Turbulent natural convection between two differentially heated vertical plates: A theoretical study

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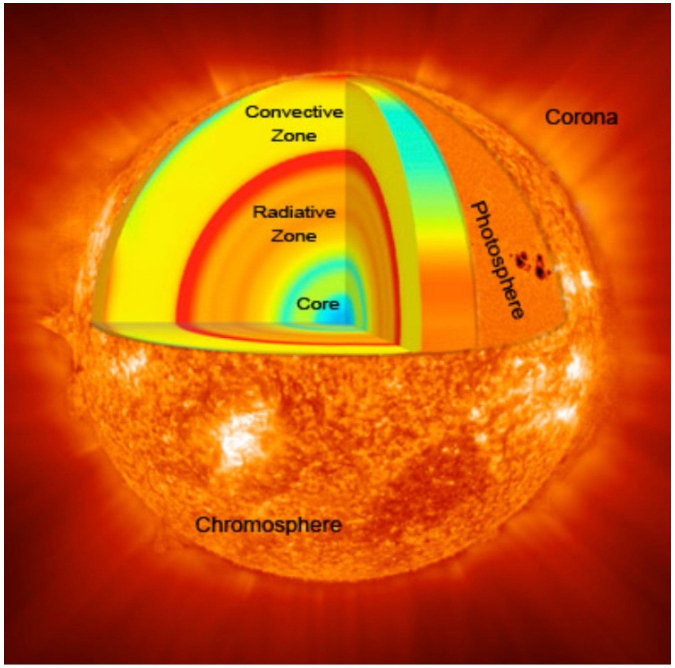
University of Rome Tor Vergata, September 18, 2023

Outline

- Introduction
- Natural Vertical Convection
- Earlier Work
- Our Theory and Results
- Summary

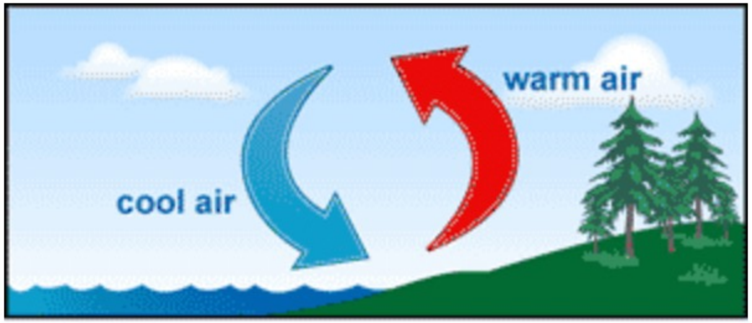
Introduction

- Thermally-driven fluid flows are ubiquitous

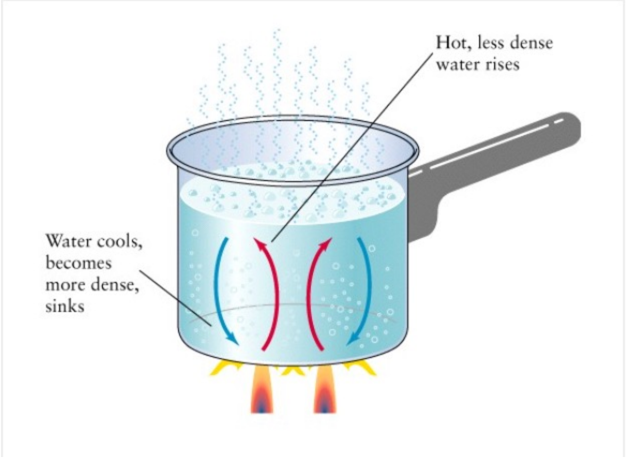


The solar interior includes the core, radiative zone and convective zone. The photosphere is the visible surface of the Sun. The solar atmosphere includes the chromosphere and corona.

SOHO (ESA & NASA)

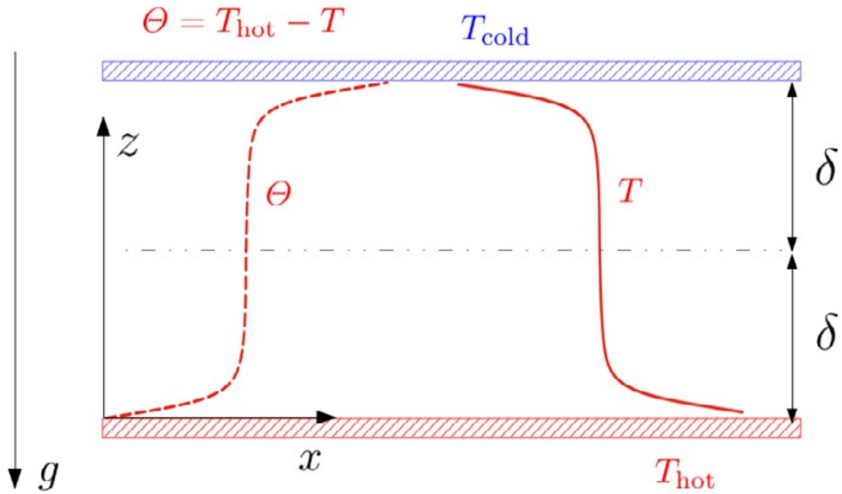


https://www.teachengineering.org/activities/view/cub_air_lesson04_activity4

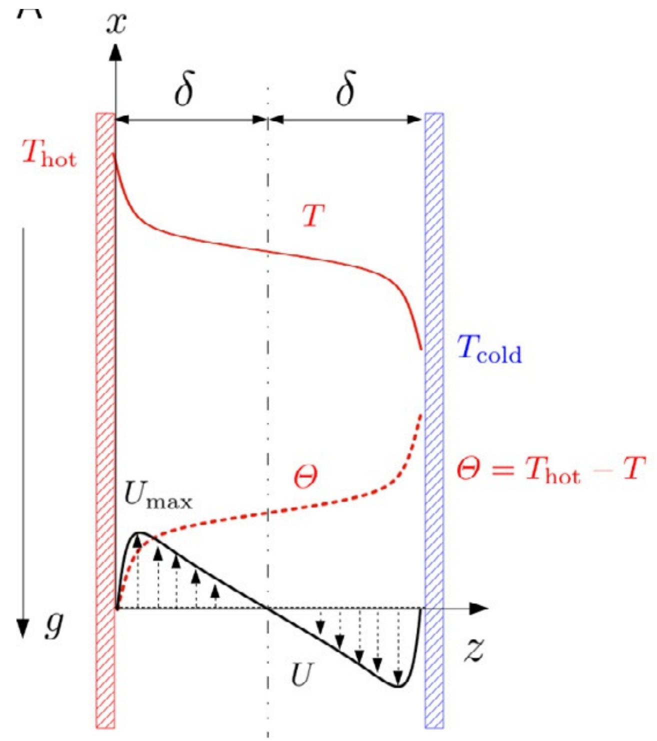


https://heatspecialists.weebly.com/heat-convection.html

- Two model systems:



