Role of helicity in transfer of energy and small-scale structures in three dimensional turbulence

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Outline

- Introduction
- 2D and 3D turbulence
- Decimated Navier-Stoke's equation
- Recent results

Turbulence everywhere

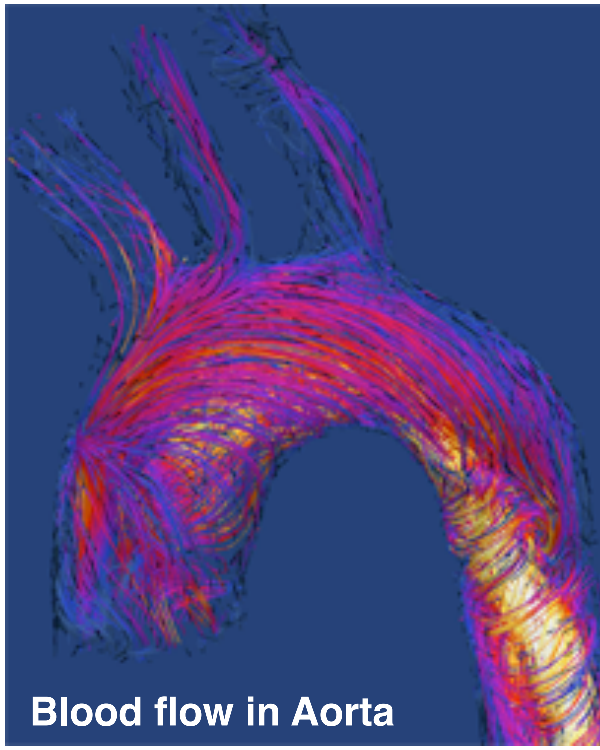

Fumes


Clouds

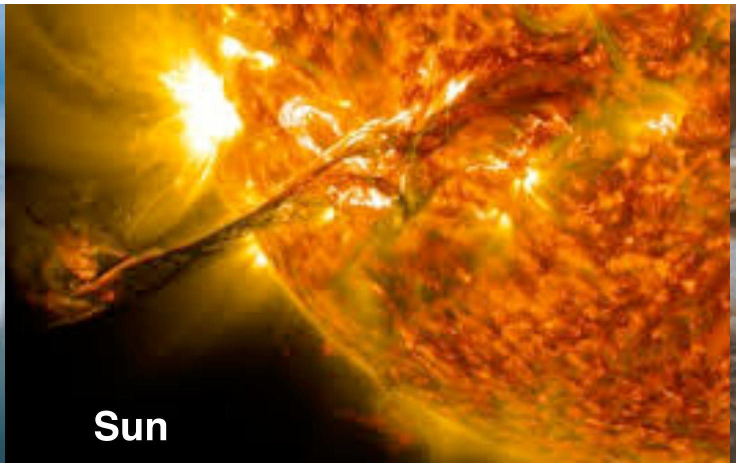

Tornado

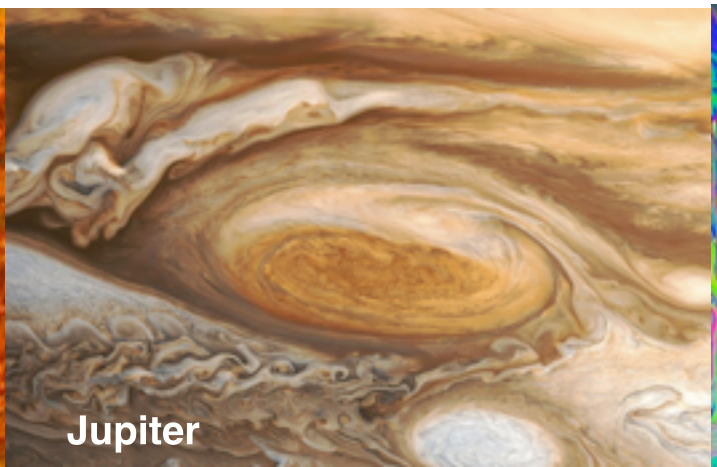

Ocean

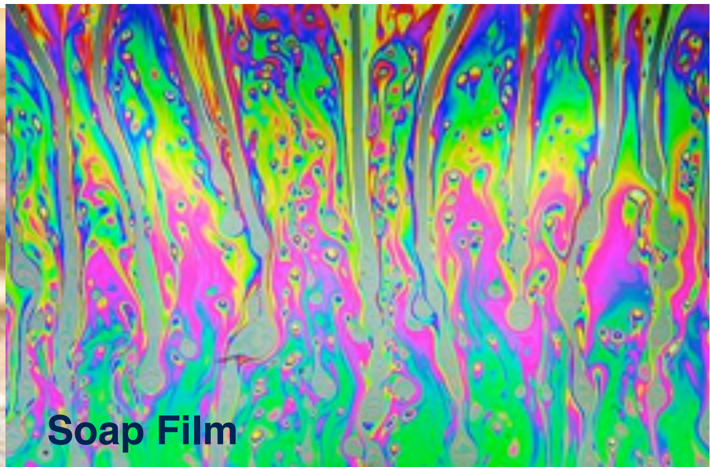
- All environmental flows are turbulent,
- Atmospheric boundary layer, Ocean Currents, interstellar clouds, flow of gas and oil in pipe lines, combustion in engines,


Blood flow in Aorta


Windmills

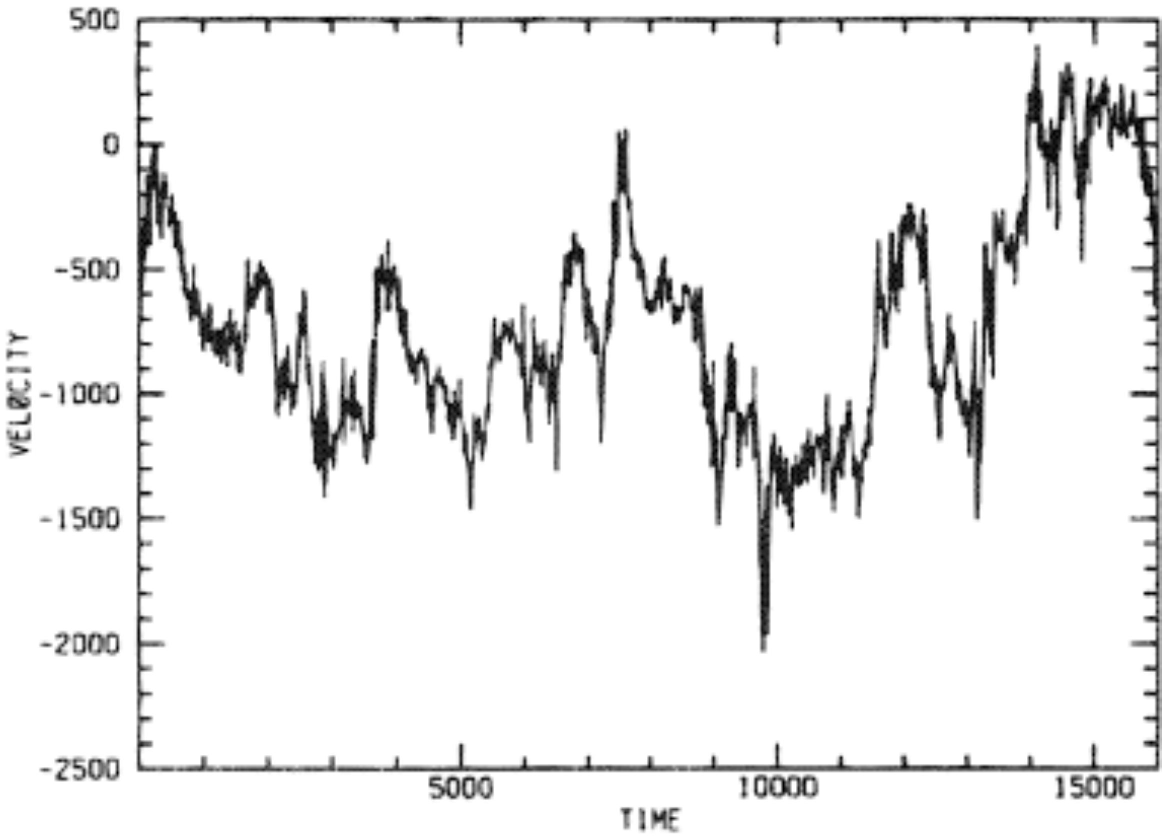

Sun


Jupiter


Soap Film

What is it?

- A turbulent flow can be interpreted as a population of many eddies (vortices), of different sizes and strengths, embedded in one another and forever changing, giving a random appearance to the flow.
- Highly irregular and intermittent.
- Multiple length and time scales.
- Diffusive: enhancement of momentum, heat, and mass transfer,
- Essentially dissipative: drag on moving body, e.g. airplanes, friction in pipe flows.
- Rotational: large vorticity fluctuations.



Reynolds Number

$$Re = \frac{UL}{\nu}$$

- U : mean velocity
- ν : kinematic viscosity
- L : characteristic length scale/ diameter of the pipe.

Transition to turbulence from laminar flow at high *Re* ~ 2000.

- Different fluid flows with same Re are similar in nature.
- Also Re describes different regimes in the same flow.

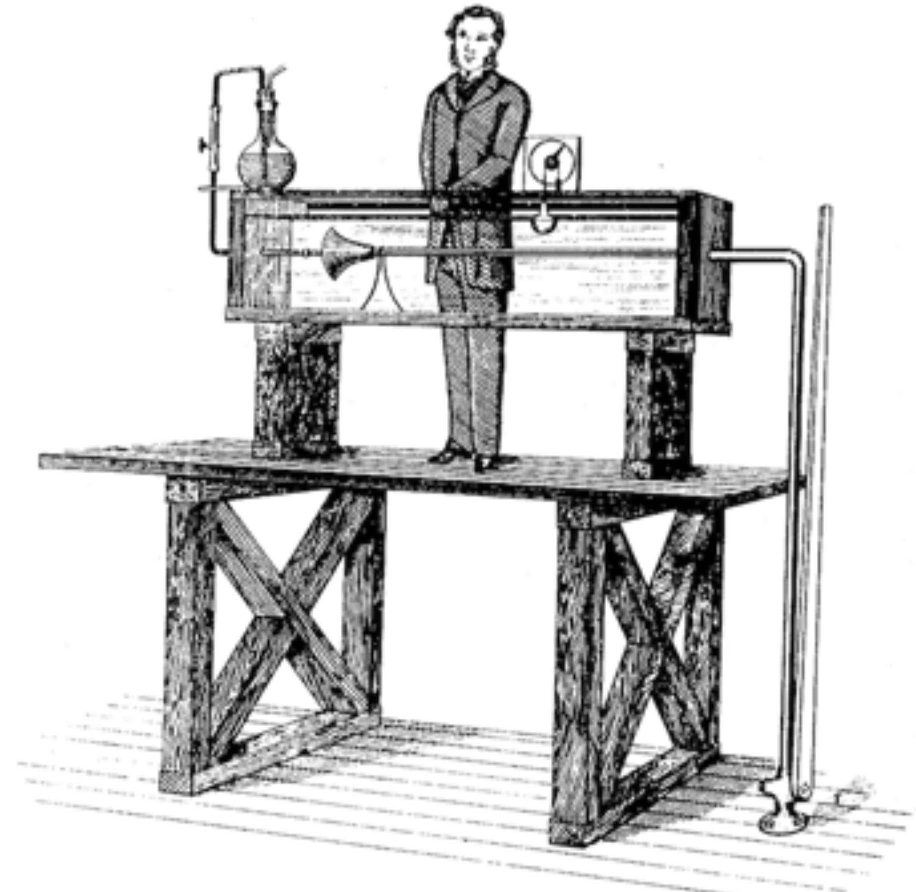
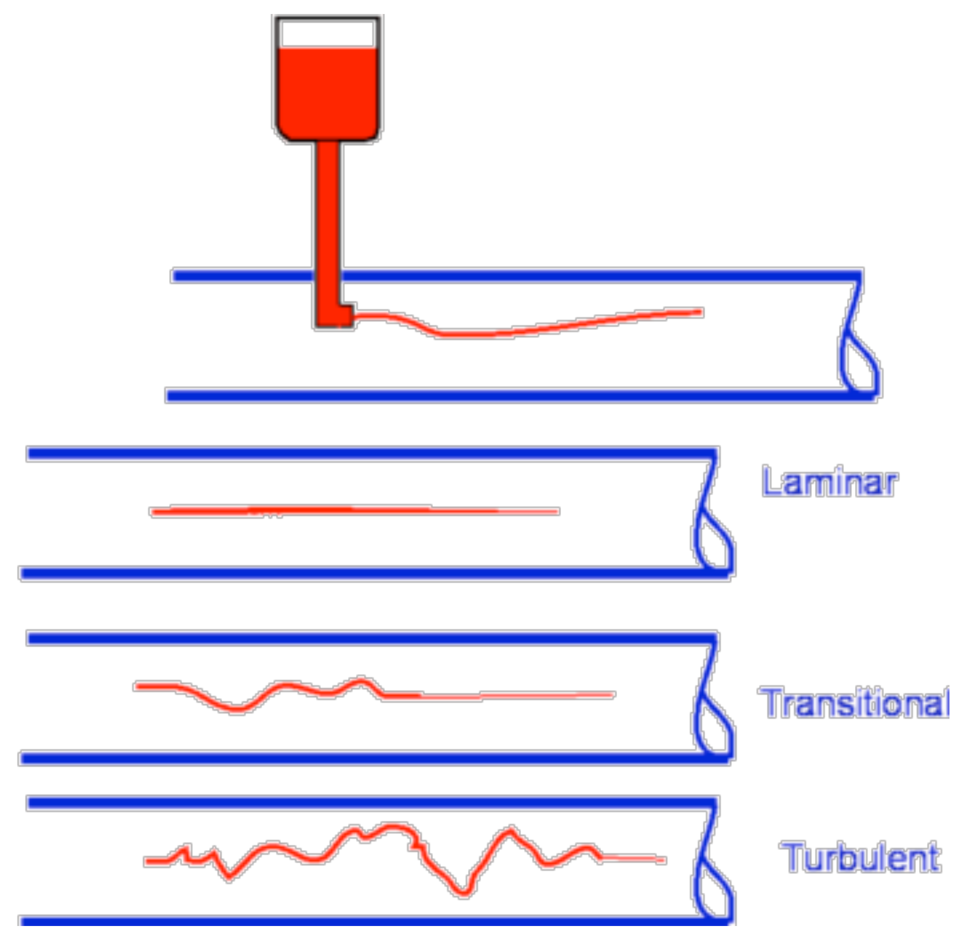
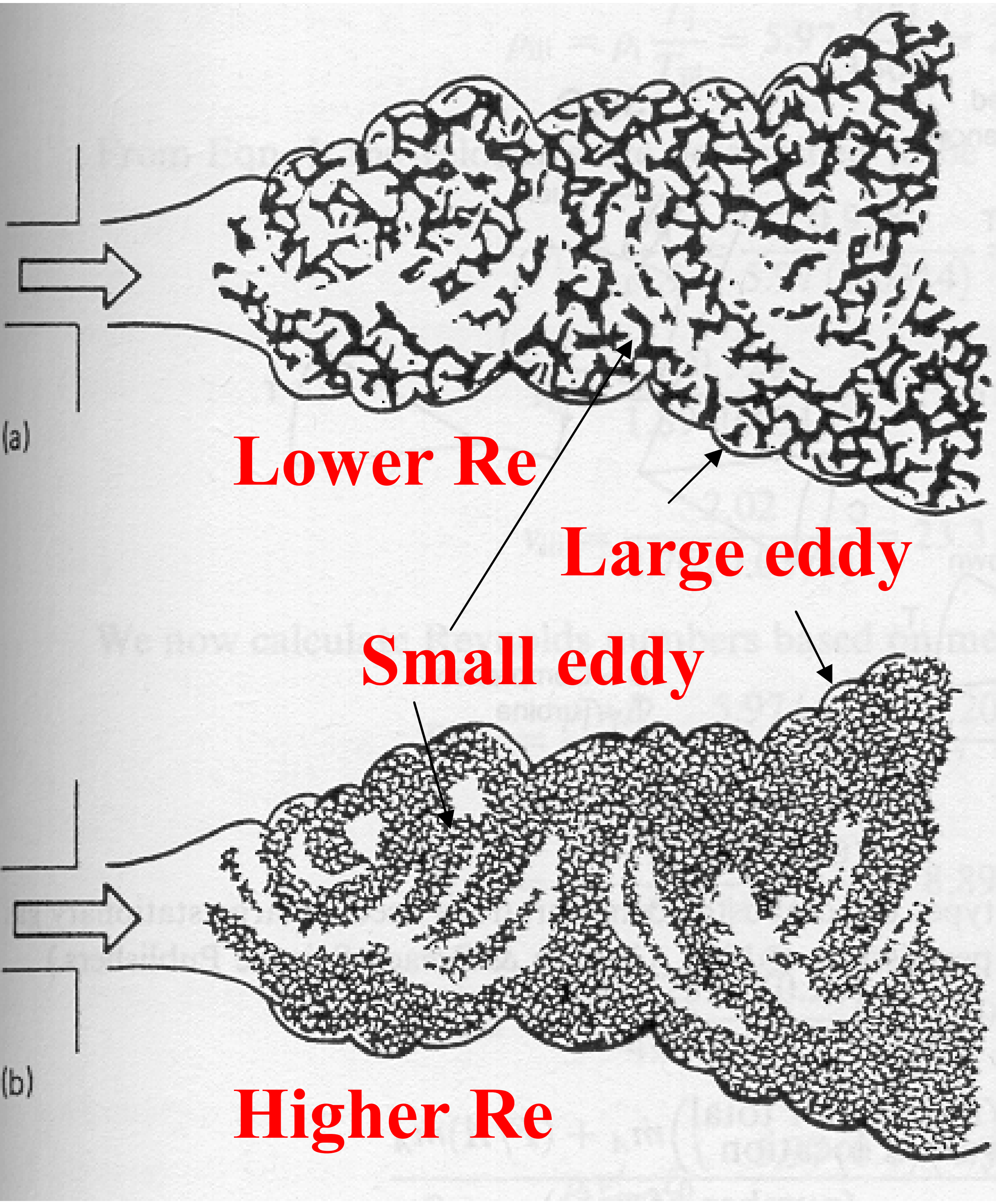


