On the statistics of backscatter from sub-grid fluctuations at high Reynolds numbers

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$A - priori\ analysis\ of\ energy\ backscatter$

Large Eddy Simulations

$$\partial_t \boldsymbol{v} + \nabla(\boldsymbol{v}\boldsymbol{v}) = -\nabla p + \nu \Delta \boldsymbol{v} \qquad \text{(Navier Stokes eq.)}$$

Filtered velocity field;

$$\overline{\boldsymbol{v}}(\boldsymbol{x}, t) \equiv \int_\Omega d\boldsymbol{y}\ G(|\boldsymbol{x} - \boldsymbol{y}|)\ \boldsymbol{v}(\boldsymbol{y}, t) = \sum_{\boldsymbol{k} \in \mathbb{Z}^3} G(\boldsymbol{k})\ \hat{\boldsymbol{v}}(\boldsymbol{k}, t) e^{i\boldsymbol{k}\boldsymbol{x}}$$

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$$\partial_t \overline{\boldsymbol{v}} = ?$$

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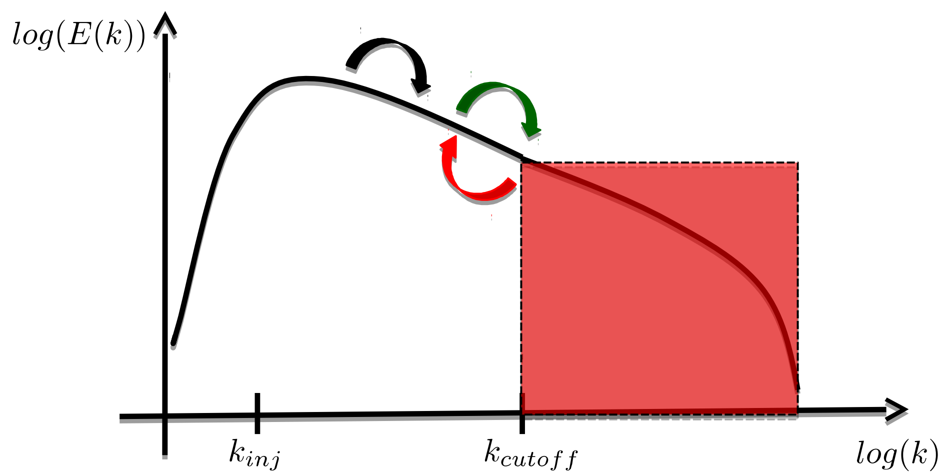

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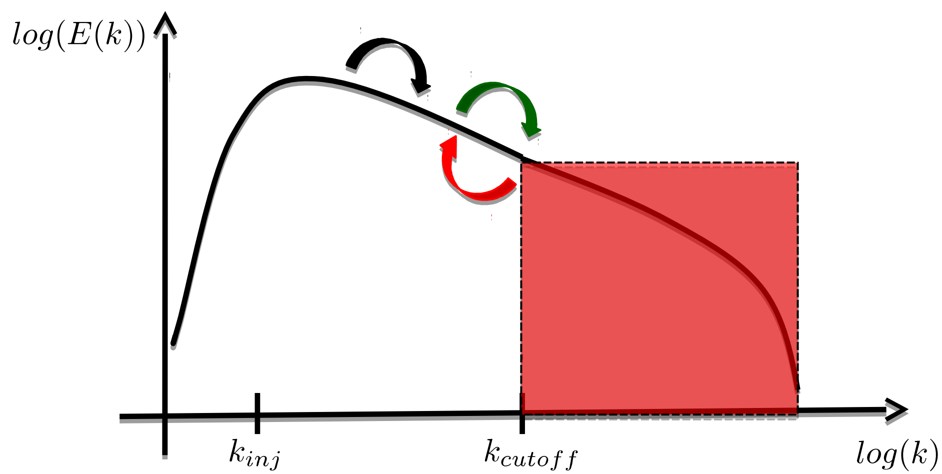


$$G = ?$$

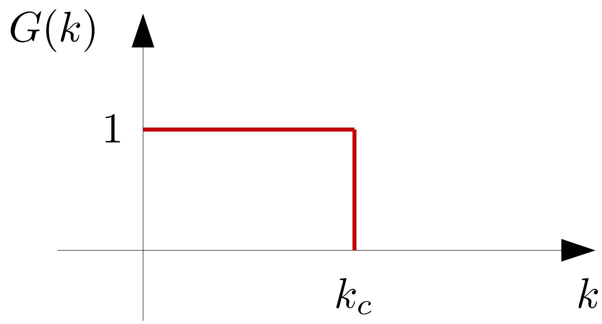
Projectors filters;

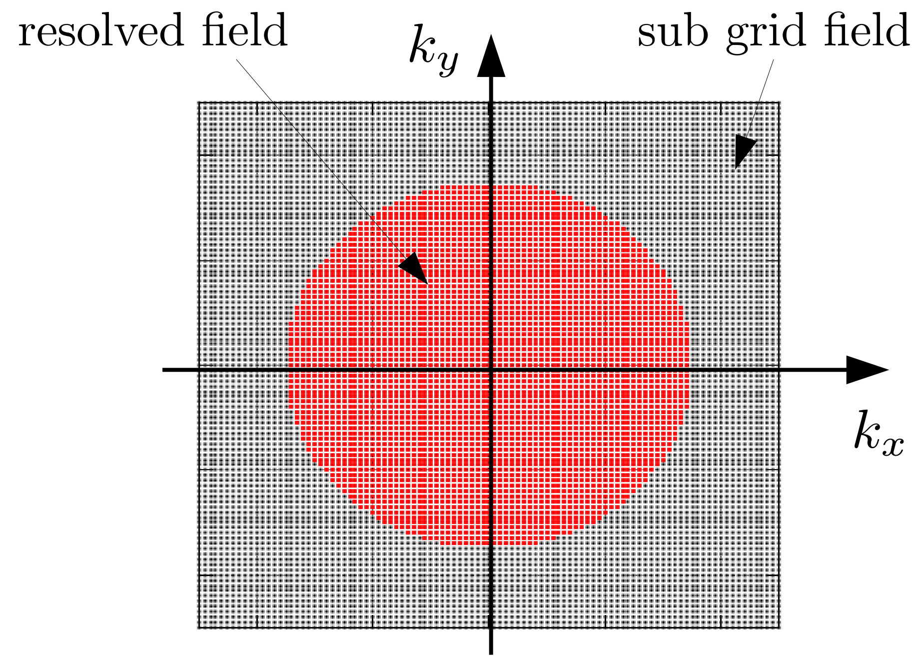
Smooth filters;

$$\overline{\boldsymbol{v}}(\boldsymbol{x},t) = \sum_{\boldsymbol{k}\in\mathbb{Z}^3} G(\boldsymbol{k})\ \hat{\boldsymbol{v}}(\boldsymbol{k},t)e^{i\boldsymbol{k}\boldsymbol{x}}$$