Rayleigh-Bénard Convection



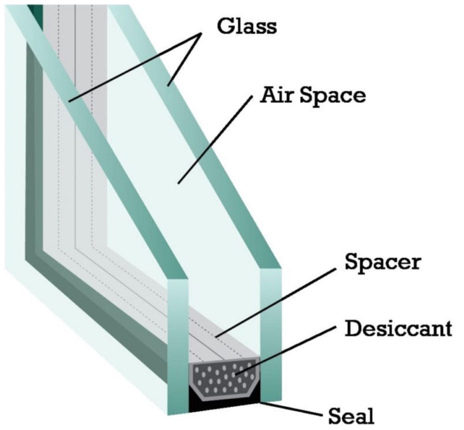
Vertical Convection

Tie Wei **Analyses of buoyancy-driven convection**

Advances in Heat Transfer, Volume 52
ISSN 0065-2717
https://doi.org/10.1016/bs.aiht.2020.09.002

Natural convection between two vertical plates

- many applications in engineering



https://glassdoctor.com/expert-tips/all-about-window-glass/double-pane-windows

- ice-ocean interactions

Antarctica - credit: Pixabay

Fluid motion driven by both temperature difference and salinity difference between melt water and salty seawater

How does heat transfer
depend on
the control parameters of the flow?

- Control parameters:

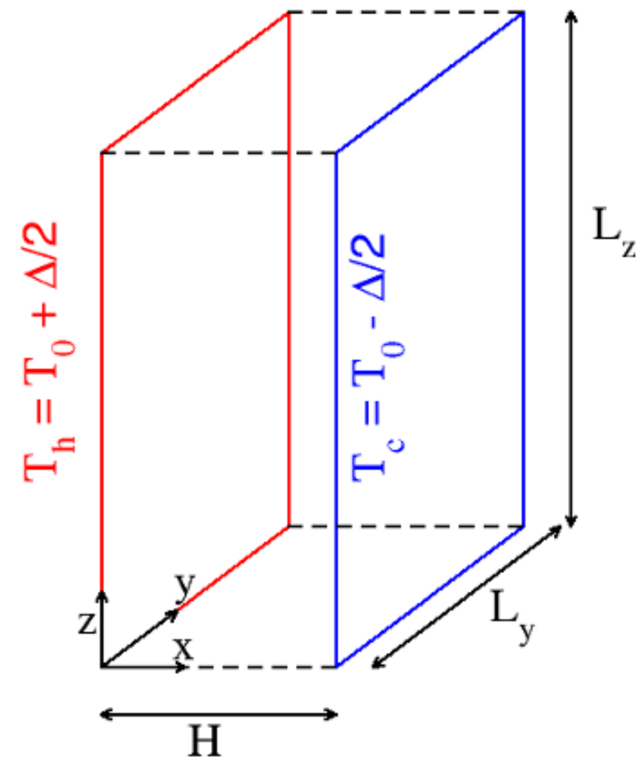
  Rayleigh (Ra) and Prandtl numbers (Pr)

$$Ra = \frac{g\alpha H^3 \Delta}{\nu\kappa} \qquad Pr = \frac{\nu}{\kappa}$$

aspect ratios L_y/H and L_z/H

α, ν, and κ are the thermal expansion coefficient, kinematic viscosity, and thermal diffusivity of the fluid, respectively

Δ = temperature difference



- Heat transfer is measured by the dimensionless Nusselt number (Nu):

$$\mathrm{Nu} \equiv \frac{Q}{k\Delta/H}$$

which is defined as the actual heat flux Q normalized by that when there were only heat conduction

k is the thermal conductivity of the fluid

- With the Oberbeck-Bousinessq approximation, the governing equations of motion are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u} + \alpha g (T - T_0)\hat{z} \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T \quad (2)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (3)$$

Earlier Work

(I) Laminar Vertical Convection

- Pioneer work by Batchelor 1954

QUARTERLY OF APPLIED MATHEMATICS

Vol. XII | October, 1954 | No. 3

HEAT TRANSFER BY FREE CONVECTION ACROSS A CLOSED CAVITY BETWEEN VERTICAL BOUNDARIES AT DIFFERENT TEMPERATURES*

BY

G. K. BATCHELOR

Trinity College, Cambridge, England

- For very small Ra, $\mathrm{Nu} \approx 1 + a\mathrm{Ra}^2$

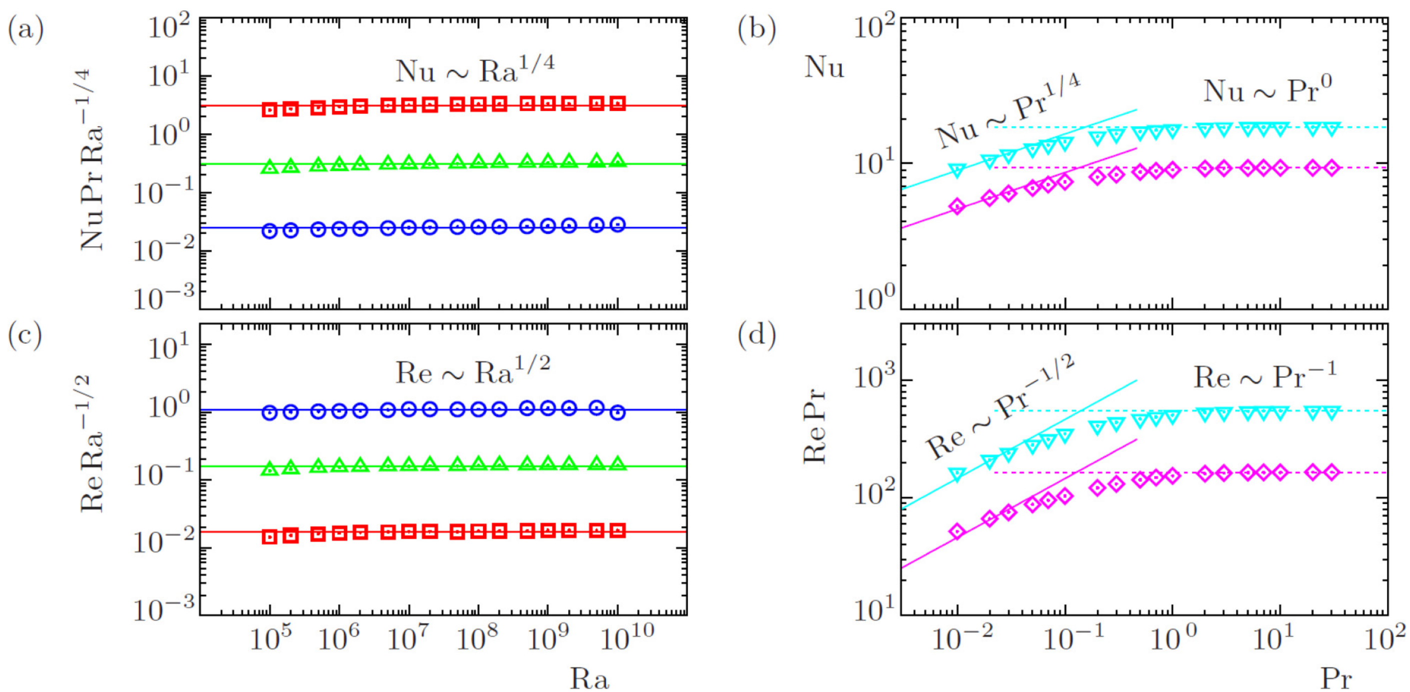
Nu(Ra, Pr) has been determined by solving steady-state boundary layer equations