Fig. 9.1. Sketch of Reynolds's dye experiment, taken from his 1883 paper



Reynolds Number

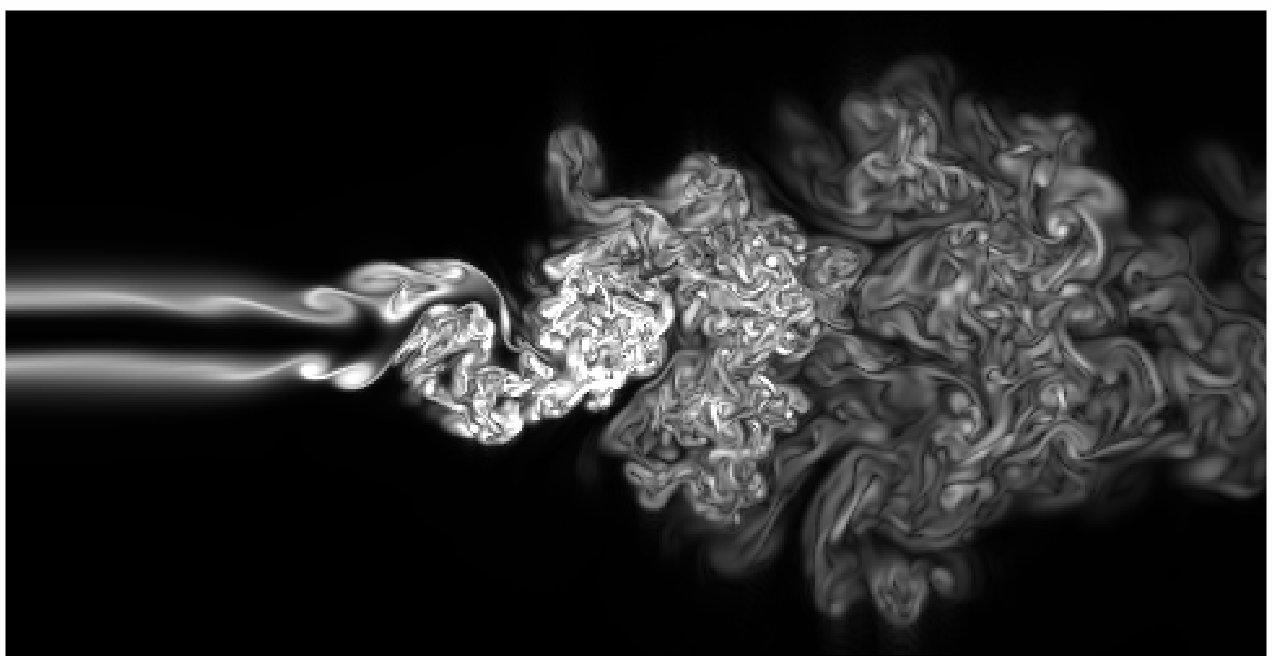


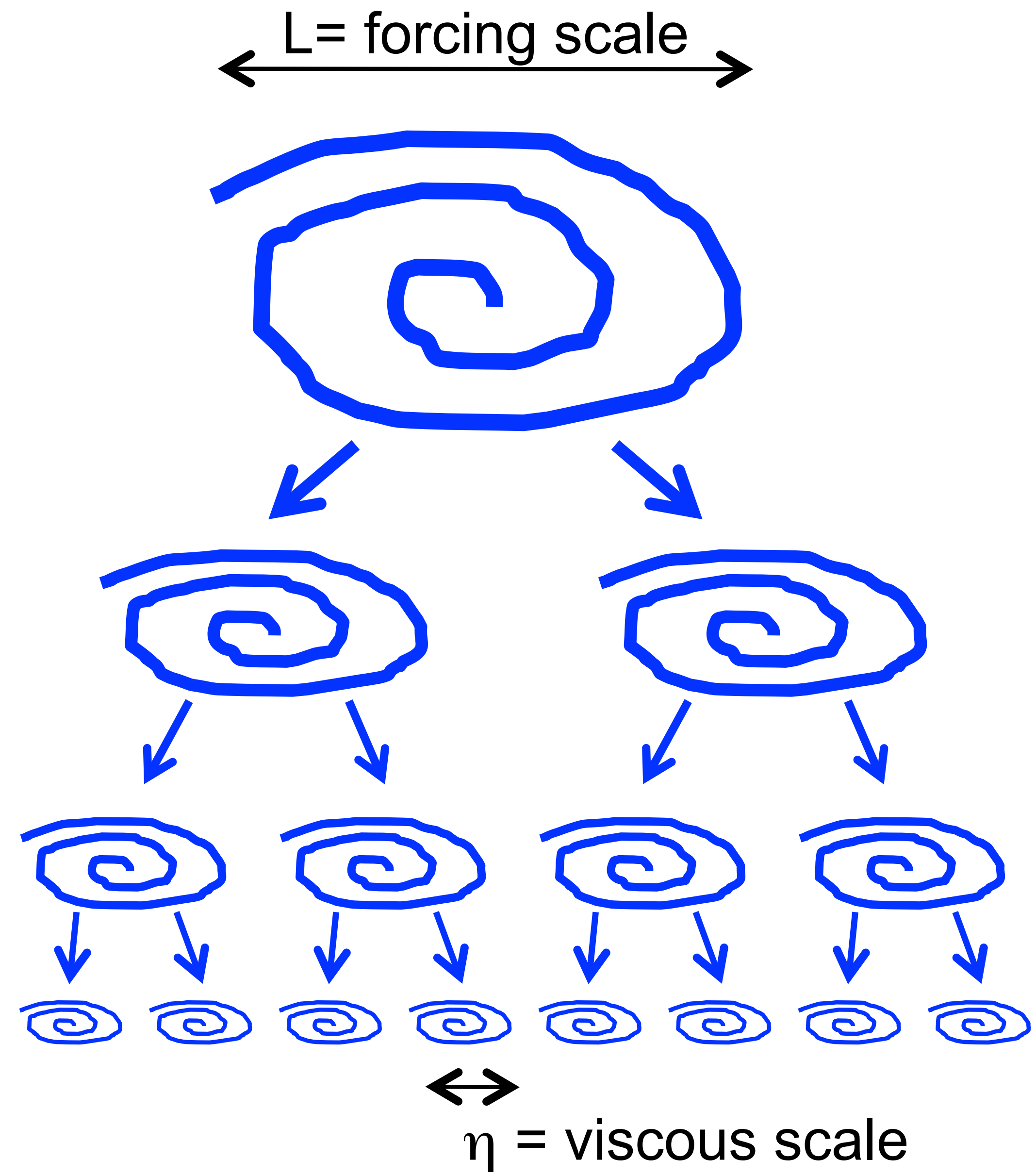
- Large scale structures are mainly independent of Reynolds number.
- Large Reynolds number produces smaller scale structures.
- Reynolds number is a measure of scale separations in the flow.

$$Re = \left(\frac{L}{\eta}\right)^{4/3}$$

Richardson cascade Picture

- Richardson's definition of turbulence eddies
  - Turbulence consists of different eddies
  - An eddy is a localized flow structure
  - Large eddies consists small eddies





- Energy is fed into the large eddies.
- Large eddies break to smaller and smaller eddies and energy gets dissipated at viscous scales.

Navier-Stoke's equation

- Navier-Stokes equation for incompressible flows,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u} + \mathbf{f},$$

$$\nabla \cdot \mathbf{u} = 0.$$

- The nonlinear term is responsible for a cascade of energy in a turbulent flow.
- Since the equations are nonlinear, generic solutions are not superposition of basic solutions.
- Navier-Stokes equations display a strong sensitivity to initial conditions. Hence, exact solutions are less interesting.
- Need for a statistical approach.
- Complex to solve analytically. We use computers!
- Large grid Direct Numerical Simulations to resolve smaller length scales.

Universality?

Turbulence is irregular or chaotic in space in time.
But is there any universal aspect?

- In a statistically stationary, homogeneous and isotropic flow, all eddies of size l behave similarly.
- They have a characteristic velocity, say u $[LT^{-1}]$.
- They transfer as much energy received from larger eddies to smaller eddies; rate of energy transfer is the same for all scales.
- Energy supplied at largest scales is equal to the energy dissipated at small scales; the rate of energy dissipation per unit mass is ε $[L^2T^{-3}]$.

Kolmogorov theory (1941)

- For very high *Re*, the statistical properties of eddies of sizes in the inertial range of scales are
  - independent of the forced and dissipative scales, and are locally homogeneous and isotropic.
  - universally and uniquely determined by the length scale l, viscosity ν, and the rate of energy dissipation ε.
- Characteristic velocity of an eddy of size l scales as $u_l \sim (l\varepsilon)^{-1/3}$.
- Energy spectrum in the inertial range $E(k) \sim \varepsilon^{2/3} k^{-5/3}$,

$$\text{for} \quad L^{-1} << k << \eta^{-1}; \quad \eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

- Self-Similarity hypothesis: Structure functions of p-th order scales as

$$S_p(l) = \langle \delta u_l^p \rangle \sim (\varepsilon l)^{p/3},$$
$$\delta u_l = [\mathbf{u}(\mathbf{r}+\mathbf{l}) - \mathbf{u}(\mathbf{r})] \cdot \frac{\mathbf{l}}{l}.$$

Role of helicity

- Like energy, helicity is also an invariant of the inviscid and unforced flow (discovered only in 1960's).
- Conservation of helicity is linked to the parity invariance of the flow.
- At a very high *Re*, there is a growth of helicity at the small scales, but total helicity remains finite, because of the symmetry.
- Energy gets distributed among scales by the nonlinear term in Navier-Stoke's equation and assuming a constant energy flux we observe the scaling behaviour $\delta u_l = [\mathbf{u}(\mathbf{r}+\mathbf{l}) - \mathbf{u}(\mathbf{r})] \cdot \frac{\mathbf{l}}{l} \sim \varepsilon^{1/3} l^{1/3}$
- By similar dimensionality argument and assuming a constant helicity flux *h* [LT^{-3}], we obtain

$$\delta u_l = [\mathbf{u}(\mathbf{r}+\mathbf{l}) - \mathbf{u}(\mathbf{r})] \cdot \frac{\mathbf{l}}{l} \sim h^{1/3} l^{2/3}$$

- But such a scaling is not observed. Why?

Role of helicity

- There is no purely helicity dominated turbulence since both energy and helicity cascade to the small scales.