-) Sharp Projector:

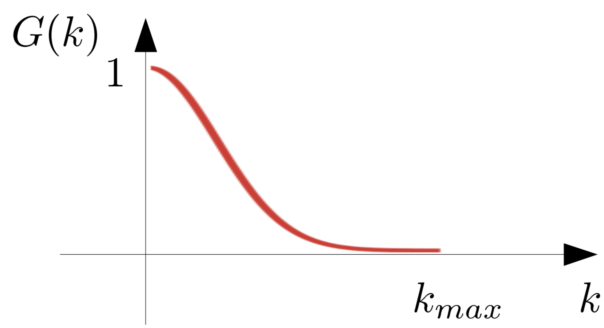
$$G_\Delta(|\boldsymbol{k}|) = \begin{cases} 1, & \text{if } |\boldsymbol{k}| < k_c \\ 0, & \text{if } |\boldsymbol{k}| \geq k_c \end{cases} \qquad \begin{cases} G^2 & = G \\ \overline{\overline{\boldsymbol{v}}} & = \overline{\boldsymbol{v}} \end{cases}$$

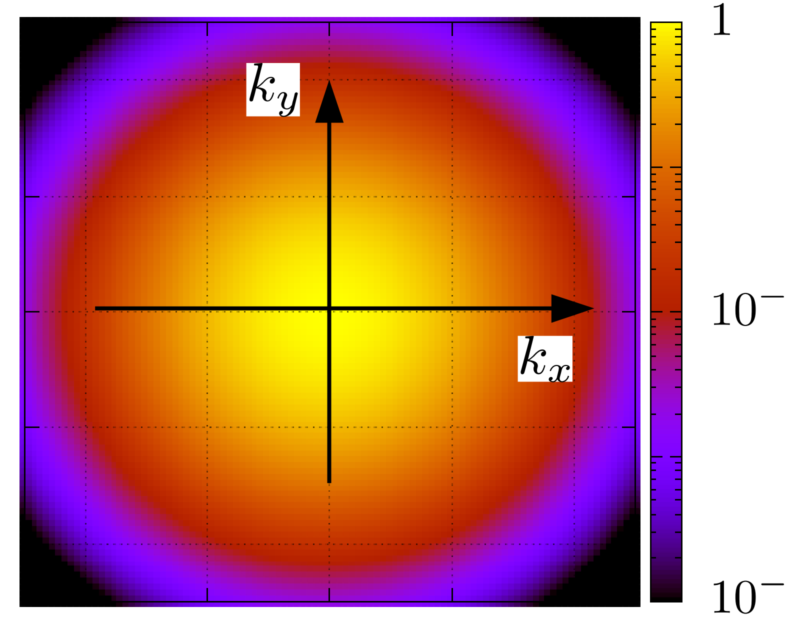




-) Gaussian smoothing:

$$G_\Delta(|\boldsymbol{k}|) = \exp\left(-\frac{|\boldsymbol{k}|^2\Delta^2}{24}\right) \qquad \begin{cases} G^2 & \neq G \\ \overline{\overline{\boldsymbol{v}}} & \neq \overline{\boldsymbol{v}} \end{cases}$$

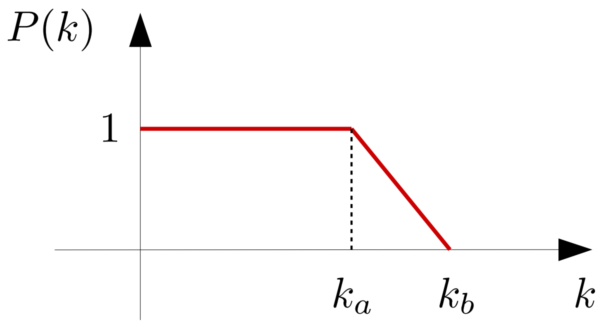


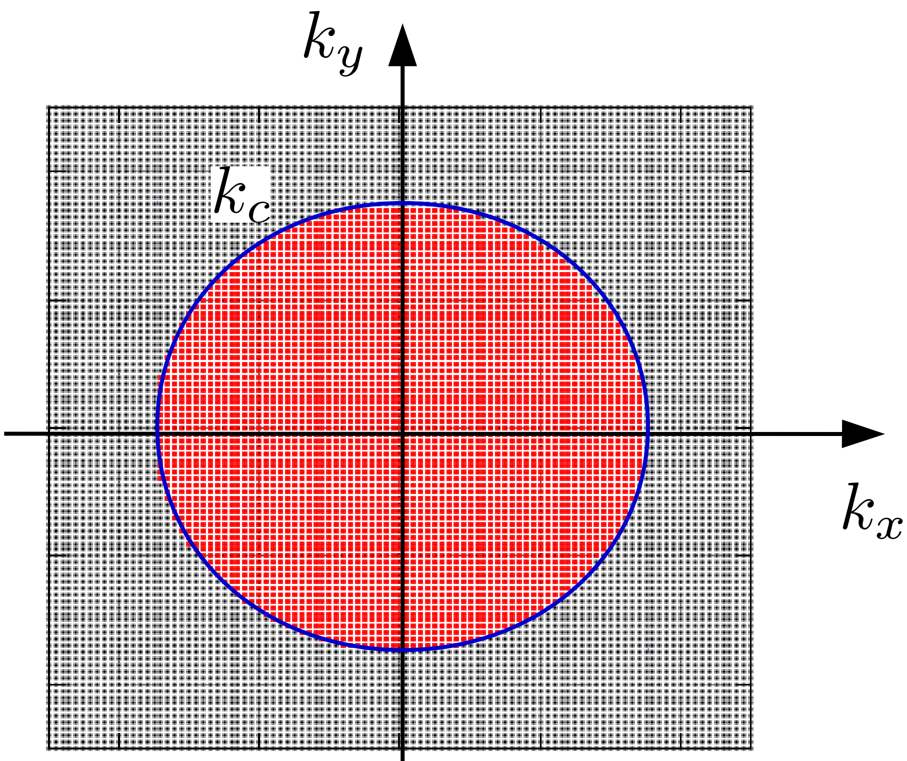


From sharp to smooth Projectors:

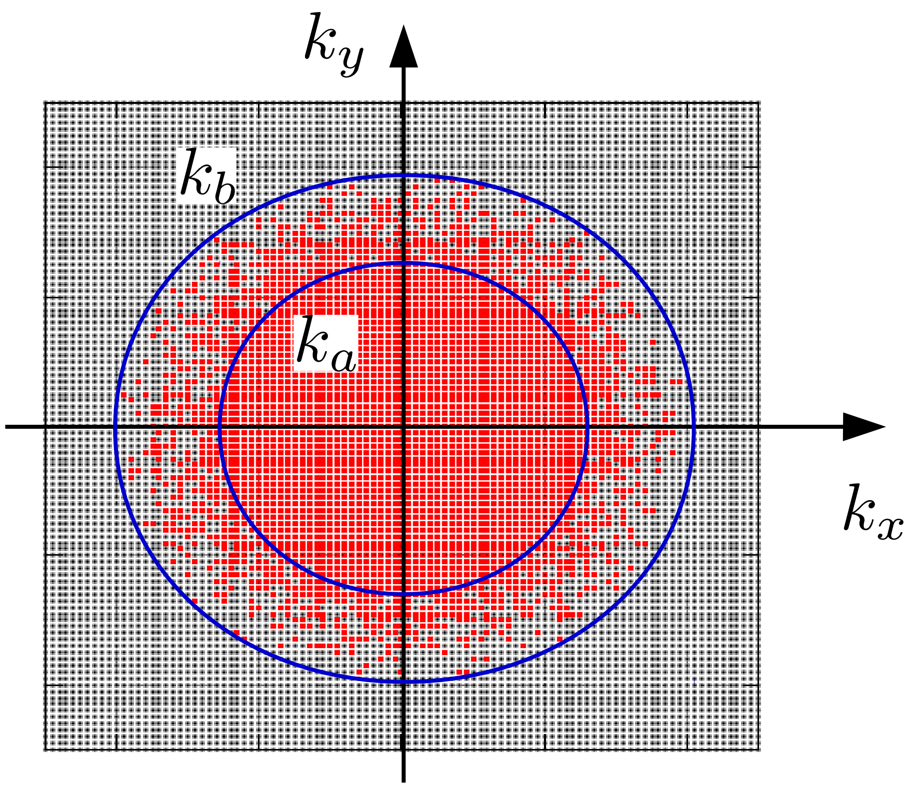
-) Linear Projector:

$$G_\Delta(|\boldsymbol{k}|) = \begin{cases} 1, & \text{if } |\boldsymbol{k}| < k_a \\ 1, & \text{with } P(|\boldsymbol{k}|) = \frac{|\boldsymbol{k}|-k_b}{k_a-k_b}, \text{ if } k_a \leq |\boldsymbol{k}| < k_b \\ 0, & \text{if } |\boldsymbol{k}| \geq k_b \end{cases}$$





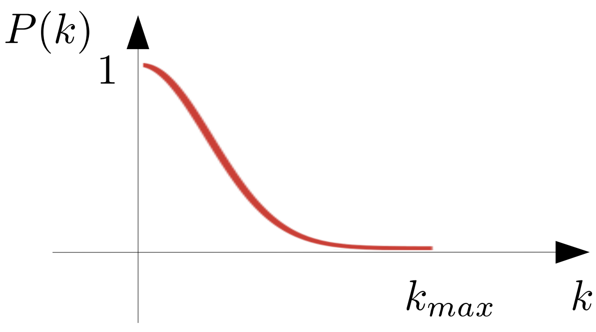
Sharp Projector

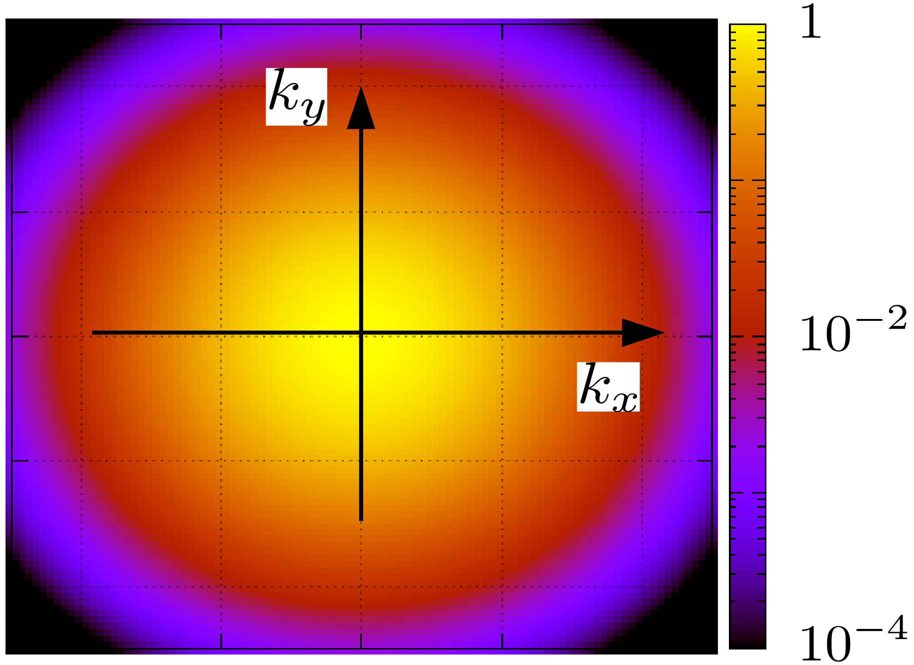


From sharp to smooth Projectors:

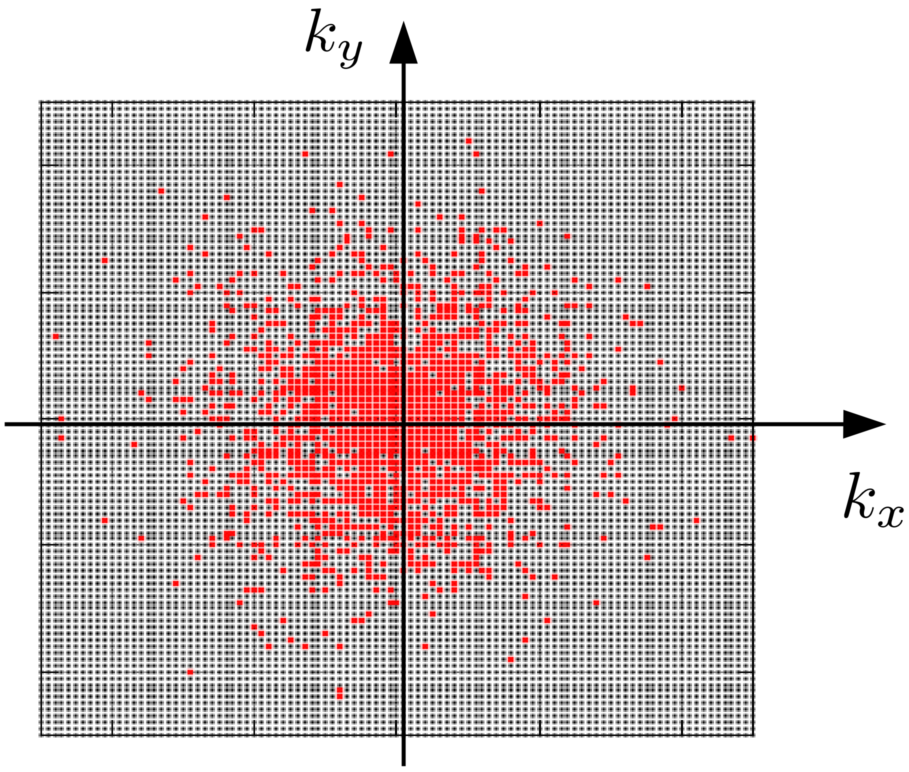
-) Gaussian Projector:

$$G_\Delta(|\boldsymbol{k}|) = \begin{cases} 1, & \text{with } P(|\boldsymbol{k}|) = exp(-\frac{|\boldsymbol{k}|^2\Delta^2}{24}) \\ 0, & 1 - P(|\boldsymbol{k}|) \end{cases}$$