$Nu \sim Ra^{1/4} Pr^{1/4}$ for Pr <<1

$Nu \sim Ra^{1/4} Pr^{0}$ for Pr >>1

S. Ostrach, NACA Rep. **1111**, 63 (1953)
H.K. Kuiken, J. Engg. Math. **2**, 355 (1968)
O. Shishkina, Phys. Rev. E **93**, 051102(R) (2016)

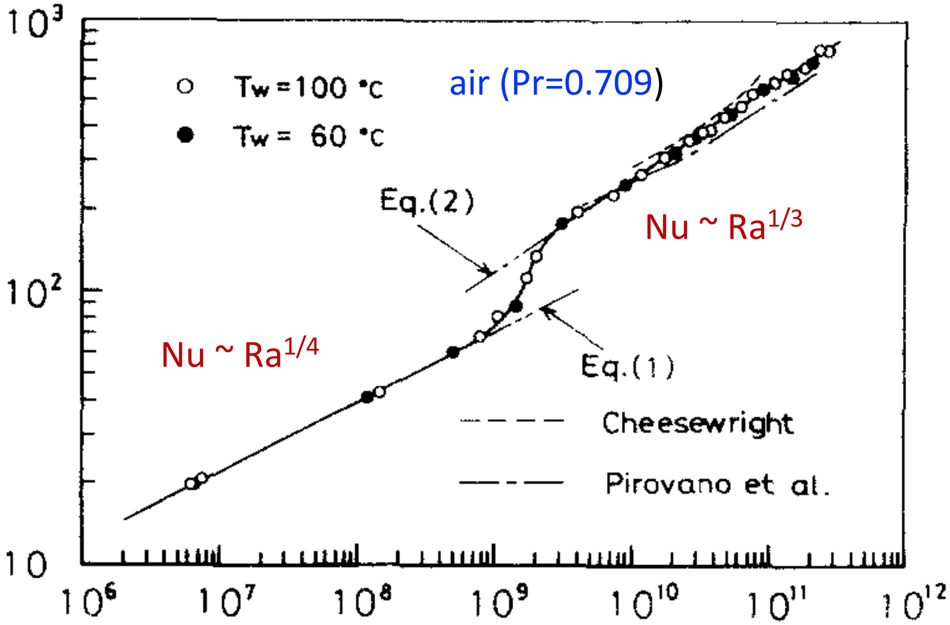
Olga Shishkina PHYSICAL REVIEW E **93**, 051102(R) (2016)



$Re = V_{max} H/\nu$

Earlier Work
(II) Turbulent Vertical Convection

Experimental studies:



T. Tsuji and Y. Nagano, Int. J. Heat Mass Transfer 31, 1723 (1988)

Similar results were reported for water and ethanol

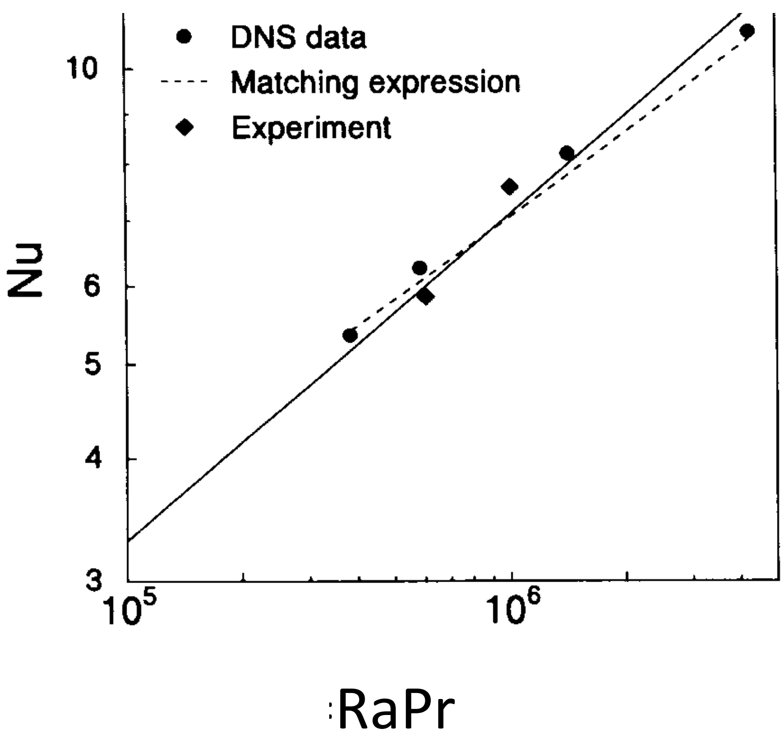
R.K. MacGregor and A.F. Emery, Trans. ASME, J. Heat Transfer 93, 253 (1969)

Direct numerical simulations (DNS):