- For the joint cascade of energy and helicity we expect

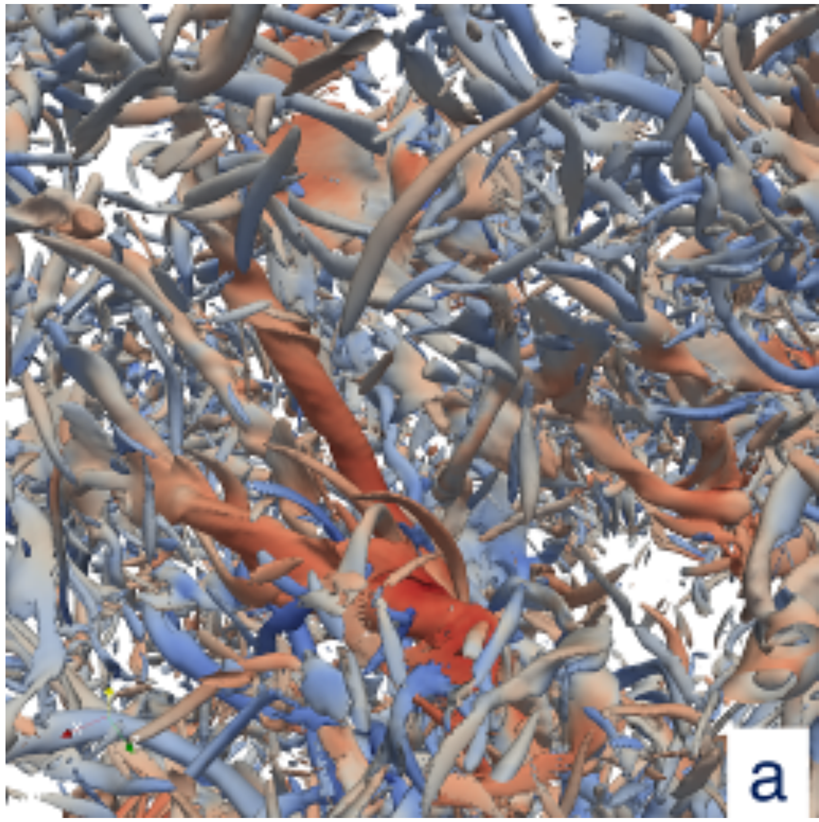
$$\delta u_l = [\mathbf{u}(\mathbf{r}+\mathbf{l}) - \mathbf{u}(\mathbf{r})] \cdot \frac{\mathbf{l}}{l} \sim \varepsilon^{\beta} h^{\gamma} l^{\delta}$$

- But then, we can not determine the exponents, uniquely, from dimensionality argument.

- Presence of helicity changes the geometrical structure in a subtle way, which could not be captured by simple dimensional analysis.

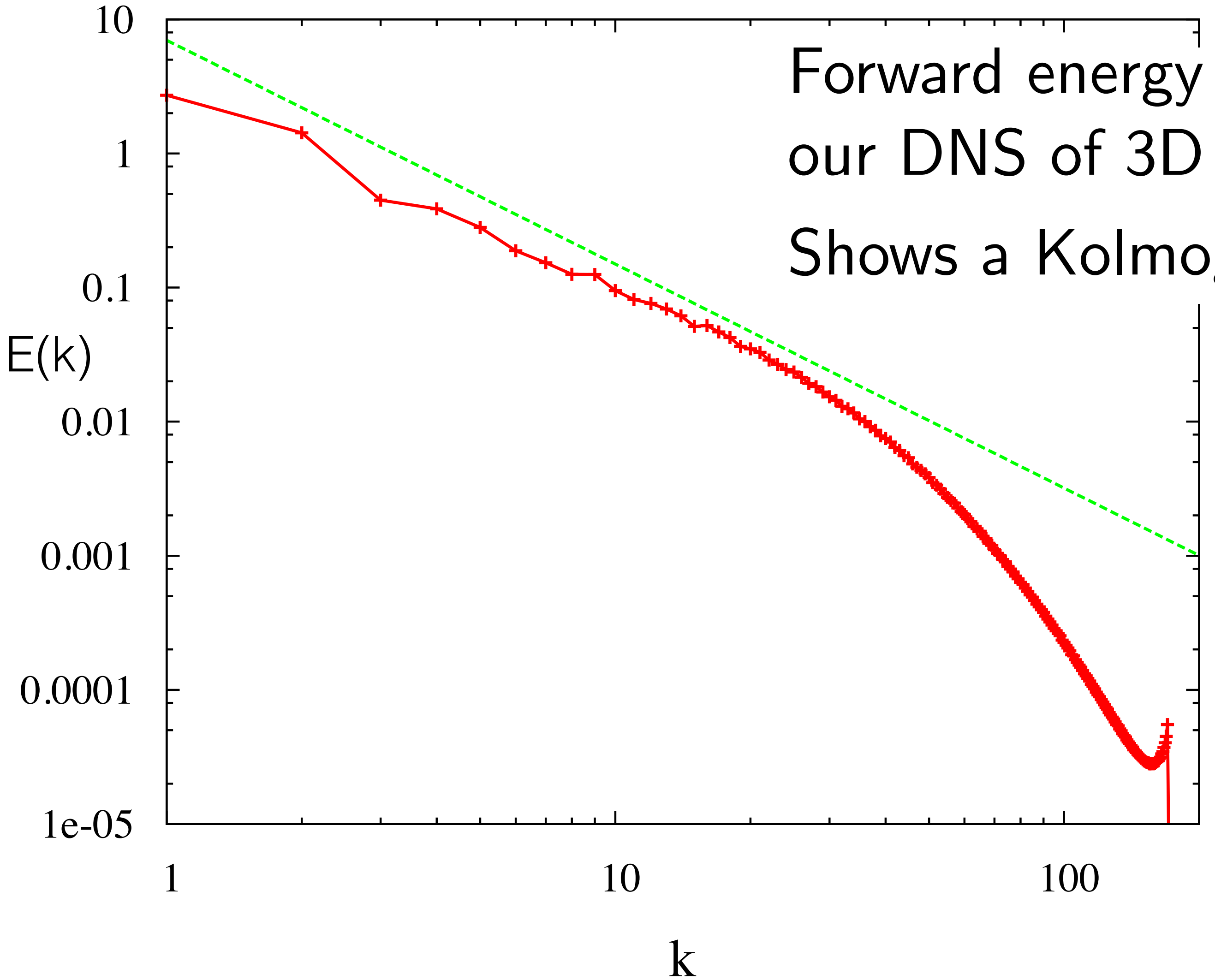
Role of helicity

$$H = \int_V \mathbf{u} \cdot \omega \; d^3x$$ is a pseudoscalar.



- Tells us if the instantaneous streamline is close to right-handed or left-handed screw.
- Helicity measures the knottedness of the vortex lines.
- It would help us in understanding the origin of vorticity tube and sheets.
- Change in the helicity may be associated to a certain event called vorticity reconnection.
- In presence of the viscosity, the vortex lines can touch and re-connect and produce helicity.

3D turbulence



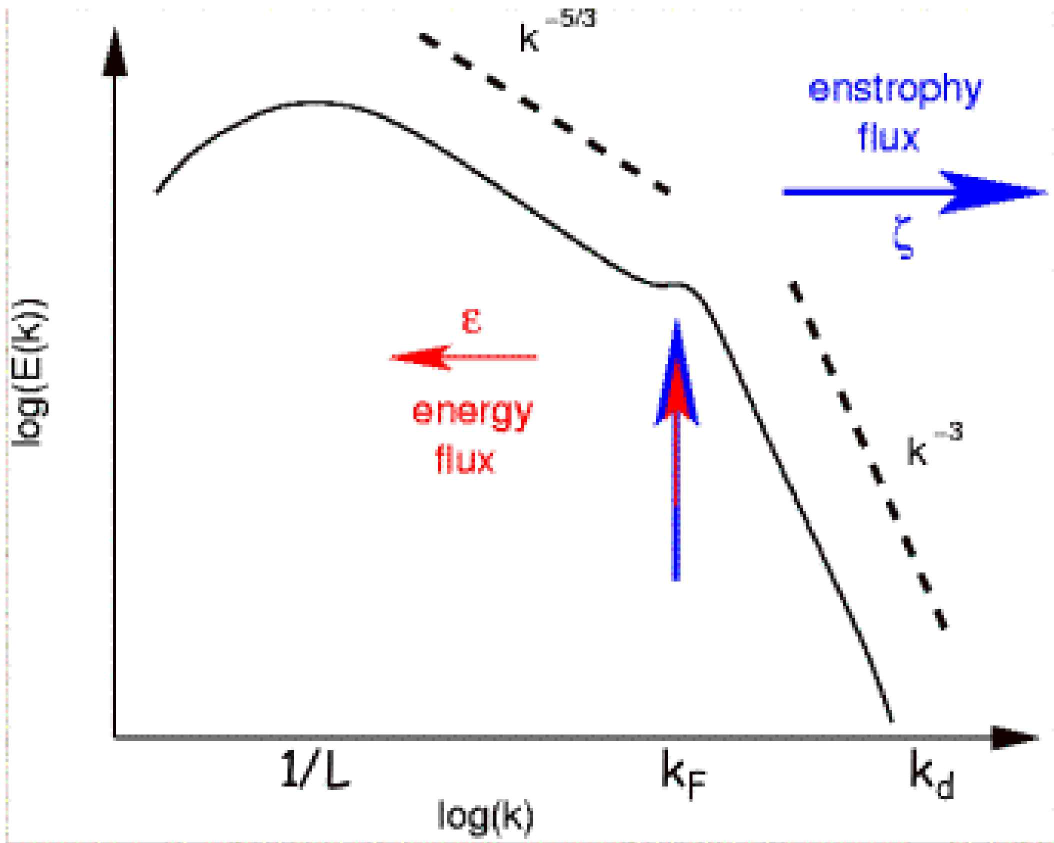
Forward energy cascade from large scales to small scales in our DNS of 3D Navier-Stokes equations.

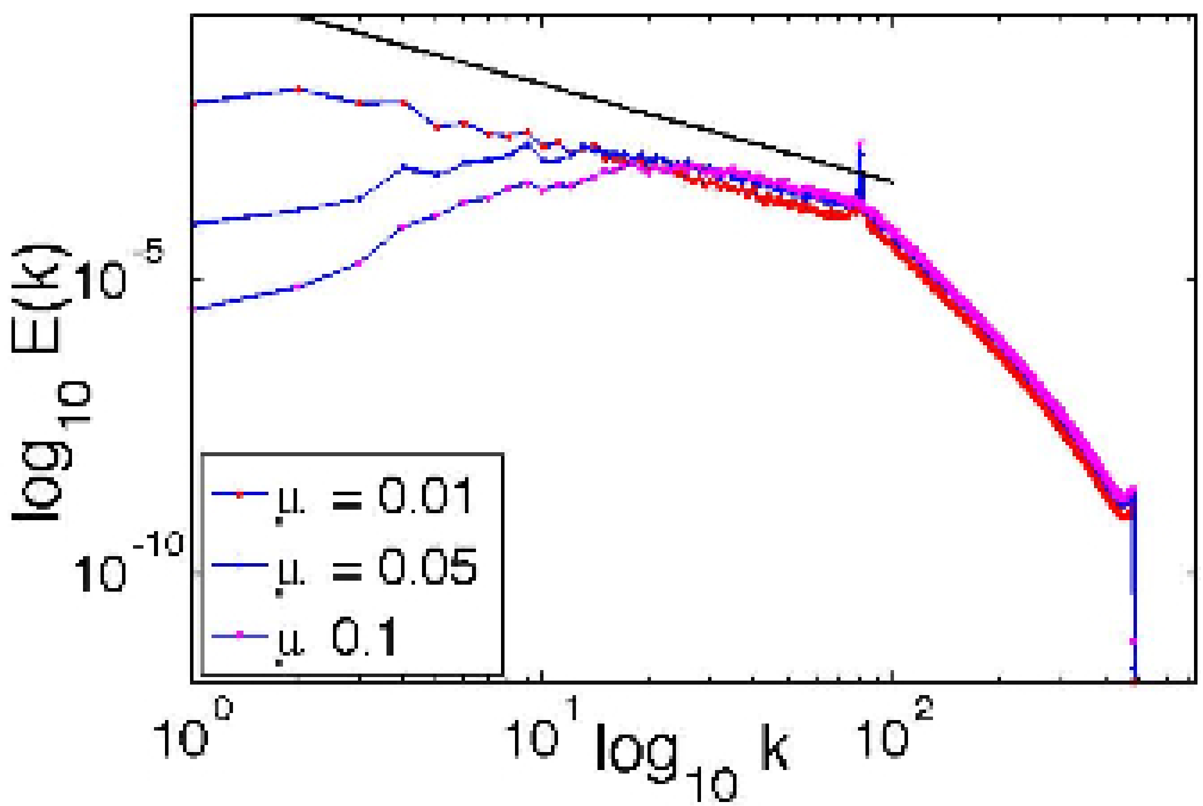
Shows a Kolmogorov $k^{-5/3}$ scaling in the inertial range.

- The invariants of 3D Navier-Stokes equations:
  Energy $E = \int d^3r\ \vec{u} \cdot \vec{u}$ and Helicity $H = \int d^3r\ \vec{u} \cdot \vec{\omega}$
- Helicity could be positive or negative.
- Both cascades forward, from large scales to small scales. (Chen, Phys. Fluids 2003)

2D turbulence

- For 2D Navier-Stokes equations two conserved quantities: Energy $E = \int d^2r\ \vec{u} \cdot \vec{u}$ and Enstrophy $\Omega = \int d^2r\ \vec{\omega} \cdot \vec{\omega}$
- Forward cascade of energy is blocked, since enstrophy is also positive and definite. (Boffetta Ann. Rev. Fluid Mech 2012)

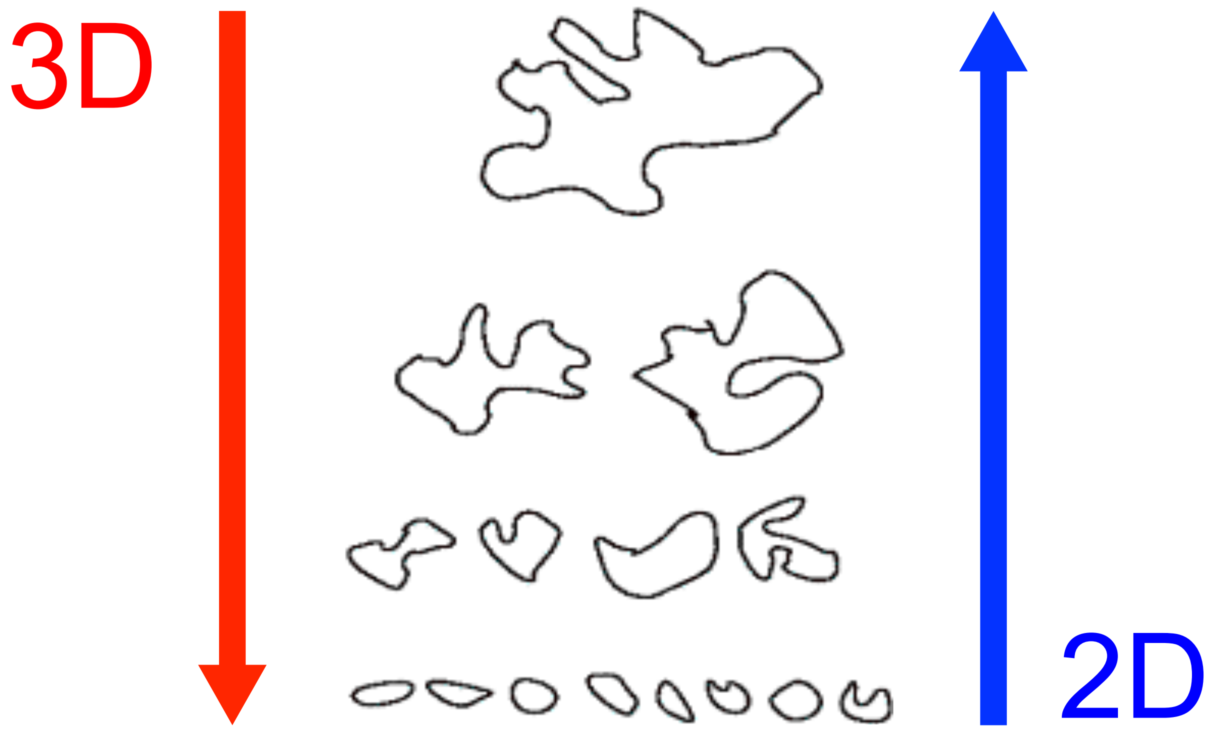




Ray et al, Phys. Rev. Lett. 107, 184503 (2011)

Dimensionality

- The direction of cascade is determined by positive-definite inviscid invariants.
- In 2D: energy and enstrophy are conserved; both positive-definite.
- In 3D: energy and helicity are conserved; helicity is not positive-definite.