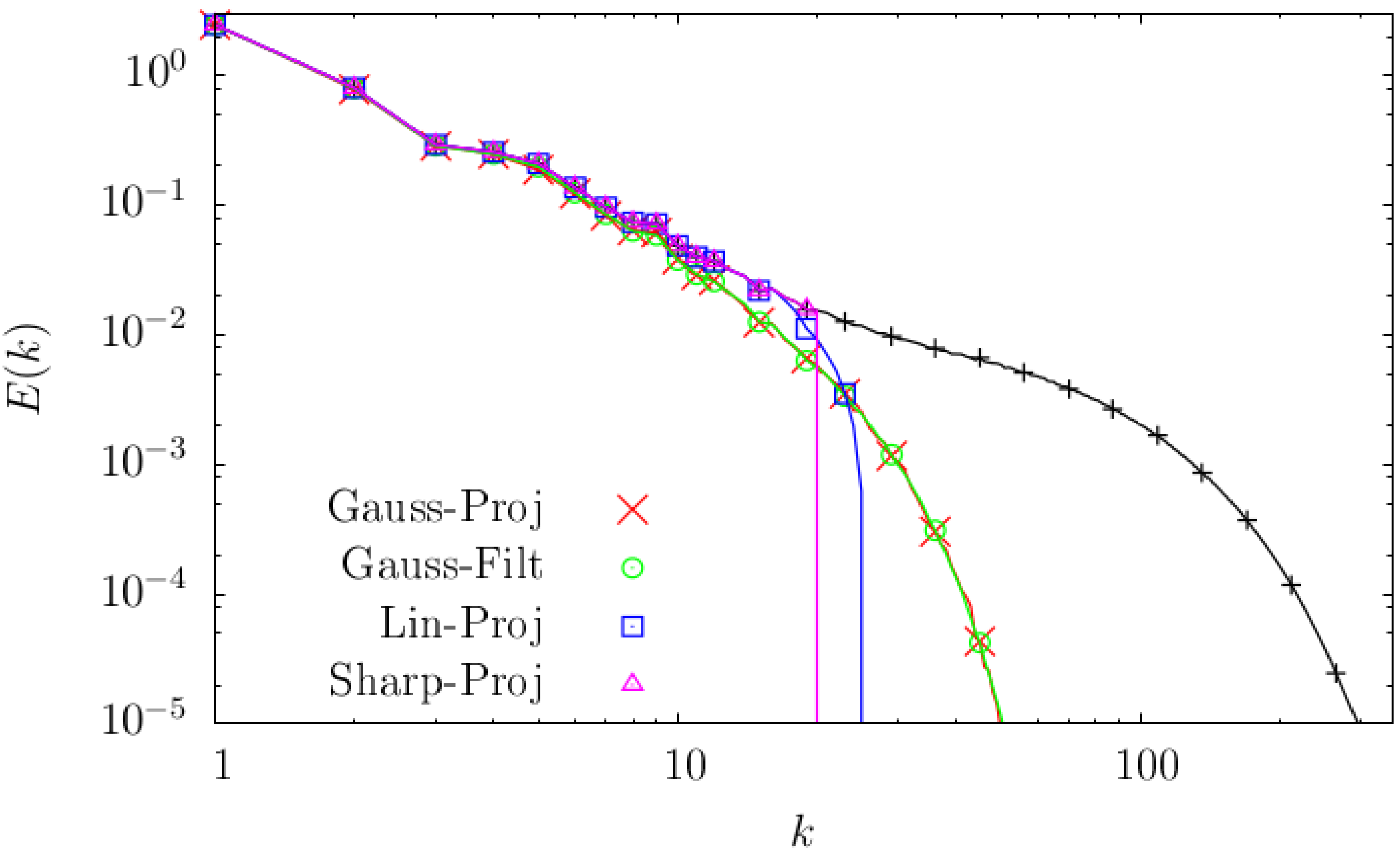


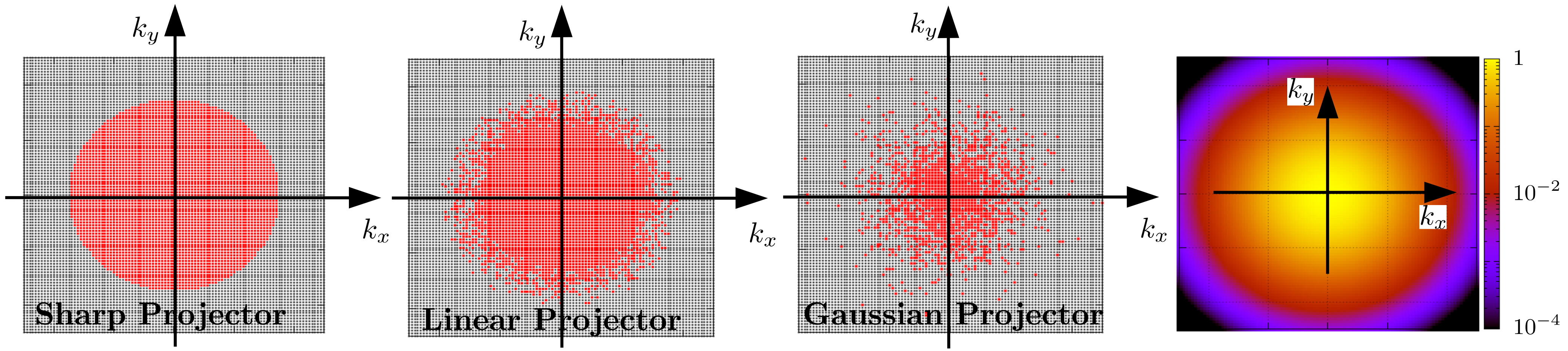


Gaussian smoothing



Filtered energy spectra:





Projected - Large Eddy Simulations; $\partial_t \overline{\boldsymbol{v}} = ?$

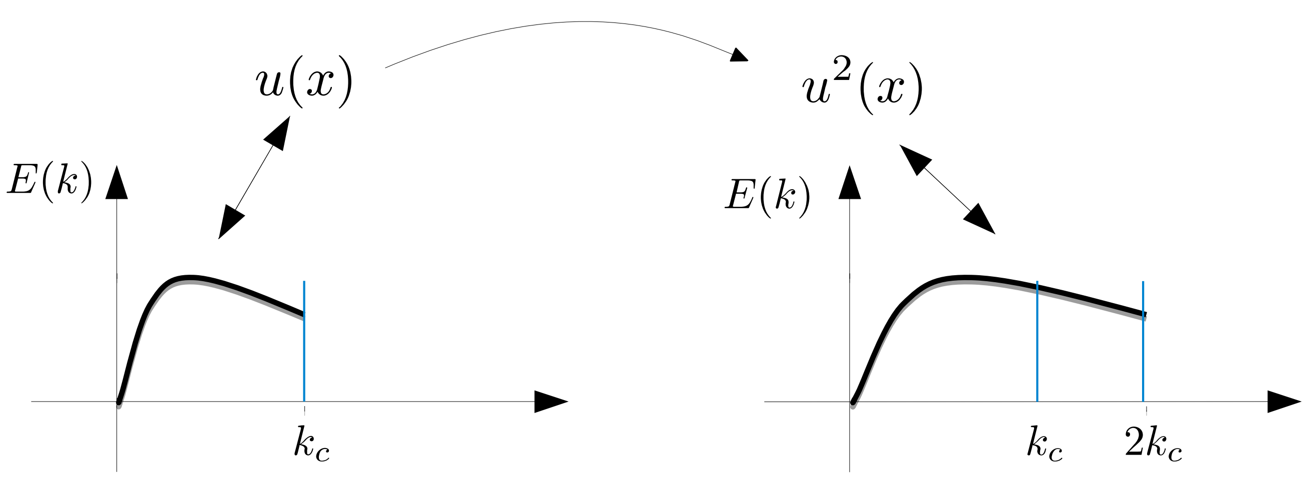
-) If $G(\boldsymbol{k})$ is a projector:

$$\begin{cases} \partial_t \overline{\boldsymbol{v}} + \nabla(\overline{\overline{\boldsymbol{v}}\ \overline{\boldsymbol{v}}}) = -\nabla \overline{p} - \nabla \cdot \overline{\tau(\boldsymbol{v}, \boldsymbol{v})} + \nu \Delta \overline{\boldsymbol{v}} \\ \overline{\tau_{ij}(\boldsymbol{v}, \boldsymbol{v})} = \overline{v_i v_j} - \overline{\overline{v}_i \overline{v}_j} \end{cases}$$

We need a double projection to ensure that $\overline{\boldsymbol{v}}$ remains projected at all times;

$$\forall\, t, \overline{\boldsymbol{v}}(\boldsymbol{k}, t) \text{ with } \boldsymbol{k} \in \left\{\mathbb{Z}^3\right\}_{projected} \Rightarrow \overline{\boldsymbol{v}}(\boldsymbol{k}, t) = \overline{\overline{\boldsymbol{v}}}(\boldsymbol{k}, t),\ \forall\ t$$

Notice that: *the product of two projected functions is not a projected function!*



Sub Grid Scales energy transport

$$\partial_t E(x,t) = \frac{1}{2}\partial_t(\overline{v}_i\overline{v}_i)$$

$$\partial_t E + \partial_j \underbrace{\left(\overline{v}_i(\frac{1}{2}\overline{v}_j\overline{v}_i + \overline{P}\delta_{ij} + \overline{\tau_{ij}} + \tau_{ij}^{Leonard})\right)}_{\substack{\text{Total gradient} \\ \text{zero on the volume average}}} = \overbrace{-\Pi^{Tot}}^{\substack{\text{Total SGS energy} \\ \text{flux}}}$$

Sub Grid Scales energy transport

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$$\Pi^{Proj} = -(\partial_i\overline{v}_j)\ \overline{\tau_{ij}} = -(\partial_i\overline{v}_j)(\overline{v_iv_j} - \overline{\overline{v}_i\overline{v}_j})$$

$$\Pi^{Leo} = -(\partial_i\overline{v}_j)\tau_{ij}^{Leo} = -(\partial_i\overline{v}_j)(\overline{\overline{v}_i\overline{v}_j} - \overline{v}_i\overline{v}_j)$$

$$\Pi^{Tot}(x,t) = \Pi^{Proj}(x,t) + \Pi^{Leo}(x,t)$$

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$$\Pi^{Tot}(x,t) = \Pi^{Proj}(x,t) + \Pi^{Leo}(x,t)$$

$$\langle\Pi^{Tot}\rangle = \langle\Pi^{Proj}\rangle + \langle\underbrace{\cancel{\Pi^{Leo}}}_{=0}\rangle = \langle\Pi^{Proj}\rangle$$

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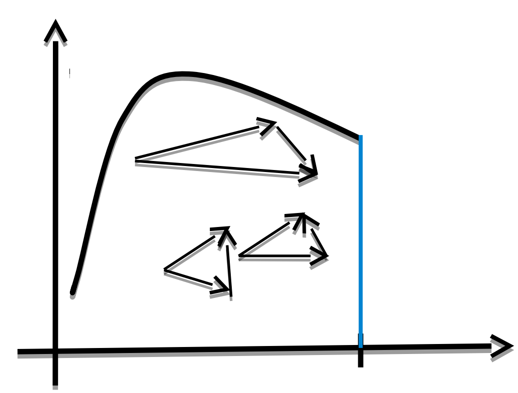
$$\Pi^{Tot}(x,t) = \Pi^{Proj}(x,t) + \Pi^{Leo}(x,t)$$

$$\langle\Pi^{Tot}\rangle = \langle\Pi^{Proj}\rangle + \underset{=0}{\langle\cancel{\Pi^{Leo}}\rangle} = \langle\Pi^{Proj}\rangle$$

(∗) The Leonard tensor is composed only by resolved triads, hence on avarage it gives zero SGS energy transfer!

