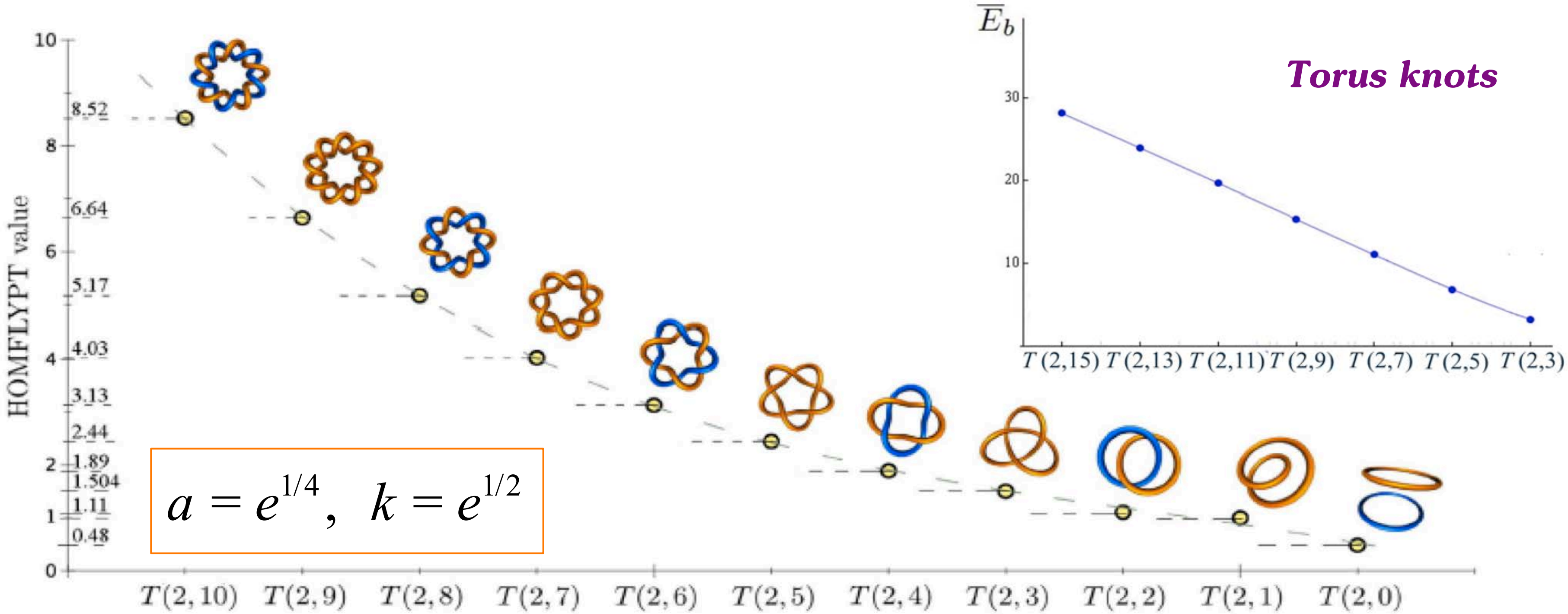
Vortex trefoil cascade process in water (Kleckner & Irvine 2013)



Vortex link cascade in BECs (Zuccher & Ricca, IUTAM 2016)



HOMFLYPT quantifies topological complexity



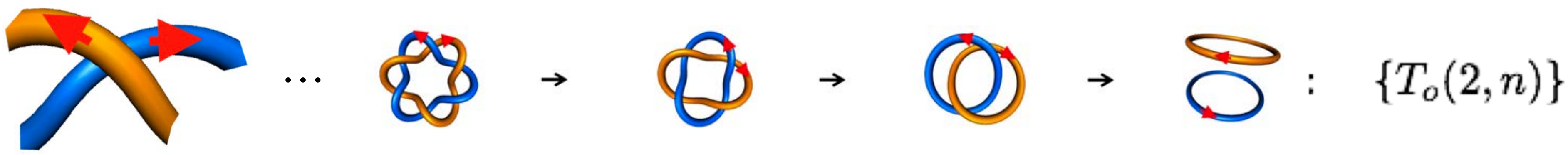
- ***Jones polynomial:*** $a = k^{-2} \quad (a = e,\ k = e^{-1/2})$;

$$V_{T(2,n)} = \frac{e^{-\frac{3}{2}n+4} + (-1)^{n-1} e^{-\frac{1}{2}n+2}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}} V_{T(2,3)} + \frac{e^{-\frac{1}{2}(3n-7)} + (-1)^{n-2} e^{-\frac{1}{2}(n-1)}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}} V_{T(2,2)} .$$