3D: Kinetic energy is transferred from large to small eddies

2D: Kinetic energy is transferred from small to large eddies

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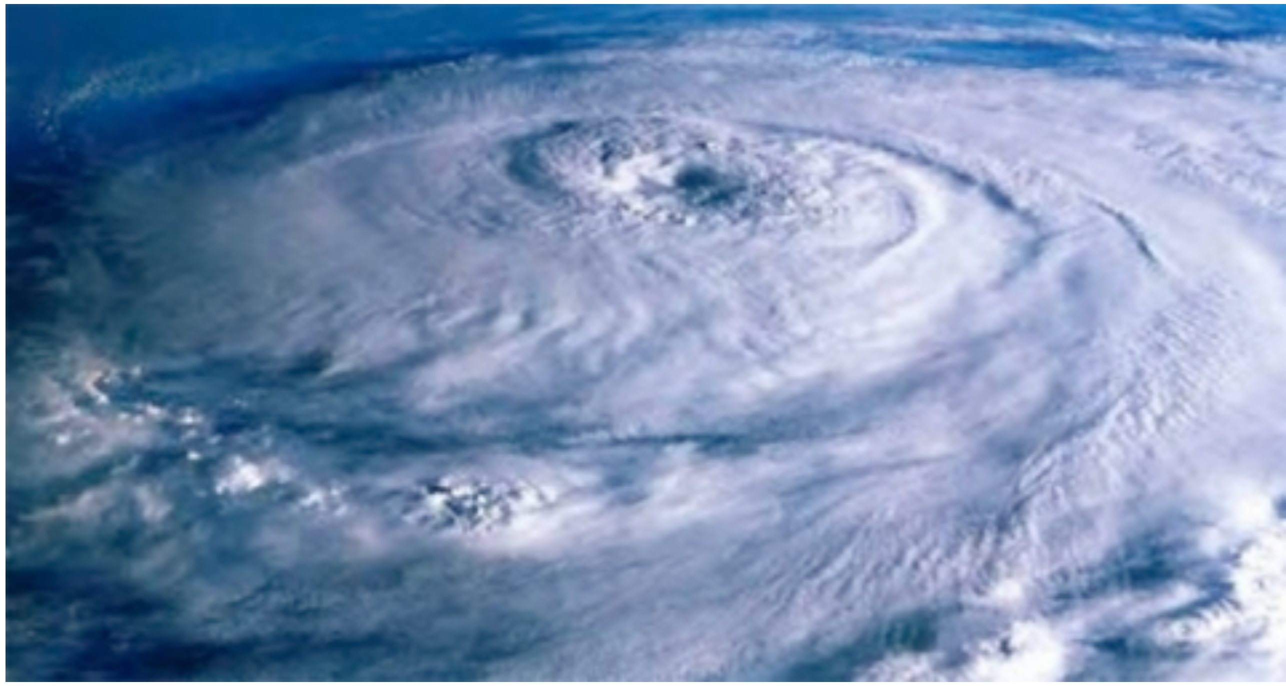
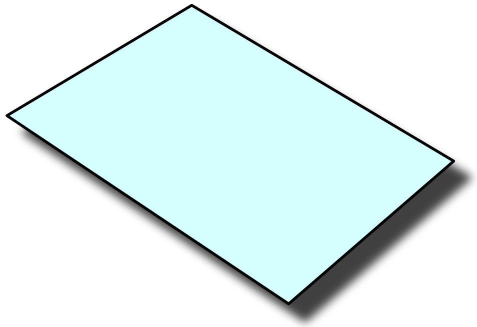
2D or 3D ?

- Many flows are quasi-2D, like thick films, geophysical flows like ocean and atmosphere.
- Physical phenomenas change the dimensionality of the system, like rotation.
- There have been evidence of inverse energy cascade in such systems.
- Also conducting fluids transfer energy to the large scales.



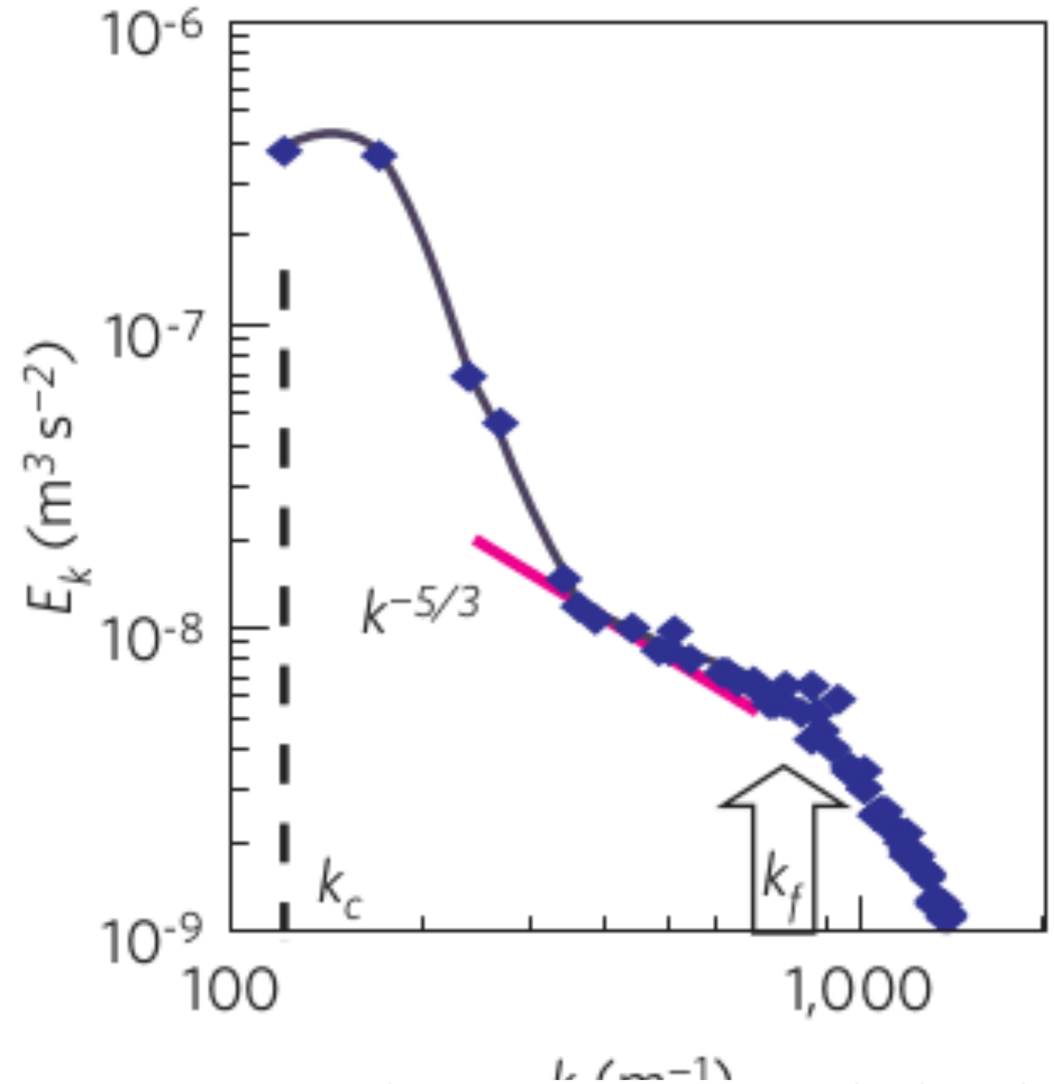


A4 paper ($80 gr/m^2$)
L_1 = 210 mm
L_2 = 297 mm
h = 0.1 mm

Pacific Ocean
N-S = 15000 km
E-W = 19800 km
average depth = 4.28 km

Transition from 3D to 2D

- Dimensional transition occurs in turbulent fluid layers from 3D direct energy cascade to 2D inverse energy cascade as we decrease the thickness of the layer.
- Depending upon the aspect ratio there is a coexistence of inverse and direct cascade.
- Enstrophy (w.w) becomes quasi-invariant as only conserved by large scale dynamics where the flow is two dimensional.
- Inverse cascade develops because of existence of another positive definite conserved quantity.



Upscale energy transfer in thick turbulent fluid layers

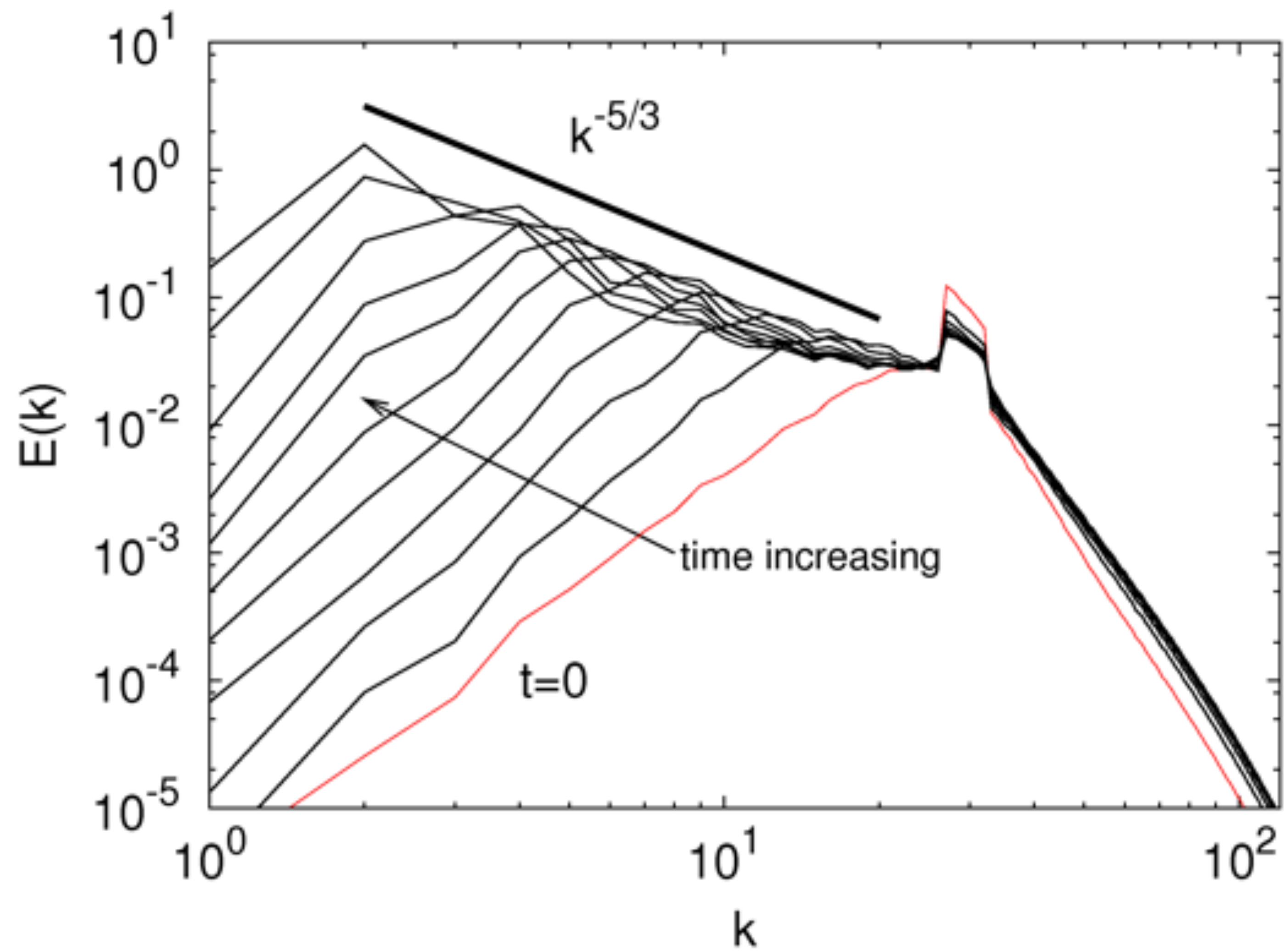
H. Xia[1], D. Byrne[1], G. Falkovich[2] and M. Shats[1]*

Nat. Phys. 7, 321 (2011)

- **If we make helicity positive definite, do we see inverse energy transfer in 3D?**

Inverse energy cascade in 3D

- Making the helicity sign-definite, we observe inverse cascade of energy.



Inverse energy cascade in three-dimensional isotropic turbulence,

Biferale, L., Musacchio, S., Toschi, F., Phys. Rev. Lett. 108, 164501 (2012)

Pseudospectral method for DNS

- We solve the Navier-Stokes equations on a triply periodic box of size 2π.
- Initial velocity field is in Fourier space on a grid of size N^3.
- The nonlocal terms like $\vec{\nabla} \times \vec{u}$, $\nabla^2 \vec{u}$ are evaluated in in Fourier space.
- Terms like $\vec{u} \times \vec{\omega}$ are calculated in real space.
- Switch between real and Fourier space by using the FFT algorithm FFTW.
- For the first step of evolution a Runge-Kutta scheme is used.
- Then an Adams-Bashforth second-order scheme is used.

For an equation of the form

$$\frac{dq}{dt} = -\alpha q + f(t) \tag{1}$$

A second-order Adams-Bashforth scheme

$$q(t + \delta t) = e^{-2\alpha\delta t} q(t - \delta t) + \frac{1 - e^{-2\alpha\delta t}}{2\alpha} \times [3f(t) - f(t - \delta t)]. \tag{2}$$

Navier-Stokes equations

- 3D Navier-Stokes equations in Fourier-space

$$\dot{u}_i(k) + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) N_j(k) = -\nu k^2 u_i(k),$$

$$\text{where} \quad N_i(q) = \sum_{\mathbf{q}=\mathbf{k}+\mathbf{p}} i k_j u_i(k) u_j(p)$$

- In Fourier space, $\mathbf{u}(\mathbf{k}, t)$ has two degrees of freedom since $\mathbf{k} \cdot \mathbf{u}(\mathbf{k}, t) = 0$.
- We chose projection on orthonormal helical waves with definite sign of helicty.