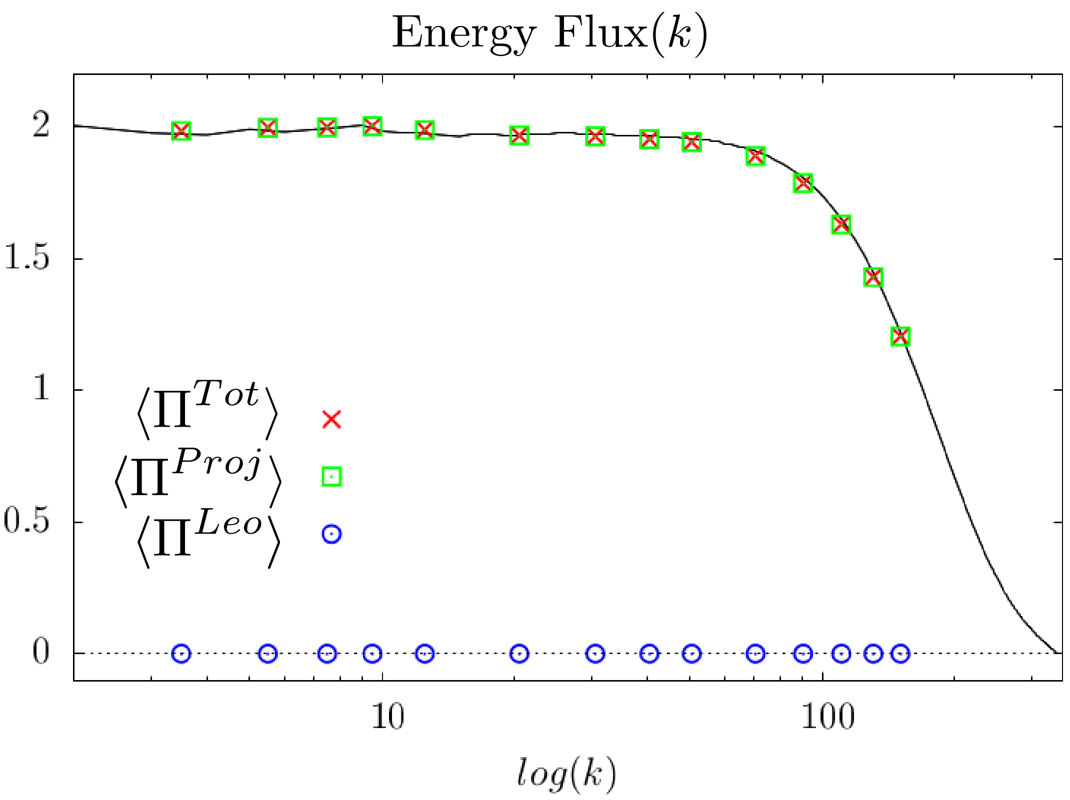


A-priori analysis and comparison of the three different SGS energy transfer;

$$\Pi^{Tot};\ \Pi^{Proj};\ \Pi^{Leo}$$

1024^3 High Reynolds DNS with Hyperviscosity

SGS transfer measured with a Sharp-Projector filter



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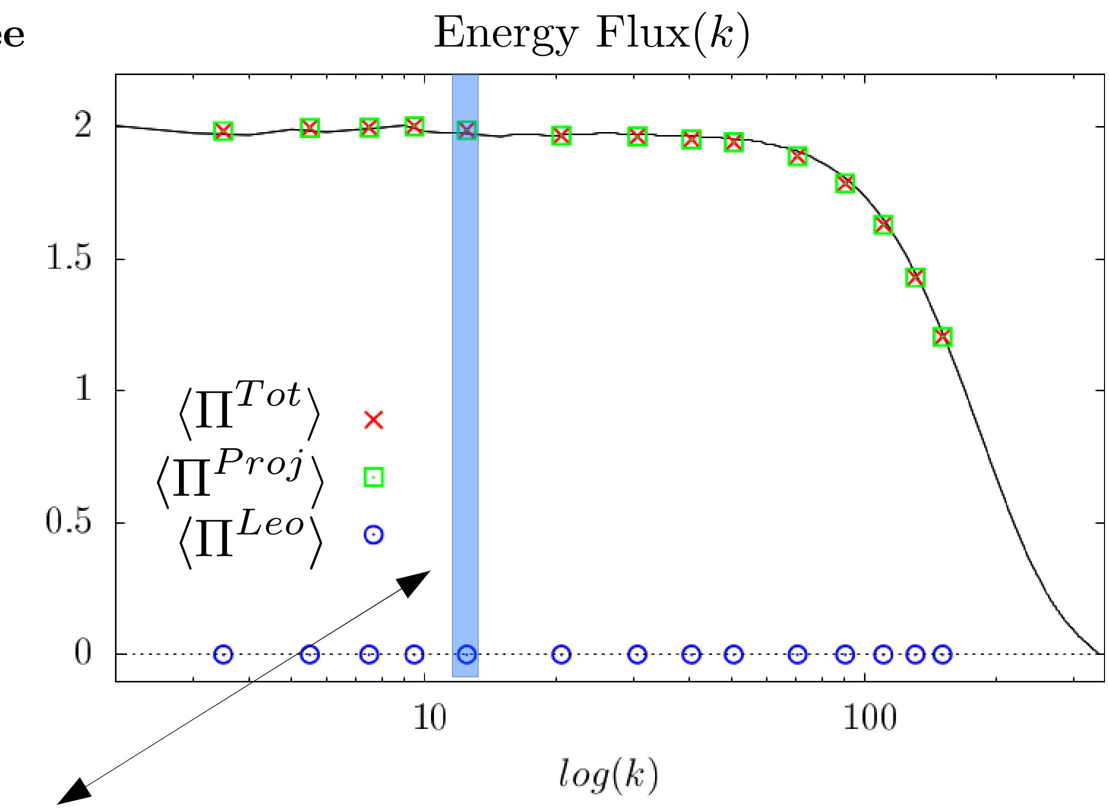
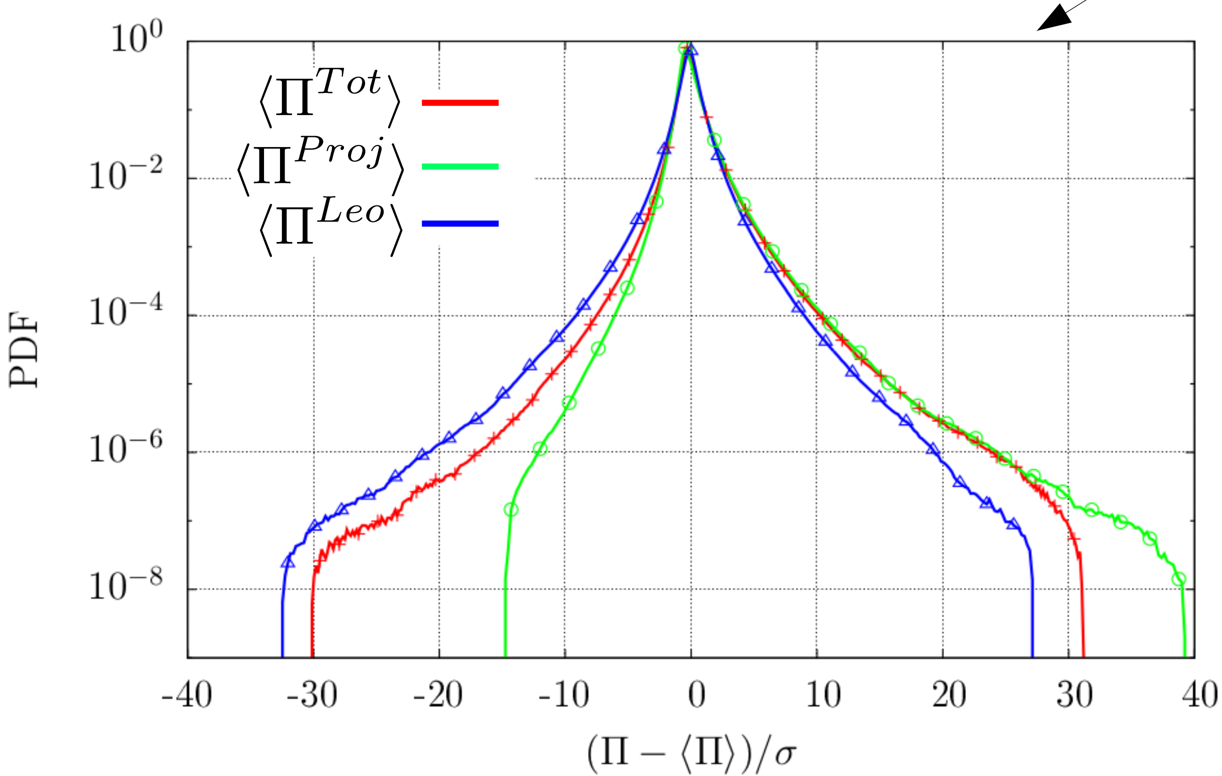