- ***Alexander-Conway polynomial:*** $a = 1 \quad (a = 1,\ k = e^{-1/2})$;

$$\Delta_{T(2,n)} = \frac{e^{-\frac{n-2}{2}} + (-1)^{n-1} e^{\frac{n-2}{2}}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}} \Delta_{T(2,3)} + \frac{e^{-\frac{n-3}{2}} + (-1)^{n-2} e^{\frac{n-3}{2}}}{e^{\frac{1}{2}} + e^{-\frac{1}{2}}} \Delta_{T(2,2)} .$$

Cascade of oppositely oriented components (negative crossings)

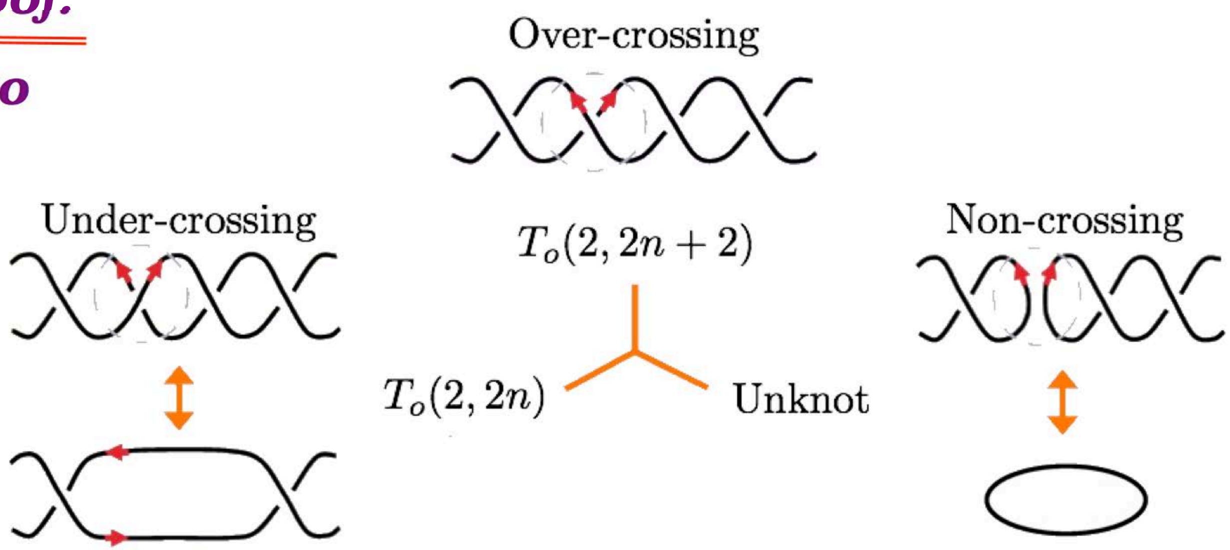


- **Lemma** (Ricca & Liu, *FDR* 2017): *Let us consider the ordered set of oppositely oriented torus links* $\{T_o(2,2n)\}$ (n *integer,* $n \geq 1$). *The HOMFLYPT polynomial* $P_{T_o(2,2n)}$ *is given by*

$$P_{T_o(2,2n)} = \frac{a^2-1}{az}\,a^{2n} + \frac{1-a^{2n}}{a^2-1}\,az\ .$$

Sketch of proof.

Apply (P.2) to



to obtain

$$aP_{T_o(2,2n)} - a^{-1}P_{T_o(2,2n+2)} = z \ ,$$

that is

$$P_{T_o(2,2n+2)} = a^2P_{T_o(2,2n)} - az \ .$$

By applying the same relation recursively, we have

$$a^2P_{T_o(2,2n)} = a^4P_{T_o(2,2(n-1))} - a^3z \ ,$$

$$a^4P_{T_o(2,2(n-1))} = a^6P_{T_o(2,2(n-2))} - a^5z \ ,$$

$$\vdots$$

$$a^{2(n-1)+2}P_{T_o(2,2)} = a^{2(n-1)+4}P_{T_o(2,0)} - a^{2(n-1)+3}z \ ,$$

and by recursive substitution, we obtain

$$\begin{aligned} P_{T_o(2,2n+2)} &= a^{2(n-1)+4}P_{T_o(2,0)} - az(1 + a^2 + a^4 + \cdots + a^{2n}) \\ &= a^{2n+2}P_{T_o(2,0)} - az\frac{1 - a^{2(n+1)}}{1 - a^2} \qquad (n \geq 1) \ . \end{aligned}$$