- 3D with periodic boundary conditions in x- and z-directions for Pr=0.709

T.A.M. Versteegh, F.T.M. Nieuwstadt

International Journal of Heat and Mass Transfer 42 (1999) 3673–3693



$$Nu^{3/4}\left[c_3(Nu\,Ra\,Pr)^{-1/12}-c_2\right]=\frac{1}{2}(Ra\,Pr)^{1/4}$$

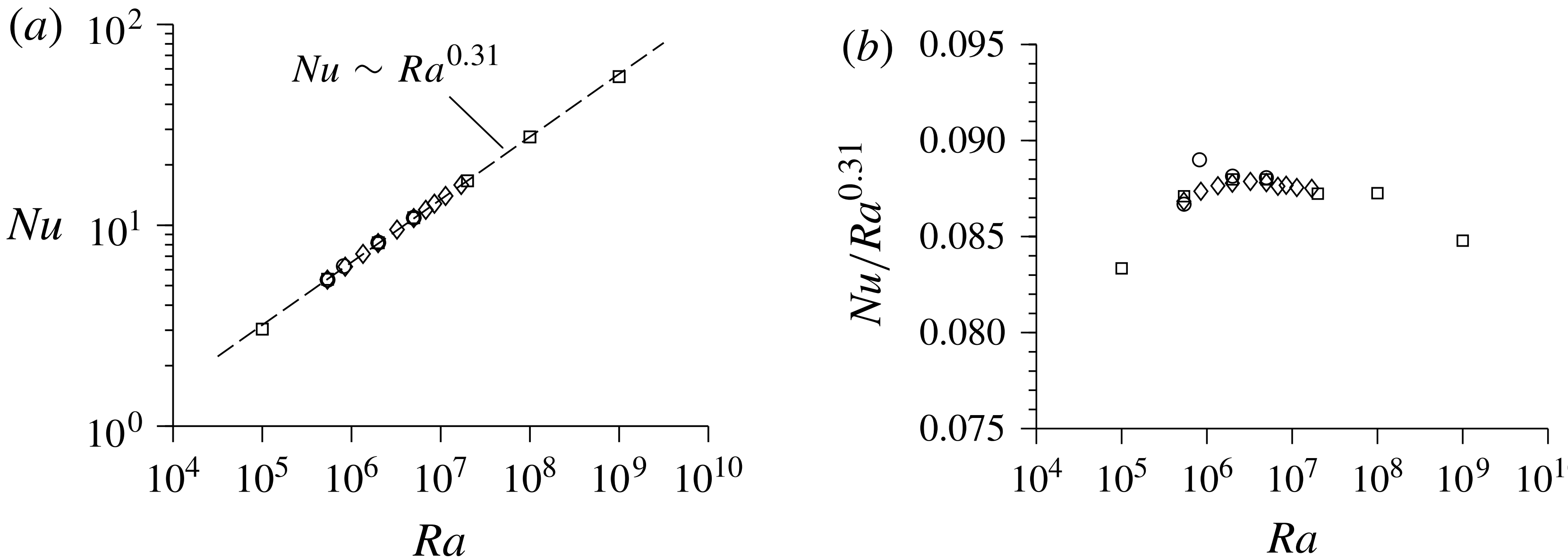
P. Kiš, H. Herwig

International Journal of Heat and Mass Transfer 55 (2012) 2625–2635

$Nu \sim Ra^{1/c}$, c=3.2

Chong Shen Ng[1,†], Andrew Ooi[1], Detlef Lohse[2] and Daniel Chung[1]

J. Fluid Mech. (2015), *vol.* 764, *pp.* 349–361



The dependence of Nu on Ra might not be best represented by a pure power law

Howland et al. *J. Fluid Mech.* (2022), *vol*. 930, A32, doi:10.1017/jfm.2021.952

Prandtl number Pr	Rayleigh numbers Ra
1	10^6–10^8
2	10^6–10^8
5	10^6–10^8
10	10^6–10^9
100	10^7–10^9

Response parameters	Two parameter regression
Nusselt number Nu	$Ra^{0.321\pm0.006}Pr^{-0.083\pm0.010}$
Reynolds number Re	$Ra^{0.489\pm0.007}Pr^{-0.738\pm0.010}$
Shear Reynolds number Re_τ	$Ra^{0.362\pm0.002}Pr^{-0.446\pm0.003}$

Re = $W_{\max}H/\nu$

$$\mathrm{Re}_\tau \equiv u_\tau H/\nu$$

$$u_\tau \equiv \sqrt{\nu dW/dx|_{x=0}}$$

$\tau_w = \rho\nu dW/dx|_{x=0}$ is the mean wall shear stress

- 2D DNS with adiabatic boundary condition for temperature in the horizontal direction

SHIHE XIN AND PATRICK LE QUÉRÉ

J. Fluid Mech. (1995), *vol.* 304, *pp.* 87–118

F. X. TRIAS, M. SORIA, A. OLIVA AND C. D. PÉREZ-SEGARRA

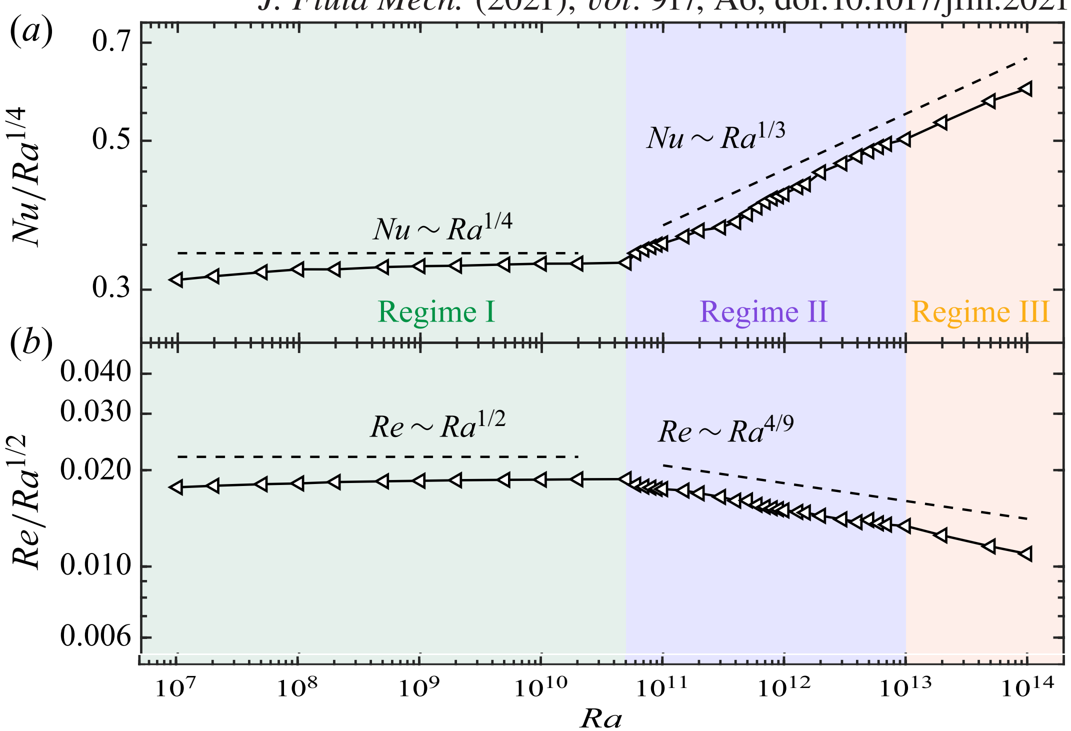
J. Fluid Mech. (2007), *vol.* 586, *pp.* 259–293.