Helical decomposition

- Following Waleffe Phys. Fluids (1992)

$$\mathbf{u}(\mathbf{k}, t) = a^+(\mathbf{k}, t)\mathbf{h}^+(\mathbf{k}) + a^-(\mathbf{k}, t)\mathbf{h}^-(\mathbf{k})$$

  where $\mathbf{h}^{\pm}(\mathbf{k})$ are the complex eigenvectors of the curl operator $i\mathbf{k} \times \mathbf{h}^{\pm}(\mathbf{k}) = \pm k\mathbf{h}^{\pm}(\mathbf{k})$.
- $\mathbf{h}_s^* \cdot \mathbf{h}_t = 2\delta_{st}$; $\mathbf{h}_s^* = \mathbf{h}_{-s}$,
  where s and t could be $+1$ or -1

- Choose $\mathbf{h}^{\pm}(\mathbf{k}) = \hat{\mu}(\mathbf{k}) \times \hat{\mathbf{k}} \pm i\hat{\mu}$,
  where $\hat{\mu}$ is an arbitrary unit vector orthogonal to $\mathbf{k}$
- reality of the velocity field requires $\hat{\mu}(\mathbf{k}) = -\hat{\mu}(-\mathbf{k})$
- Such requirement is satisfied, e.g., by the choice $\hat{\mu}(\mathbf{k}) = \mathbf{z} \times \mathbf{k}/||\mathbf{z} \times \mathbf{k}||$, with $\mathbf{z}$ an arbitrary vector.

Helically decimated Navier-Stokes equations

- Decimated Navier-Stokes equations in Fourier space:

$$\partial_t \mathbf{u}^{\pm}(\mathbf{k},t) = \mathcal{P}^{\pm}(\mathbf{k})\mathbf{N}_{u^{\pm}}(\mathbf{k},t) + \nu k^2 \mathbf{u}^{\pm}(\mathbf{k},t) + \mathbf{f}^{\pm}(\mathbf{k},t)$$

  where ν is kinematic viscosity and $\mathbf{f}$ is external forcing.
- The nonlinear term containing all triadic interactions

$$\mathbf{N}_{u^{\pm}}(\mathbf{k},t) = \mathcal{F}T(\mathbf{u}^{\pm}\cdot\nabla\mathbf{u}^{\pm} - \nabla p)$$

- Projection operator:

$$\mathcal{P}^{\pm}(\mathbf{k}) \equiv \frac{\mathbf{h}^{\pm}(\mathbf{k}) \otimes \mathbf{h}^{\pm}(\mathbf{k})^*}{\mathbf{h}^{\pm}(\mathbf{k})^* \cdot \mathbf{h}^{\pm}(\mathbf{k})}$$

$$\mathbf{u}^{\pm}(\mathbf{k},t) = \mathcal{P}^{\pm}(\mathbf{k})\mathbf{u}(\mathbf{k},t)$$

$$\mathbf{u}(\mathbf{k},t) = \mathbf{u}^{+}(\mathbf{k},t) + \mathbf{u}^{-}(\mathbf{k},t)$$

- Eneregy $E(t) = \sum_{\mathbf{k}} |\mathbf{u}^{+}(\mathbf{k},t)|^2 + |\mathbf{u}^{-}(\mathbf{k},t)|^2$.
- Helicity $\mathcal{H}(t) = \sum_{\mathbf{k}} k(|\mathbf{u}^{+}(\mathbf{k},t)|^2 - |\mathbf{u}^{-}(\mathbf{k},t)|^2)$.

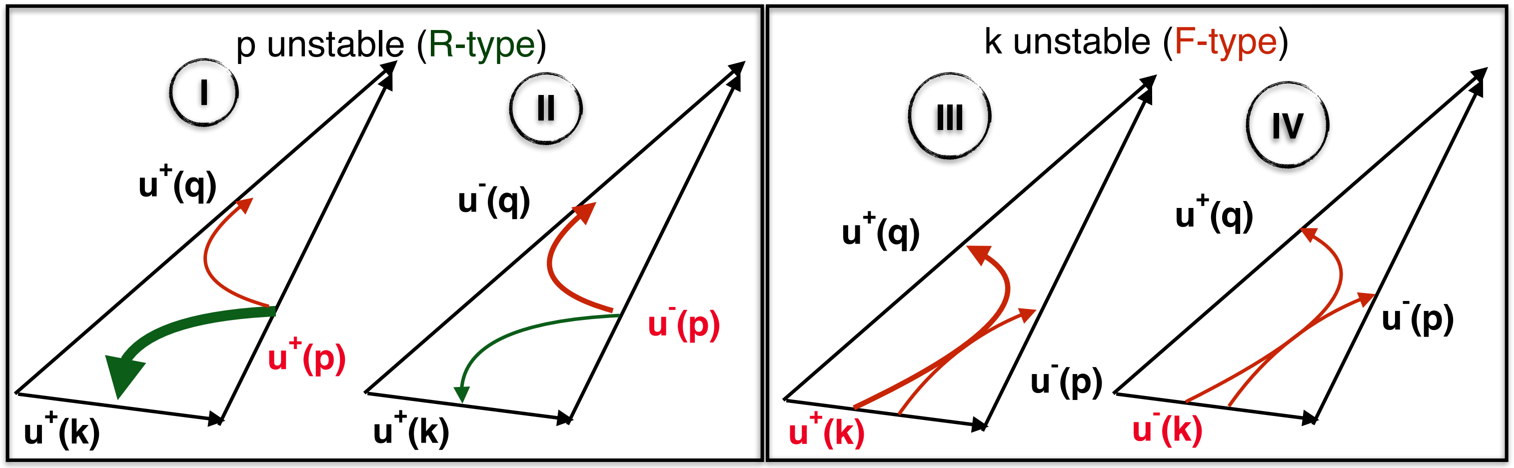
Classes of triadic interactions in NS equations

R-type: When large wavenumbers have same sign, middle one is unstable and could transfer energy to both small and large wavenumbers;

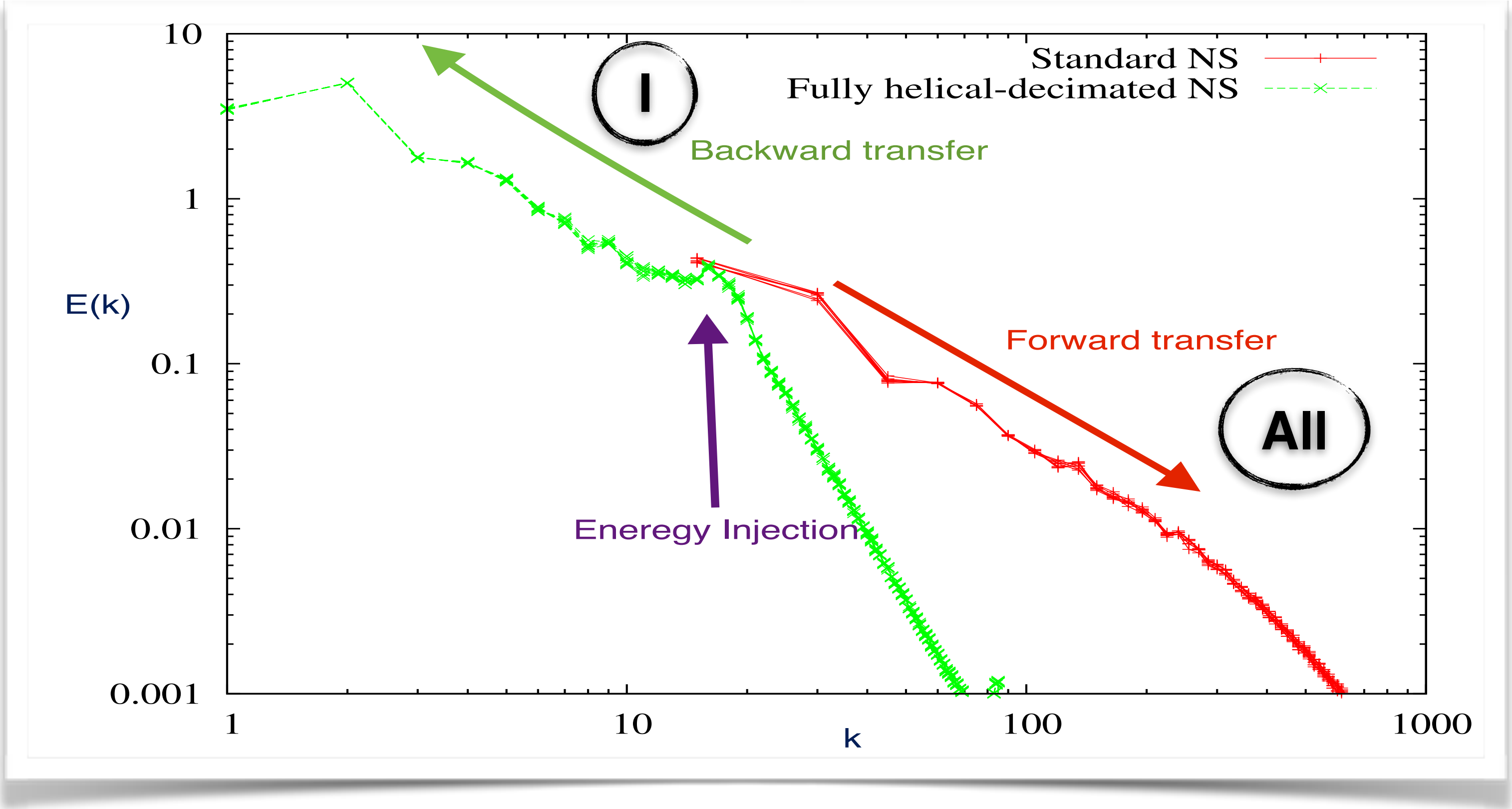
- predominantly to the smallest wavenumber if it has the same sign **[Class-I (+, +, +)]**.
- mixed transfer if smallest wavenumber has the opposite sign [Class-II (+, -, -)].

F-type: When large wavenumbers have opposite sign, smallest one is unstable and could transfer energy only to large wavenumbers, for both Class-III (+, -, +) and Class-IV (-, -, +).

$$\mathbf{N}_{\mathbf{u}^{\pm}}(\mathbf{q}) = \mathcal{FT}\left[\mathbf{u}^{\pm}(\mathbf{k}) \cdot \nabla \mathbf{u}^{\pm}(\mathbf{p})\right]; \mathbf{q} = \mathbf{k} + \mathbf{p};\ k \leq p \leq q$$



- Energy and helicity are conserved for each individual triad.
- Triads with only u^+, i.e. Class-I, lead to reversal of energy cascade.
- Energy spectra in the inverse cascade regime shows a $k^{-5/3}$ slope.



Partial Helical-decimation

What happens in between??

when we give different weights to different class of triads…

- Modified projection operator:

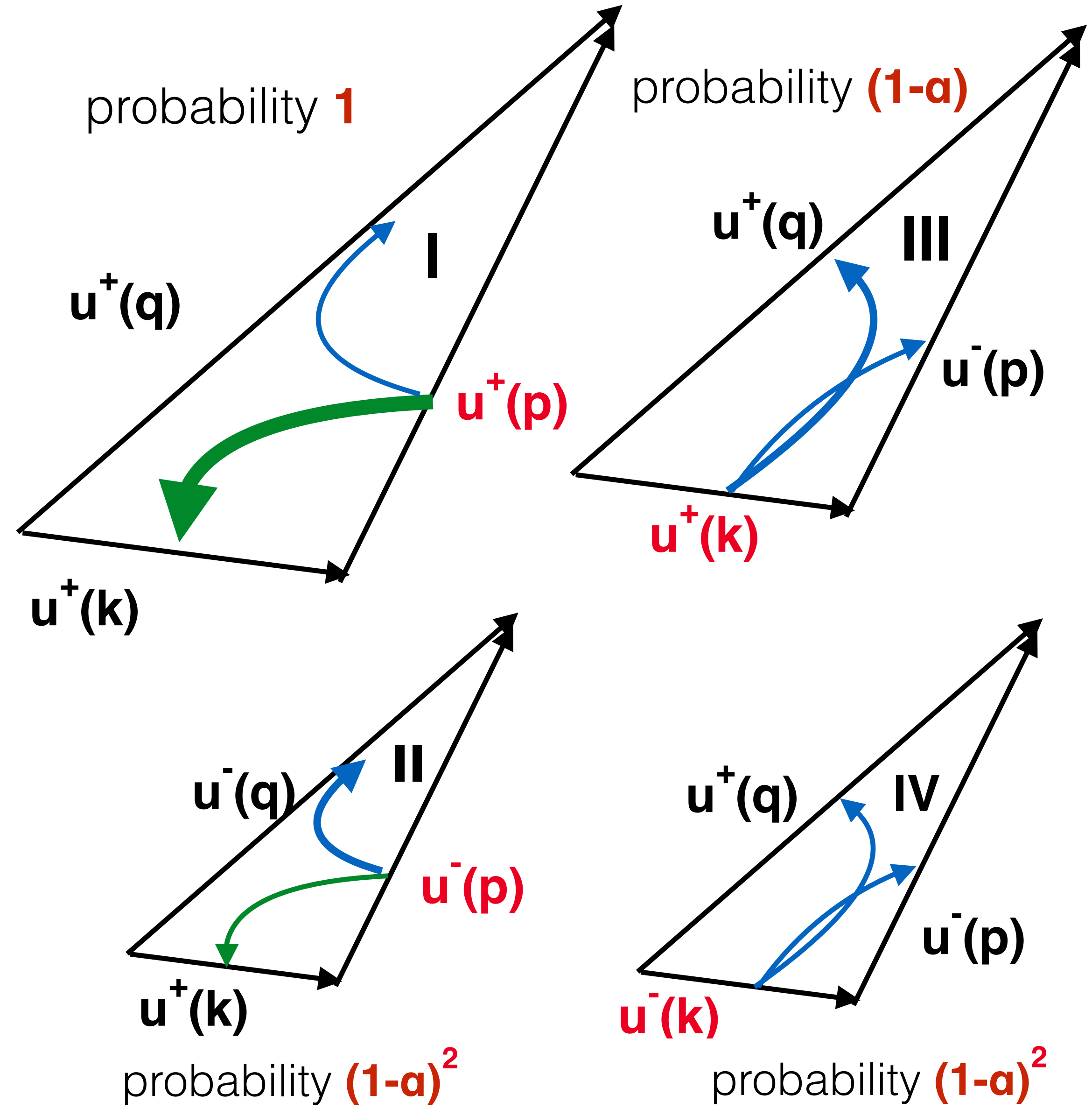
$$\mathcal{P}^{+}_{\alpha}(\mathbf{k})\mathbf{u}(\mathbf{k},t) = \mathbf{u}^{+}(\mathbf{k},t) + \theta_{\alpha}(\mathbf{k})\mathbf{u}^{-}(\mathbf{k},t)$$

  where $\theta_\alpha(\mathbf{k})$ is 0 with probability α and is 1 with probability $1-\alpha$.