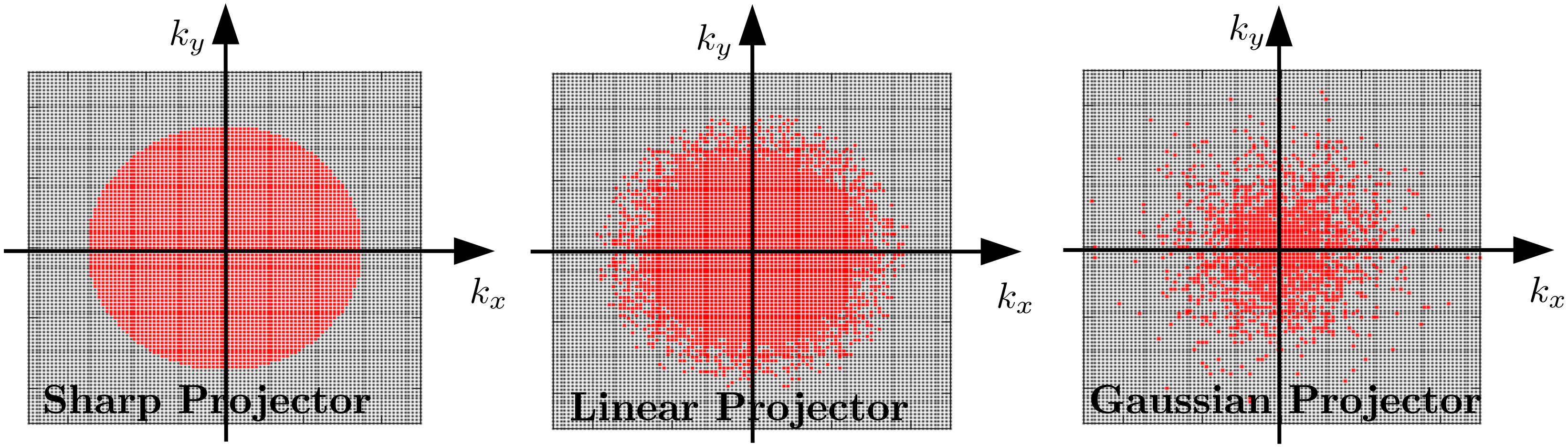
-)Π^{Proj} **is the most skewed** in the direction of the energy flux

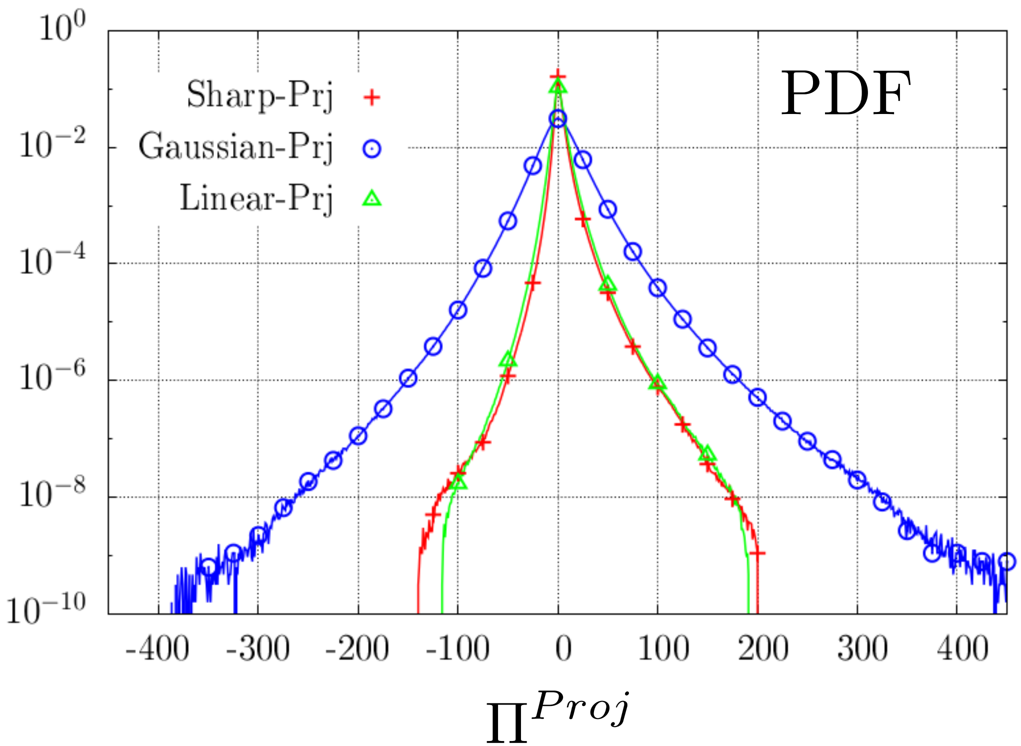
-) Π^{Leo}**doesn't contain** physiscs - of **SGS energy transport**