$Nu \sim Ra^{\beta}$

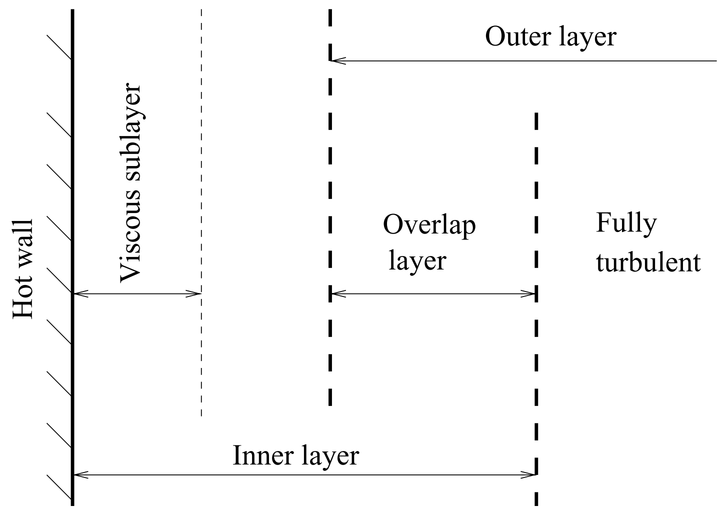
with β closer to 1/4 than 1/3 for Ra up to 10^{10}

Qi Wang[1,2], Hao-Ran Liu[1], Roberto Verzicco[1,3,4], Olga Shishkina[5,†] and Detlef Lohse[1,5,†]

J. Fluid Mech. (2021), *vol.* 917, A6, doi:10.1017/jfm.2021.262



Scaling theory:



(GC1979)

W.K. George and S.P. Capp

Int. J. Heat Transfer 22, 813-826 (1979)

M. HÖLLING AND H. HERWIG

J. Fluid Mech. (2005), *vol.* 541, *pp.* 383–397

(HH2005)

$$\frac{\Theta_w - \Theta(x)}{\Theta_I} = F_I(x/\eta, \mathrm{Pr}), \quad \text{inner layer} \quad \eta = \frac{\kappa}{(\kappa \alpha g q_w)^{1/4}}$$

$$\frac{\Theta(x) - \Theta_\infty}{\Theta_O} = F_O(x/H), \quad \text{outer layer} \quad q_w \equiv \frac{Q}{\rho c} = \mathrm{Nu}\frac{\kappa \Delta}{H}$$

- Assume there is an overlap layer

$$-\frac{\Theta_I F_I'(\hat{x},\mathrm{Pr})}{\eta} = \frac{\Theta_O F_O'(\tilde{x})}{H}$$

$\hat{x} \equiv x/\eta$ or $\tilde{x} \equiv x/H$

GC1979: $\Theta_I = \frac{q_w}{(\kappa\alpha g q_w)^{1/4}}, \qquad \Theta_O = \frac{q_w}{(\alpha g H q_w)^{1/3}}$

HH2005: $\Theta_I = \Theta_O = \frac{q_w}{(\kappa\alpha g q_w)^{1/4}}$

$$\frac{\Theta_I}{\Theta_O} = \left(\frac{\eta}{H}\right)^{\gamma}$$

$$\gamma = \begin{cases} -1/3 & \text{(GC1979)} \\ 0 & \text{(HH2005)} \end{cases}$$

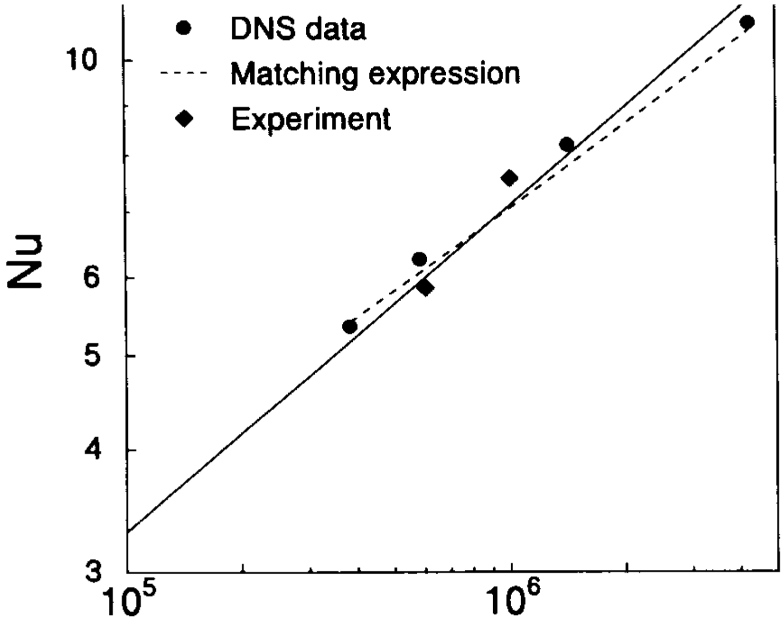
- In Ra → ∞ limit,

$$F_I'(\hat{x},\mathrm{Pr})\hat{x}^{1-\gamma} = -F_O'(\tilde{x})\tilde{x}^{1-\gamma} = K, \qquad K \neq 0$$

$$\mathrm{Nu}^{-2/3}(\mathrm{RaPr})^{1/3} = 2A(\mathrm{Pr})(\mathrm{NuRaPr})^{1/12} + 2B \qquad \text{(GC1979)}$$

$$\mathrm{Nu}^{-3/4}(\mathrm{RaPr})^{1/4} = \frac{K}{2}\ln(\mathrm{NuRaPr}) + 2A(\mathrm{Pr}) + 2B \qquad \text{(HH2005)}$$