Since $P_{T_o(2,0)}$ is the polynomial of the disjoint union of two unlinked unknots, given by

$$P_{T_o(2,0)} = \frac{a - a^{-1}}{z} = \delta \ ,$$

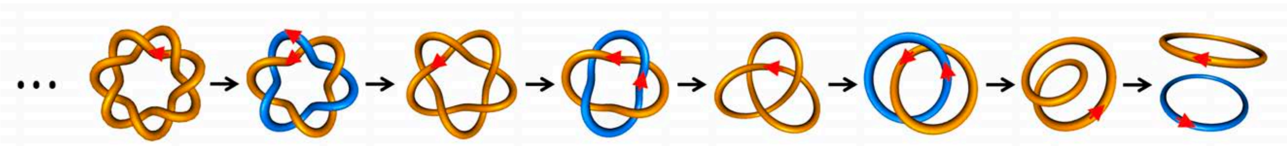
we have the statement:

$$P_{T_o(2,2n)} = \frac{a^2 - 1}{az}\ a^{2n} + \frac{1 - a^{2n}}{a^2 - 1}\ az \ .$$

■

Table of numerical values: comparative analysis

$\{T(2,n)\}$: …



Numerical values for torus knots and *co-oriented* torus links ($Wr = Tw = 1/2$)											
	$T(2,10)$	$T(2,9)$	$T(2,8)$	$T(2,7)$	$T(2,6)$	$T(2,5)$	$T(2,4)$	$T(2,3)$	$T(2,2)$	$T(2,1)$	$T(2,0)$
HOMFLYPT: $a = e^{1/4}, k = e^{1/2}$	8.52	6.64	5.17	4.03	3.13	2.44	1.89	1.50	1.11	1	0.48
Jones: $\tau = e^{-1}$	–0.01	0.02	–0.03	0.05	–0.09	0.15	–0.25	0.40	–0.69	1	–2.26
Alexander-Conway: $t = e^{-1}$	–65.81	39.92	–24.20	14.70	–8.88	5.44	–3.22	2.08	–1.04	1	-

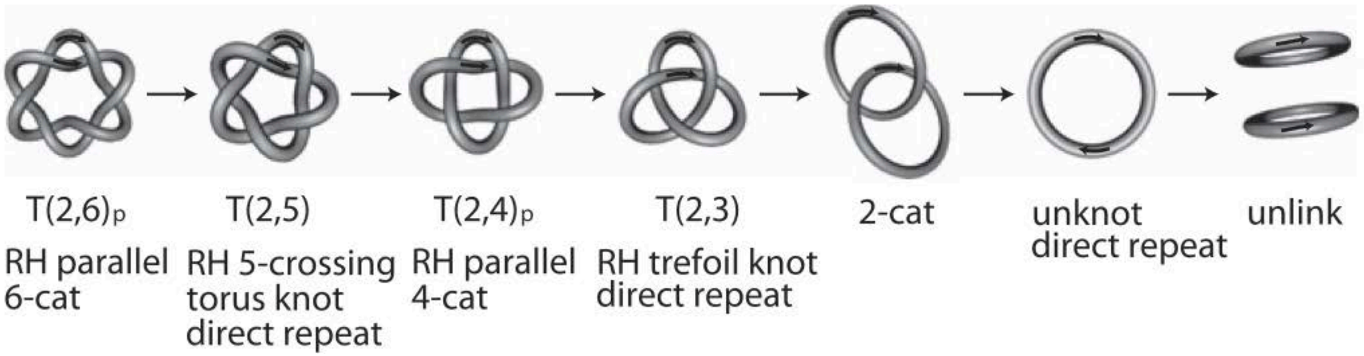
$\{T_o(2,n)\}$: …

Numerical values for *oppositely oriented* torus links ($Wr = Tw = -1/2$)											
	$T_o(2,10)$	-	$T_o(2,8)$	-	$T_o(2,6)$	-	$T_o(2,4)$	-	$T_o(2,2)$	-	$T_o(2,0)$
HOMFLYPT: $a = e^{-1/4}, k = e^{-1/2}$	1.93		1.85		1.71		1.48		1.11		0.48
Jones: $\tau = e$	–0.44		–0.45		–0.45		–0.48		–0.69		–2.25
Alexander-Conway: $t = e^{1/2}$	0		0		0		0		0		-

Conclusions and outlook

- ***Adapted HOMFLYPT is the best quantifier of cascade processes:***
  - P_K ***provides monotonic behavior consistently;***
  - ***numerical values more robust and reliable markers for diagnostics;***
  - $P_{T(2,2n)}/c_{\min} \approx 0.5$ ***,*** $(0 \leq n \leq 6)$ ***(except for the unknot).***

- ***Same cascade in recombinant DNA plasmids (Shimokawa et al., 2013):***



- ***Optimal path to cascade?***