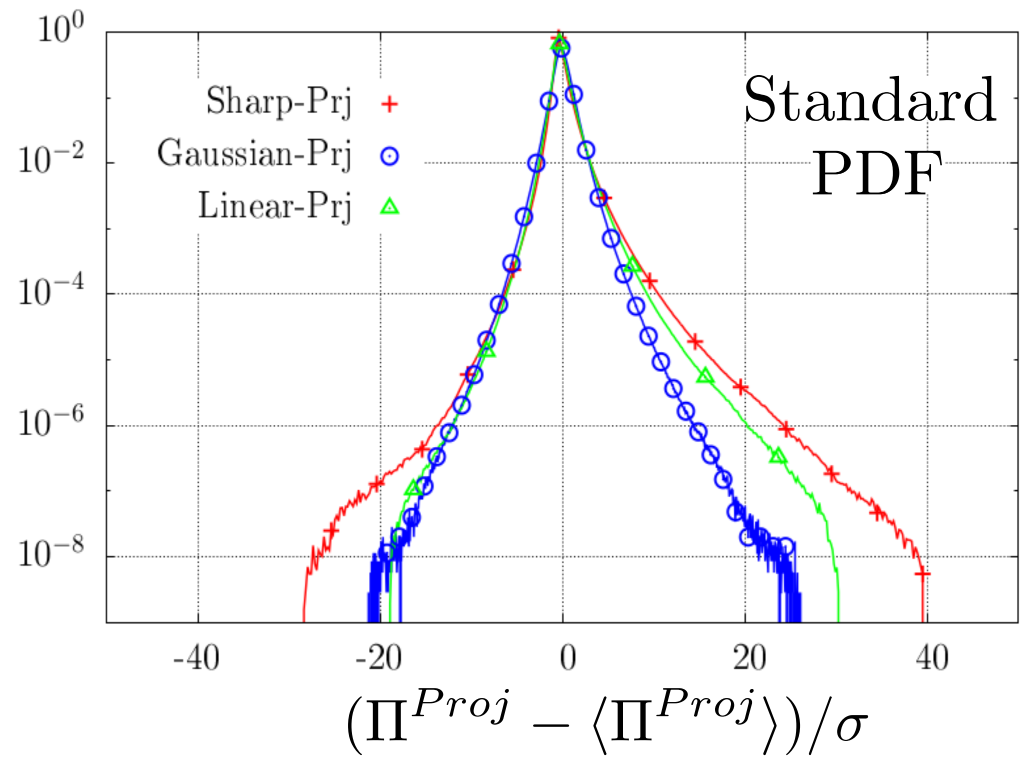
-) Π^{Tot} **is affected** by resolved scales

Effects of different Projectors:

$$\Pi^{Proj} = -(\partial_i \overline{v}_j)\overline{\tau_{ij}}$$

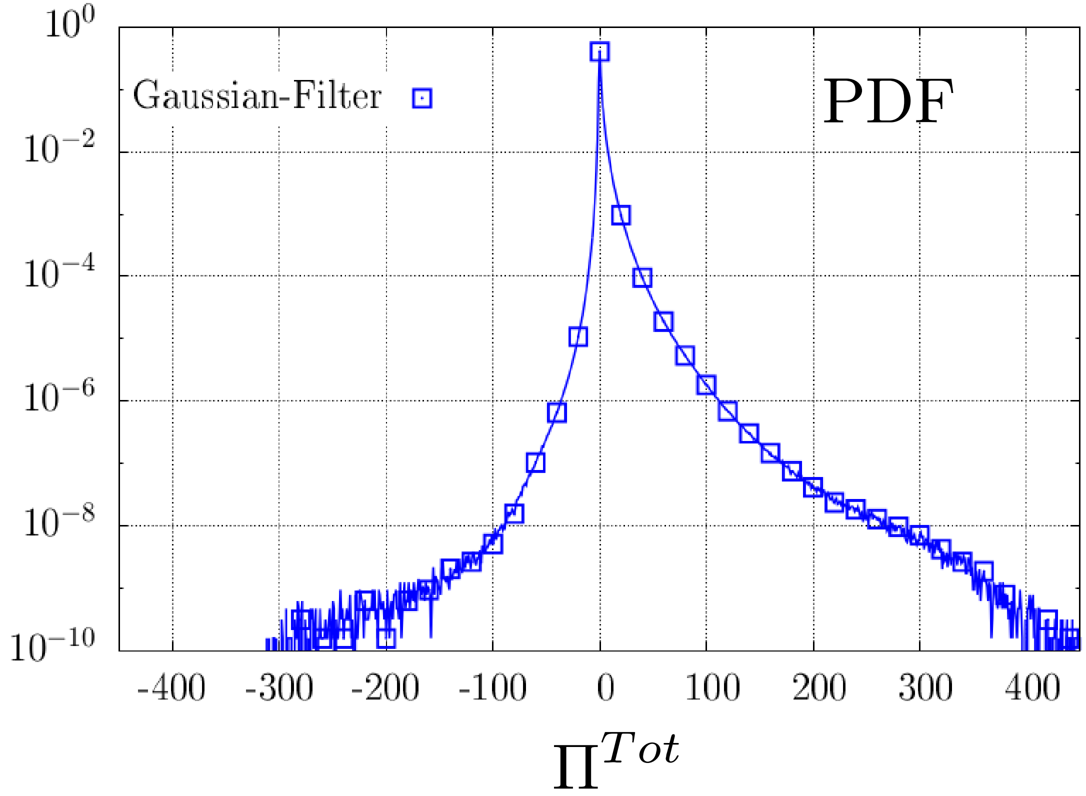






Non-Projector filter:

Gaussian Smooth filter..
..with the same spectrum of the
Gaussian Projected field



Decomposition of Π^{Tot} is not meaningful,
because Π^{Proj} and Π^{Leo} are not Galilean invariant

Advantage;

-) **Right Skewness**

-) **All phases are available** in the non linear interactions

Disadvantage;

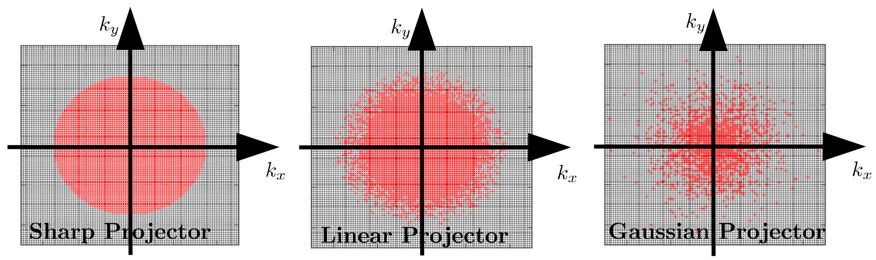
-) The equations of motion are **filter independent**

$$\partial_t \overline{\boldsymbol{v}} + (\overline{\boldsymbol{v}} \cdot \nabla)\overline{\boldsymbol{v}} = -\nabla \overline{p} - \nabla \cdot \tau_{model}(\overline{\boldsymbol{v}}, \overline{\boldsymbol{v}}) + \nu \Delta \overline{\boldsymbol{v}}$$

-) **How to compare** with a-priori analysis?

Conclusions

We perform an a-priori analysis of the SGS energy transfer by changing the projector properties



-) We introduced **Projector filters** with a non sharp Fourier profile

-) Using projector filters we can rewrite the total SGS energy transfer in two different contributions: Π^{Proj} and Π^{Leo}

-) We have shown that only Π^{Proj} contains information about the SGS energy transfer

-) Smooth filters preserve better the original physics, but

· do not define a closed evolution on a given sub-space

· difficult to validate using a-priori analysis