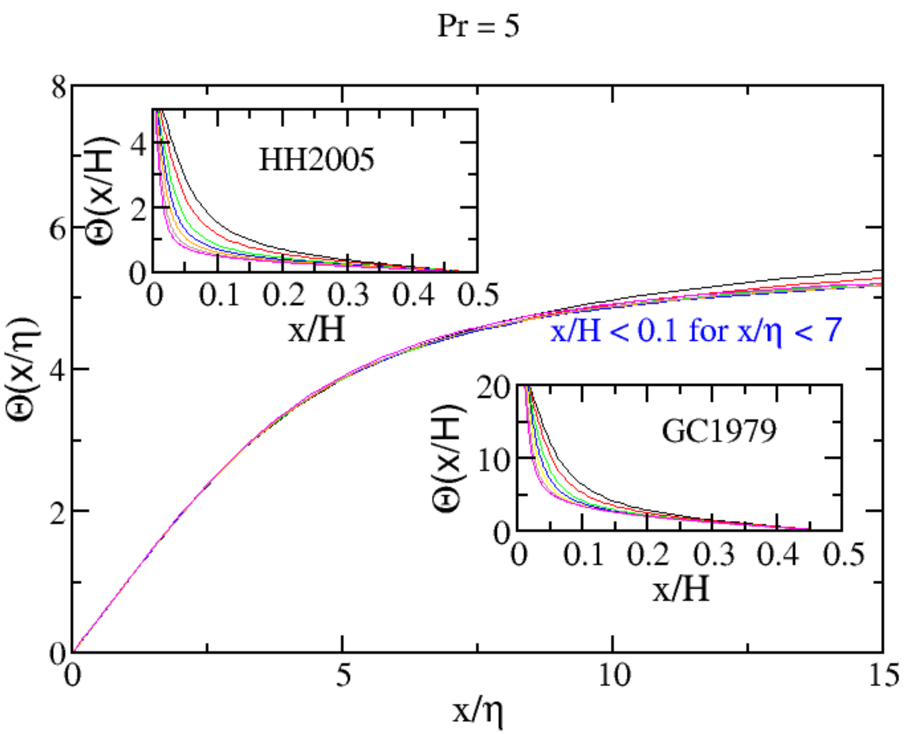
$K \neq 0$



RaPr

$$Nu^{3/4}\left[c_3(Nu\,Ra\,Pr)^{-1/12} - c_2\right] = \frac{1}{2}(Ra\,Pr)^{1/4}$$

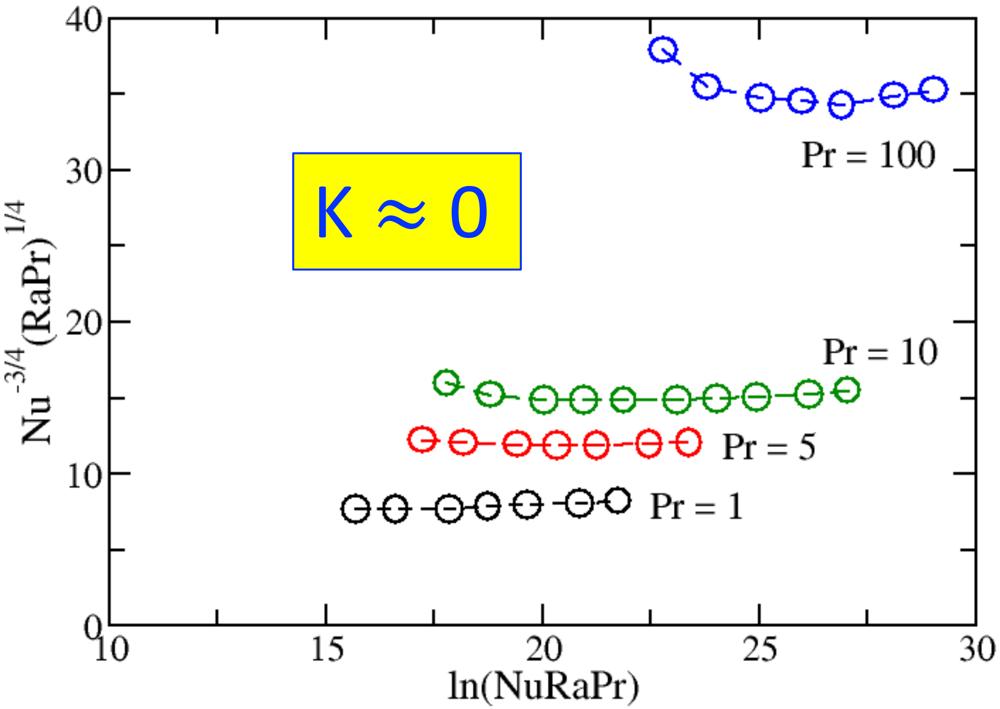
- DNS data (Howland et al., JFM 2022)



overlap layer does not exist for Pr ≥ 5

- DNS data (Howland et al., JFM 2022)

$$\mathrm{Nu}^{-3/4}(\mathrm{RaPr})^{1/4} = \frac{K}{2}\ln(\mathrm{NuRaPr}) + 2A(\mathrm{Pr}) + 2B \quad \text{(HH2005)}$$



Scaling theory is not supported by DNS data

Another theoretical attempt:

- Extend theory of Grossmann and Lohse for Rayleigh-Bénard convection to vertical convection

Chong Shen Ng[1,†], Andrew Ooi[1], Detlef Lohse[2] and Daniel Chung[1]

J. Fluid Mech. (2015), *vol.* 764, *pp.* 349–361

- However, unlike Rayleigh-Bénard convection, the exact relations involving Nu and Re are not closed

$$\langle \varepsilon_u \rangle = \frac{\nu^3}{H^4} \frac{\langle -u_g T \rangle}{\kappa \Delta T / H} \frac{Ra}{Pr^2}, \qquad \langle \varepsilon_T \rangle = \kappa \frac{\Delta T^2}{H^2} Nu$$

Our Theory

Emily S.C. Ching, Phys. Rev. Fluids 8, L022601 (2023)

- Focus on the large aspect-ratio limit
- Using Reynolds decomposition

$$w(x,y,z,t) = W(x) + w'(x,y,z,t) \qquad T(x,y,z,t) - T_0 = \Theta(x) + \theta'(x,y,z,t)$$

and taking time average of the equations of motion

$$\frac{d}{dx}\langle u'w'\rangle_t = \nu\frac{d^2}{dx^2}W + \alpha g\Theta$$ mean momentum balance equation

$$\frac{d}{dx}\langle u'\theta'\rangle_t = \kappa\frac{d^2}{dx^2}\Theta$$ mean thermal energy balance equation

- W(0)=W(H/2)=0, Θ(0)= Δ/2 and Θ(H/2)=0
- Q = - k d Θ/dx $|_{x=0}$
- Mean flow equations are not closed

- Integrating the mean momentum equation gives

$$\langle u'w'\rangle_t = \nu \frac{dW}{dx} - u_\tau^2 + \alpha g \int_0^x \Theta(x')dx'$$