- We consider triads of Class-I with probability 1, Class-III with probability $1-\alpha$ and Class-II and Class-IV with probability $(1-\alpha)^2$.

- $\alpha = 0 \to$ Standard Navier-Stokes.
  $\alpha = 1 \to$ Fully helical-decimated NS.

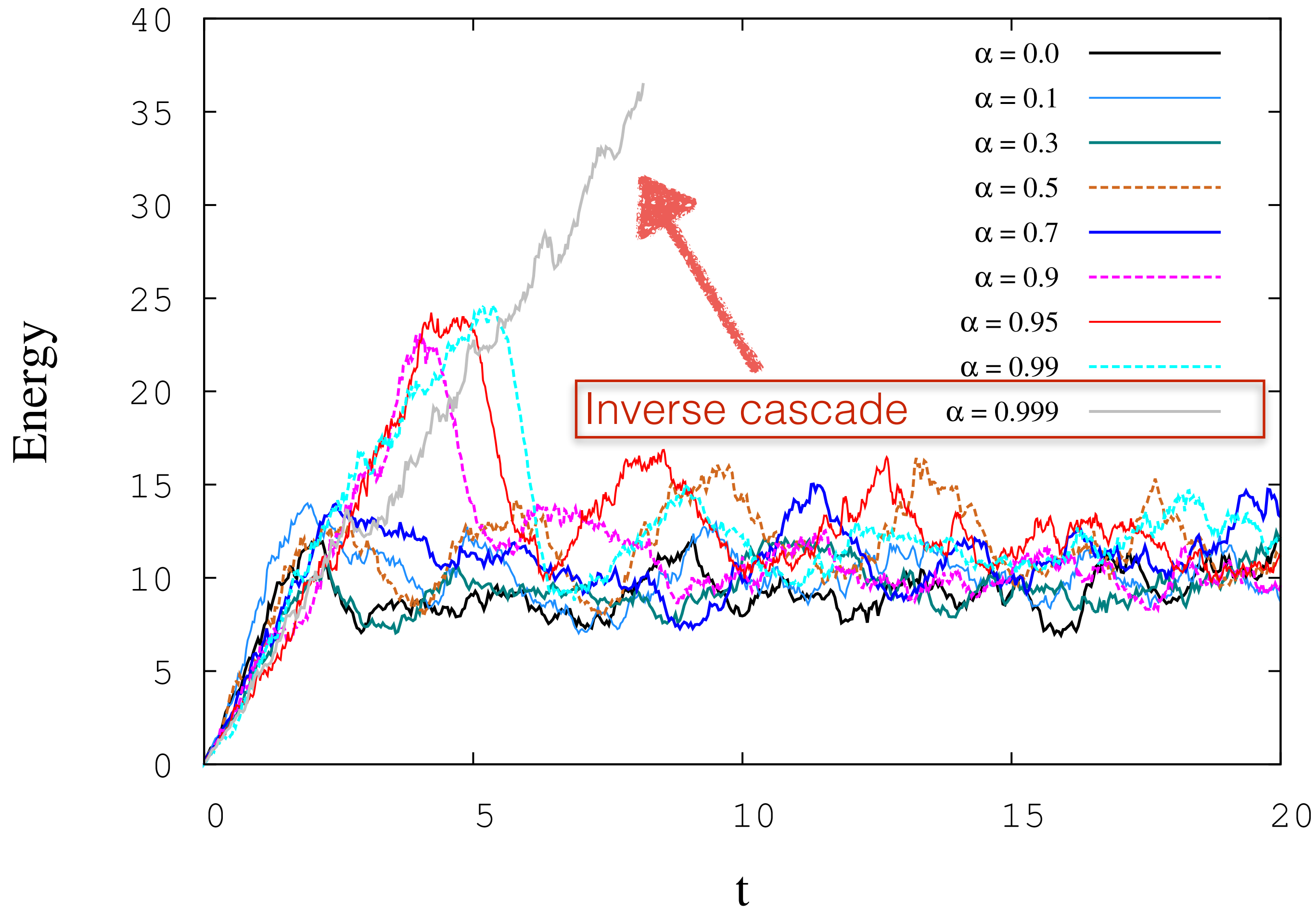
- Critical value of α at which forward cascade of energy stops?
  alternatively, inverse cascade of energy stops if forced at small scales.

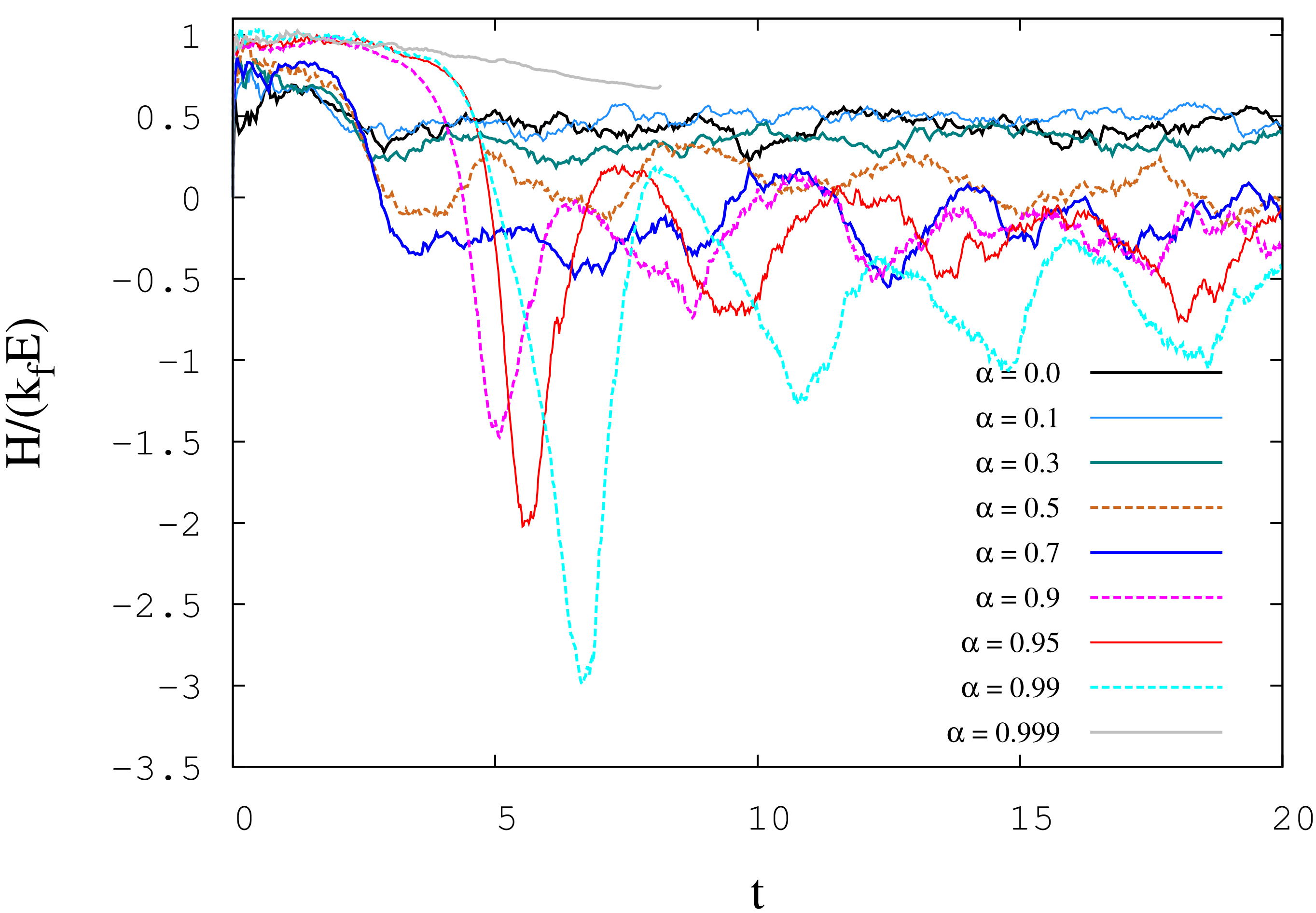


$$\mathbf{N}_{\mathbf{u}^{\pm}}(\mathbf{q}) = \mathcal{FT}\left[\mathbf{u}^{\pm}(\mathbf{k})\cdot\nabla\mathbf{u}^{\pm}(\mathbf{p})\right];\ \mathbf{q}=\mathbf{k}+\mathbf{p};\ k\le p\le q$$

Evolution of Energy and helicity

- Pseudo-spectral DNS on a triply periodic cubic domain of size $L = 2\pi$ with resolutions up to 512^3 collocation points.

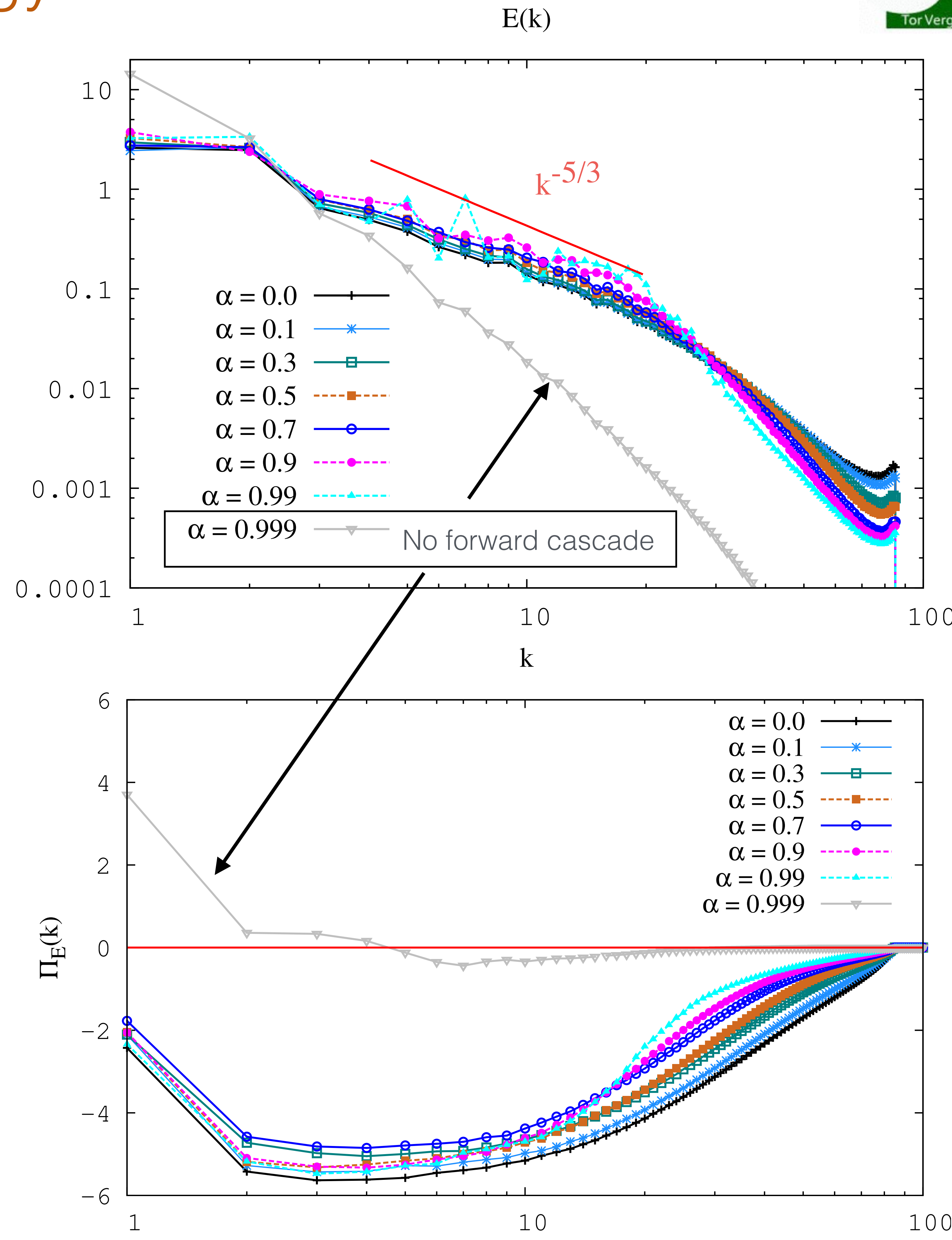




- The peaks suggest the building up of the energy at forced large scales before being able to transfer to the small scales.
- The cascade of energy starts only when helicity becomes active, i.e., modes with negative helicity becomes energetic.
- With increase in α the peak grows, a signature of inverse cascade.