- Evaluating the result at $x = x_0$, where

$$\nu dW/dx|_{x=x_0} = \langle u'w'\rangle_t(x_0)$$

$$\Theta(x/\delta_T) = \Delta\Phi(\xi)/2$$

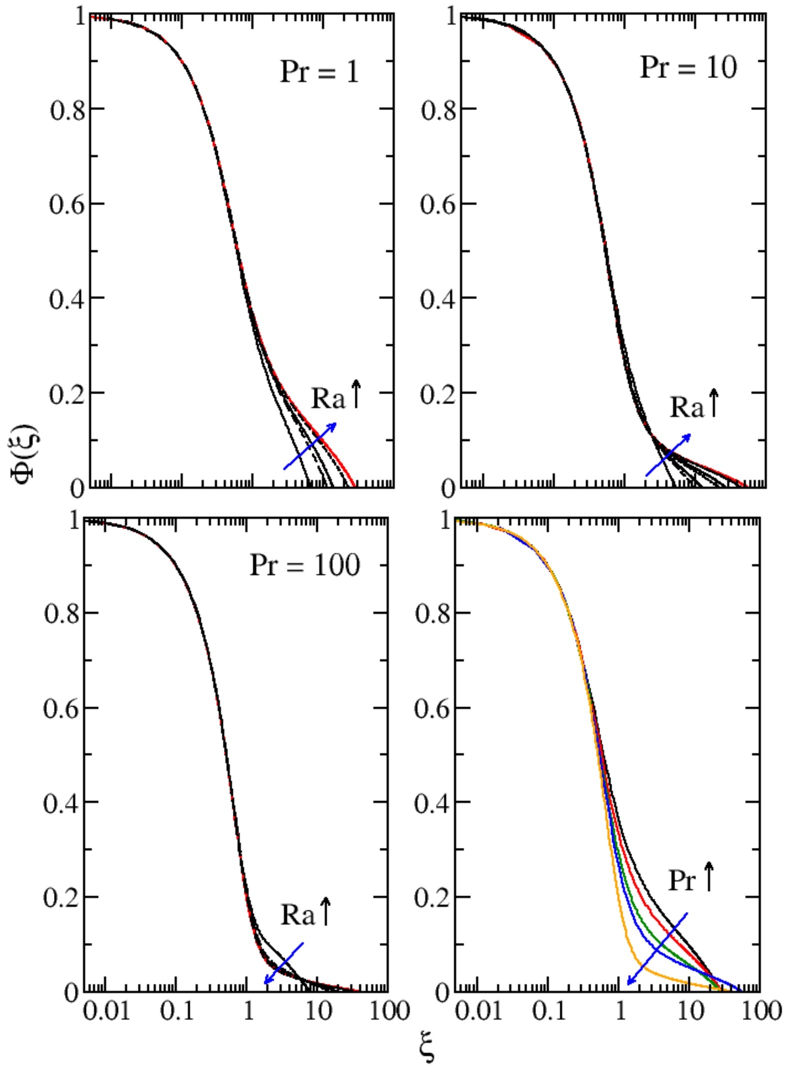
$$u_\tau^2 = \alpha g \int_0^{x_0} \Theta(x)dx$$

$\xi_0 = x_0/\delta_T$

δ_T is the thermal boundary layer thickness defined by Nu $\equiv H/2\delta_T$

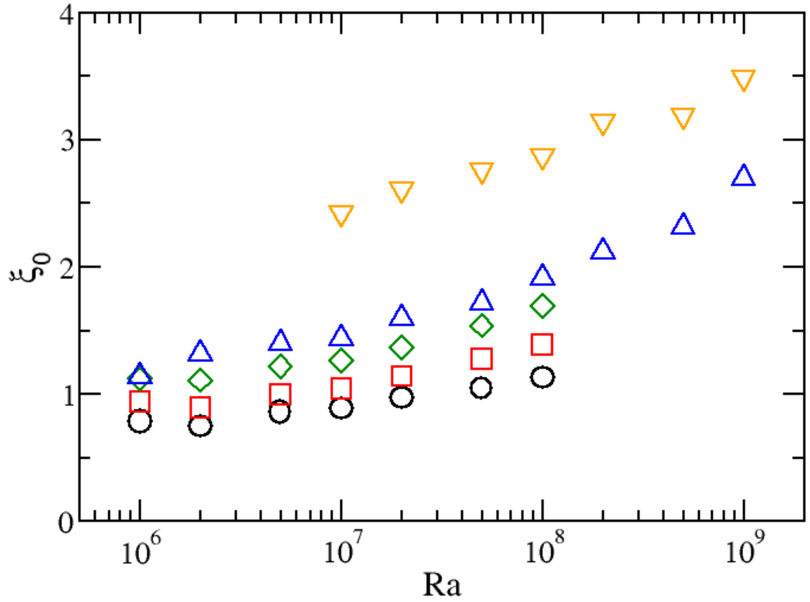
$$\mathrm{Re}_\tau^2\mathrm{NuPrRa}^{-1} = \frac{1}{4}\int_0^{\xi_0} \Phi(\xi)d\xi \equiv I(\mathrm{Ra},\mathrm{Pr})$$

Evaluate $\Phi(\xi)$ and ξ_0 using DNS data by Howland et al., JFM 930, A32 (2022)



$\Phi(\xi)$ approaches a Pr-dependent asymptotic form as Ra increases

- ξ_0 has a weak dependence on Ra for each Pr



Pr = 1 (circles), 2 (squares), 5 (diamonds), 10 (triangles) and 100 (inverted triangles)

- These results lead us to make the assumption

$$I(\mathrm{Ra}, \mathrm{Pr}) \rightarrow f(\mathrm{Pr}) \quad \text{in the high-Ra limit}$$

- This yields the first relationship

$$\mathrm{Re}_\tau^2 \mathrm{Nu} \mathrm{Pr} \mathrm{Ra}^{-1} = f(\mathrm{Pr}) \qquad \text{high-Ra limit}$$

- Integrating the mean thermal energy balance equation

$$\frac{d}{dx}\langle u'\theta'\rangle_t = \kappa \frac{d^2}{dx^2}\Theta$$

gives

$$\frac{\langle u'\theta'\rangle_t}{\nu\Delta/H} = \mathrm{NuPr}^{-1}[1+\Phi'(\xi)]$$

- Evaluating the third-order derivative of both sides:

$$\frac{3H^4}{8\nu\Delta}\mathrm{Nu}^{-3}\left\langle \frac{\partial^2 u}{\partial x^2}\frac{\partial T}{\partial x}\right\rangle_t\bigg|_{x=0} = \mathrm{NuPr}^{-1}\Phi^{(4)}(0)$$