Robustness of energy cascade

- Spectra for all values of α showing $k^{-5/3}$ suggest the forward cascade of to be strongly robust.
- Unless we kill almost all the modes of one helicity-type energy always finds a way to reach small scales.
- The energy flux also remains unaffected by the decimation until α is very close to 1.
- **Critical value of α is ~ 1 !**



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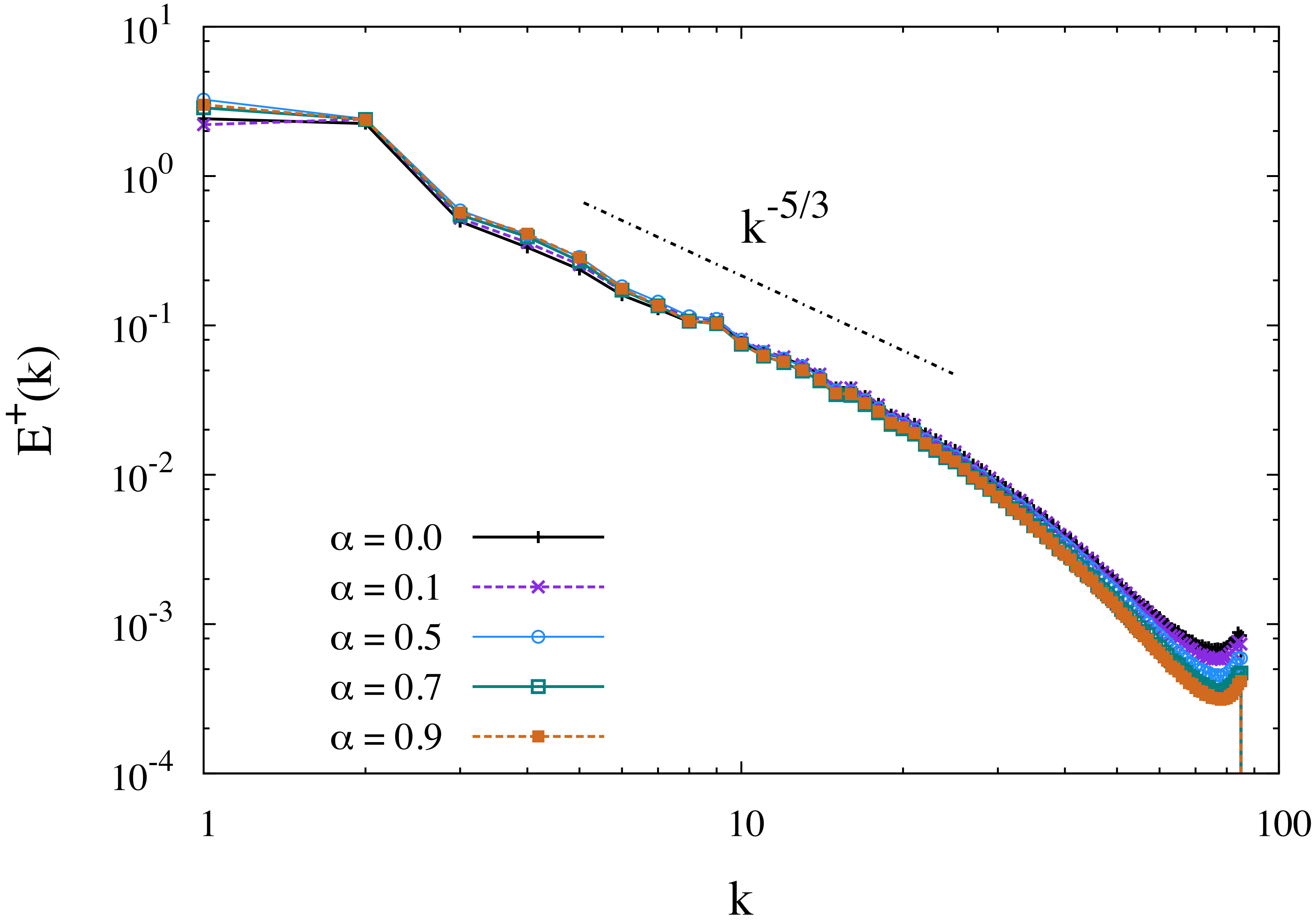
Energy in the positive helical modes

Chen, Phys. Fluids 2003

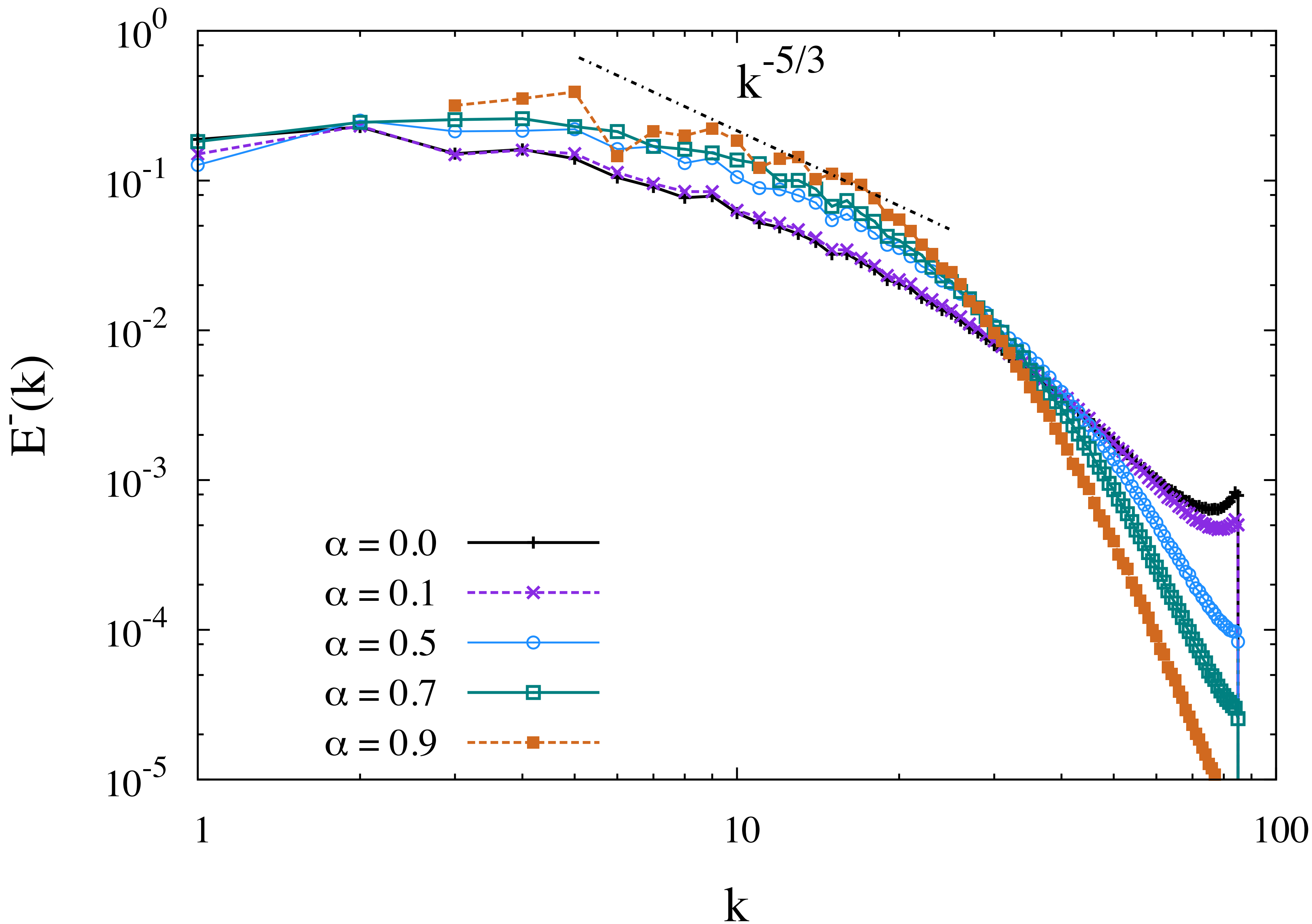
$$E^{\pm}(k) \sim C_1 \epsilon_E^{2/3} k^{-5/3} \left[1 \pm C_2 \left(\frac{\epsilon_H}{\epsilon_E} \right) k^{-1} \right],$$

where ϵ_E is the mean energy dissipation rate
and ϵ_H is the mean helicity dissipation rate.



- The $E^+(k)$ does not change with decimation.

Energy in the negative helical modes



- $E^-(k)$ shows that as we have fewer negative helical modes, they become more energetic in the inertial range of scales.

- The forward cascade of energy is though the triads of class-III where two large wavenumber modes have opposite sign of helicity.

- The energy flux is carried by correlations of type

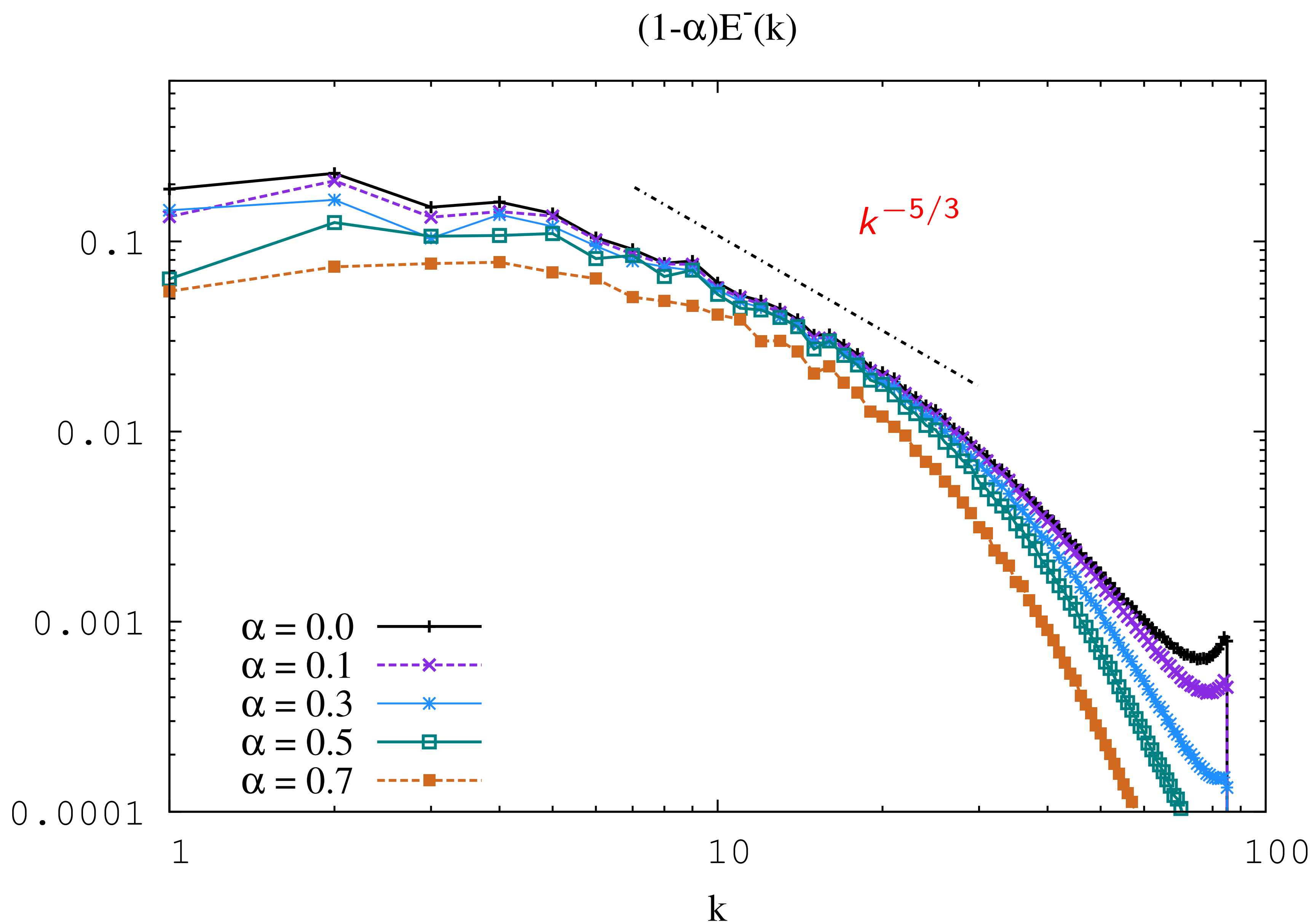
$$S(k|p,q) = \langle (\boldsymbol{k}\cdot u^-_{\boldsymbol{q}})(u^+_{\boldsymbol{k}}\cdot u^+_{\boldsymbol{p}})\rangle + \langle (\boldsymbol{k}\cdot u^+_{\boldsymbol{p}})(u^+_{\boldsymbol{k}}\cdot u^-_{\boldsymbol{q}})\rangle.$$

- This involves two positive helical modes and one negative helical modes.

- To maintain the constant flux, $u^-(k)$ must be rescaled by (1-α). since $u^-(k)$ exists with probability (1-α).

Reaction of negative helical modes

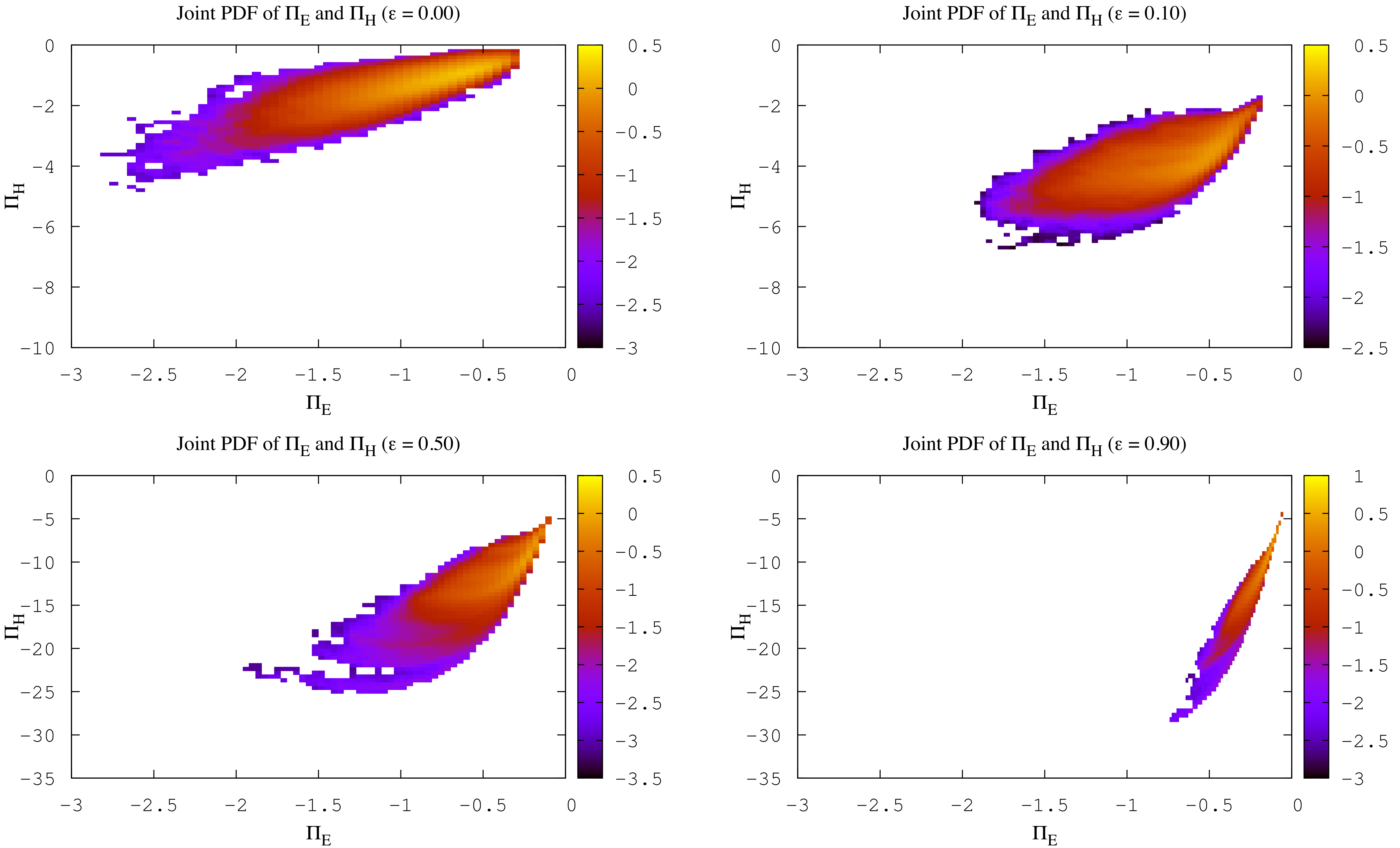
$$u^-_{\boldsymbol{k}} \to u^-_{\boldsymbol{k}}/(1-\alpha),$$
$$E^-(k) = \sum_{|\boldsymbol{k}|=k} (1-\gamma_{\boldsymbol{k}})|u^-_{\boldsymbol{k}}|^2 \to E^-(k)/(1-\alpha),$$



- Invariance of parity is restored through scaling of $E^-(k)$ by the factor $(1-\alpha)$.