- Make the closure approximation

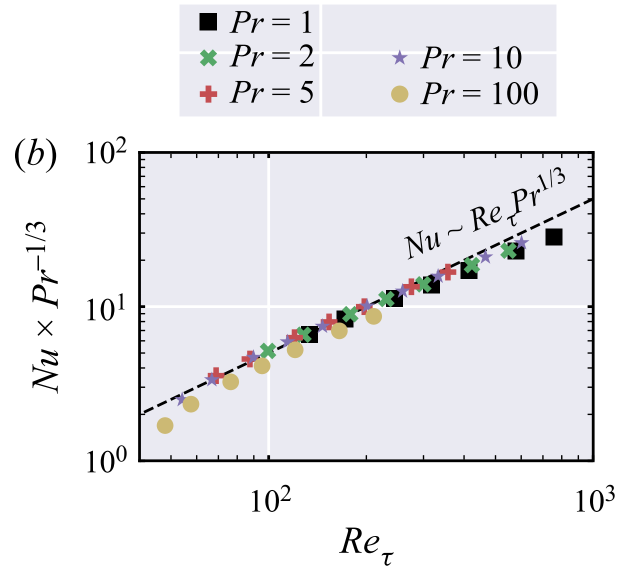
$$\left\langle \frac{\partial^2 u'}{\partial x^2}\frac{\partial \theta'}{\partial x}\right\rangle_t\bigg|_{x=0} = F(u_\tau, -d\Theta/dx|_{x=0}, l_c) \approx c_0 \frac{u_\tau}{l_c^2}\frac{\mathrm{Nu}\Delta}{H}$$

($l_c = \nu/u_\tau$ for $\mathrm{Pr} \gg 1$ and $l_c = \kappa/u_\tau$ for $\mathrm{Pr} \ll 1$)

to obtain the second relationship

$$\mathrm{Nu} \approx C\mathrm{Pr}^{\varepsilon}\mathrm{Re}_\tau, \qquad \varepsilon = \begin{cases} 1/3 & \mathrm{Pr} \gg 1 \\ 1 & \mathrm{Pr} \ll 1 \end{cases}$$

$$C = \{3c_0/[8\Phi^{(4)}(0)]\}^{1/3}$$



Howland et al., JFM 930, A32 (2022)

Heat flux and wall shear stress

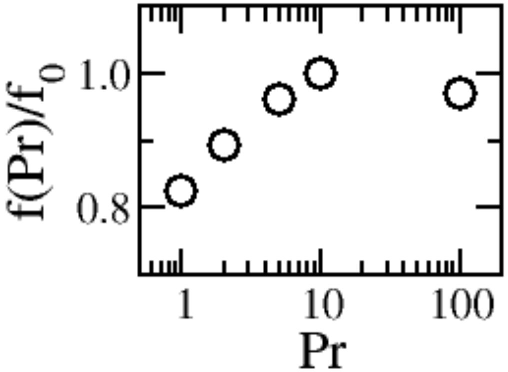
$$\mathrm{Nu} \approx [C^2 f(\mathrm{Pr})]^{1/3} \mathrm{Pr}^{-(1-2\varepsilon)/3} \mathrm{Ra}^{1/3}$$

high-Ra limit

$$\mathrm{Re}_\tau \approx [f(\mathrm{Pr})/C]^{1/3} \mathrm{Pr}^{-(1+\varepsilon)/3} \mathrm{Ra}^{1/3}$$

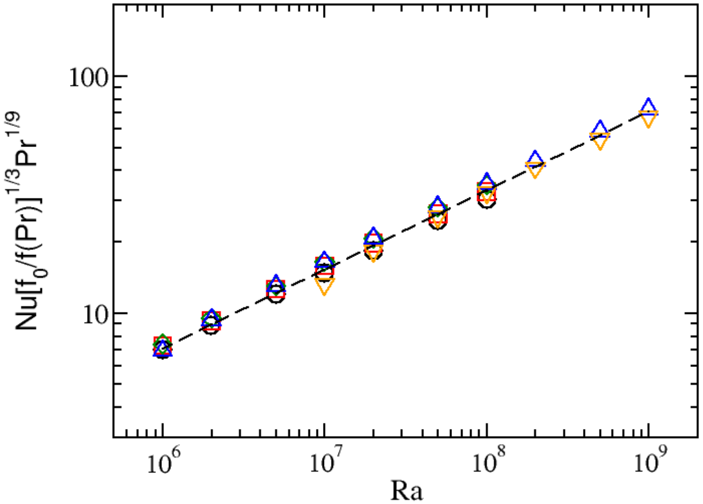
Using $\mathrm{Re}_\tau^2 \mathrm{Nu} \mathrm{Pr} \mathrm{Ra}^{-1} = f(\mathrm{Pr})$ high-Ra limit

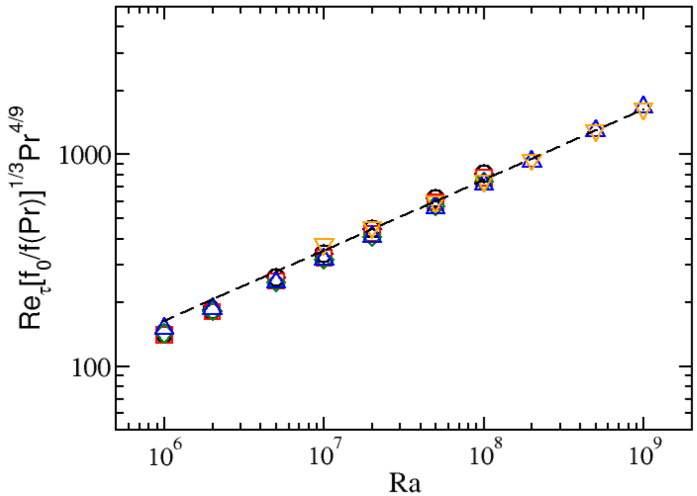
evaluate $f(\mathrm{Pr})/f_0$, where $f_0 \equiv f(\mathrm{Pr}=10)$



Excellent agreement with DNS data for Pr≥1

Pr = 1 (circles), 2 (squares), 5 (diamonds), 10 (triangles) and 100 (inverted triangles)





dashed lines:
best fits of $y = ax^{1/3}$ for $Ra \geq 5x10^7$

fitted values of a give $f_0=0.19$ and $C=0.043$

Howland et al., JFM 930, A32 (2022)

Response parameters	Two parameter regression
Nusselt number *Nu*	$Ra^{0.321\pm0.006}Pr^{-0.083\pm0.010}$
Reynolds number *Re*	$Ra^{0.489\pm0.007}Pr^{-0.738\pm0.010}$
Shear Reynolds number Re_τ	$Ra^{0.362\pm0.002}Pr^{-0.446\pm0.003}$

Summary

- We have carried out a theoretical study of large aspect-ratio turbulent vertical convection
- Our analysis is based on the mean flow equations and yields two relationships between heat flux (Nu) and wall shear stress (Re_τ).
- These two relationships give the dependence of Nu and Re_τ on Ra and Pr in the high-Ra limit and our theoretical results are in excellent agreement with the direct numerical simulation data for Pr ≥ 1.
- Our work in progress shows that the there are two contributions to the Reynolds number (Re) measuring the maximum mean vertical velocity and they have different dependence on Ra.

Thank you for your attention!