Flux in the dissipation scales

Joint PDF of helicity and energy fluxes



The helicity flux attains higher values whereas the energy flux depletes with increasing ϵ.

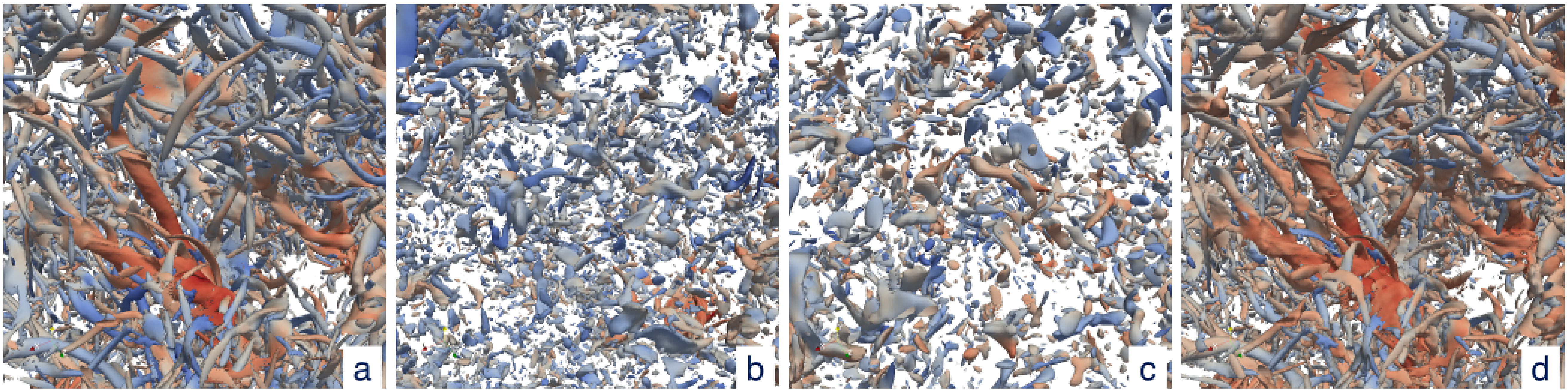


FIG. 3: (color online) iso-vorticity surfaces for: (a) $\alpha = 0$, (b) $\alpha = 0.5$, (c) $\alpha = 0.9$. Last plot (d) is obtained applying the projection with $\alpha = 0.5$ on the original NSE fields without any dynamical decimation. Color palette is proportional to the intensity of the helicity.

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Summary

- As we increase decimation of the modes with negative helicity (α), the contribution of triads leading to inverse energy cascade grows.

- The forward cascade of energy is very robust in 3D turbulence. It requires only a few negative modes to act as catalyst to transfer energy forward.

- Only when α is very close to 1, i.e., we decimate almost all modes of one helical sign, inverse energy cascade takes over the forward cascade.

- We observe a strong tendency to recover parity invariance even in the presence of an explicit parity-invariance symmetry breaking ($\alpha > 0$).

- What about abrupt symmetry breaking at some k_c?
  - can we stop the cascade by killing all negatives modes from $k > k_c$?
  - or can we start it at our wish (killing all modes up to k_c)?
- What about intermittency in the forward cascade regime at changing α?

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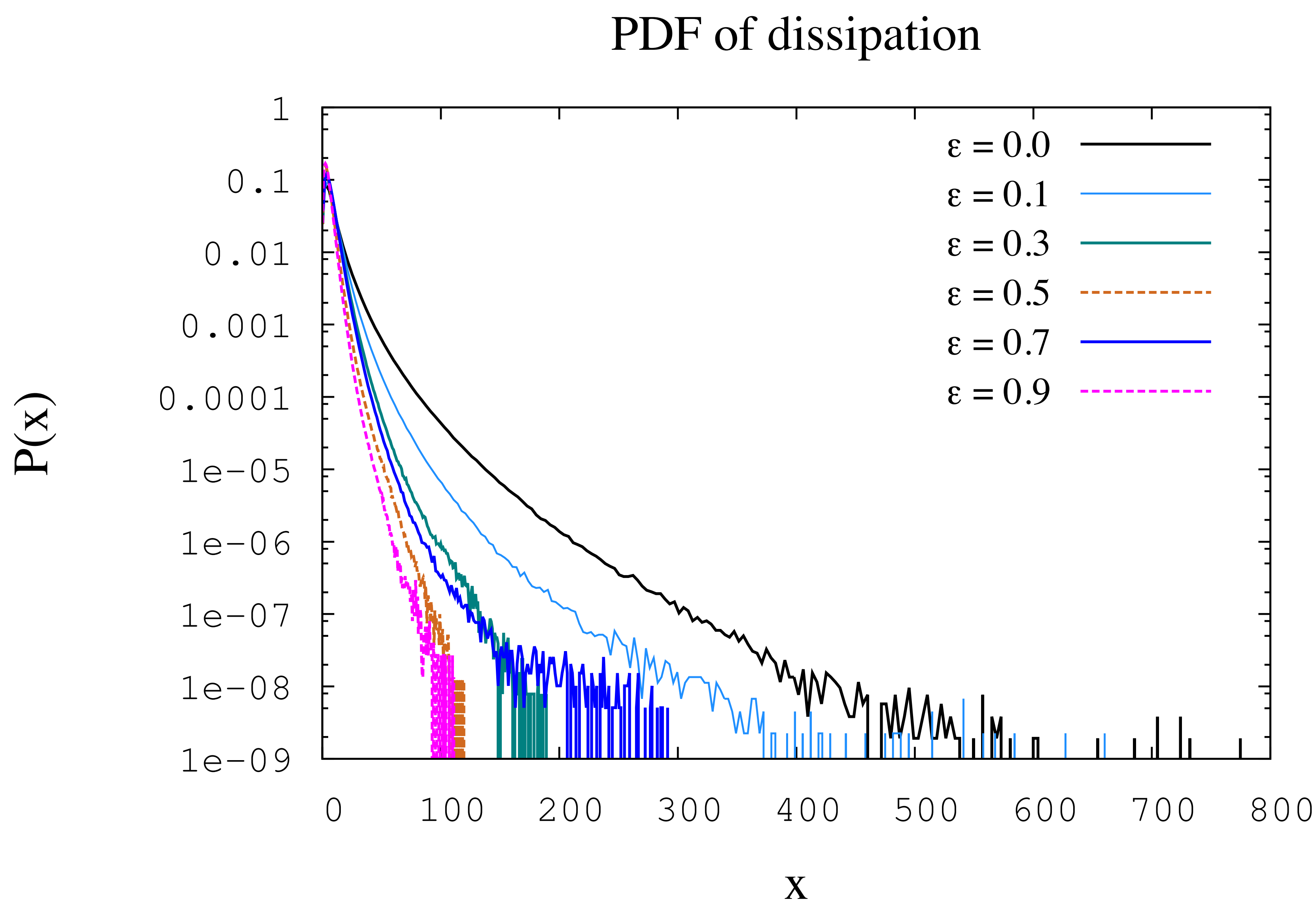
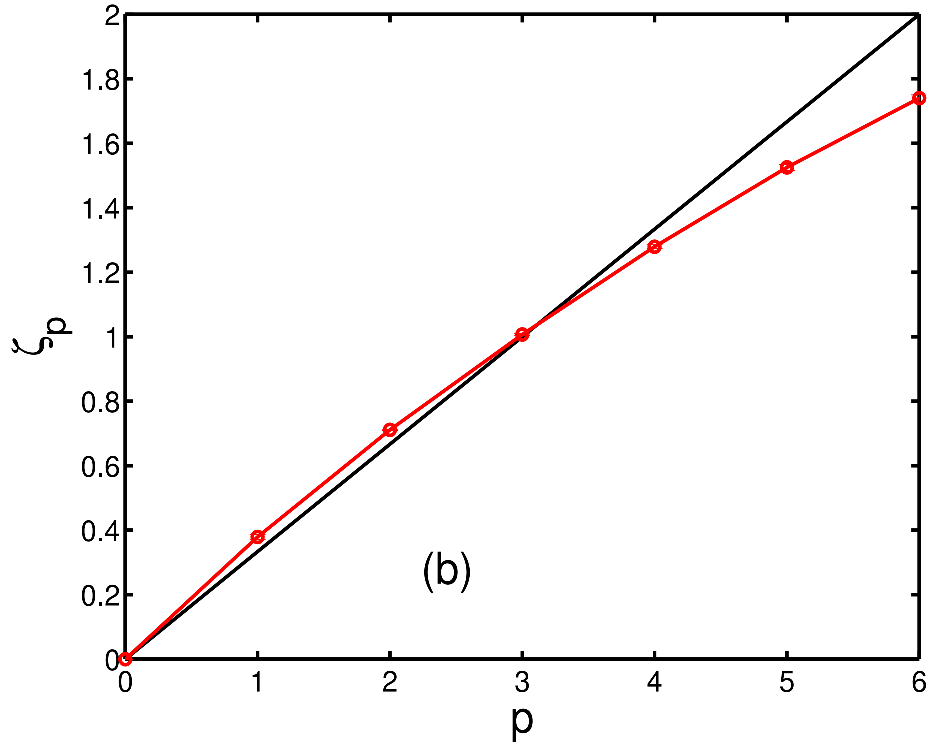
local energy dissipation rate



Comparison of PDFs of local energy dissipation rates show reduction of longer tails with increase in fraction of decimation ϵ. Less of extreme dissipation events show decrease in intermittency with increaseing ϵ

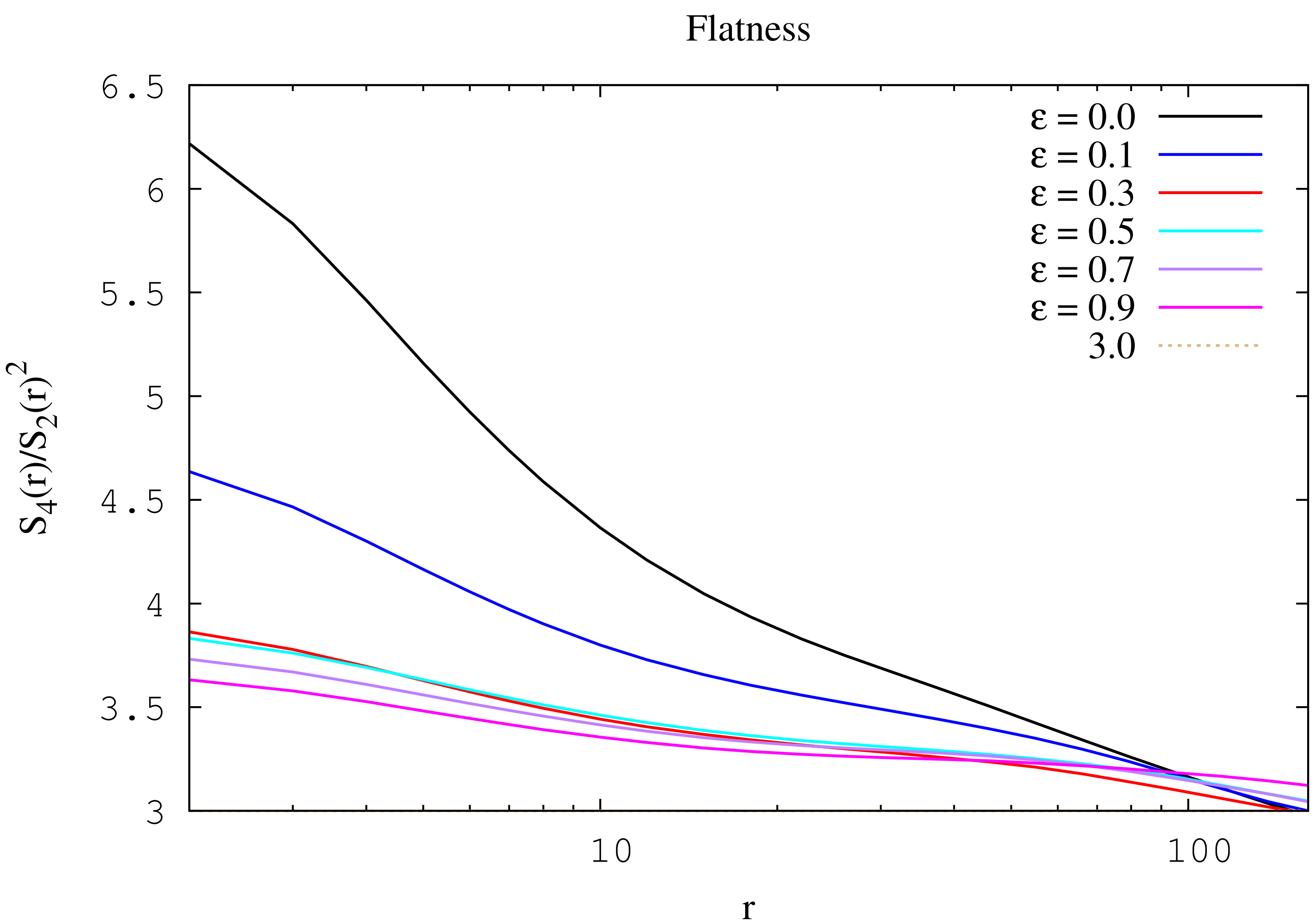
Structure functions

- Order-p equal-time, longitudinal velocity structure functions
  $S_p(r) \equiv \langle|\delta u_{\|}(\mathbf{x}, r)|^p\rangle$
  where $\delta u_{\|}(\mathbf{x}, r) \equiv [\mathbf{u}(\mathbf{x}+\mathbf{r}, t) - \mathbf{u}(\mathbf{x}, t)] \cdot \frac{\mathbf{r}}{r}$
- In the inertial range we see the universal scaling $S_p(r) \sim r^{\zeta_p}$



- Deviations from Kolmogorov scaling $\zeta_p^{K41} = p/3$ shows present intermittency.
- Extended Self-Similarity: ζ_p/ζ_3.

Measure of intermittency: Flatness $F_4(r) = S_4(r)/[S_2(r)]^2$



- Measure of flatness shows the small scale intermittency reduces significantly when 10% of $\mathbf{u}^-$ modes are killed.
- It reduces further and seems saturated with increase in ϵ

Thank you!

- *On the role of helicity for large-and small-scales turbulent fluctuations*, G Sahoo, F Bonaccorso, L Biferale - arXiv preprint arXiv:1506.04906, 2015.

- Inverse energy cascade in three-dimensional isotropic turbulence, L Biferale, S Musacchio, F Toschi, Phys. Rev. Lett. 108, 164501 (2012)