Helicity in three dimensional turbulence

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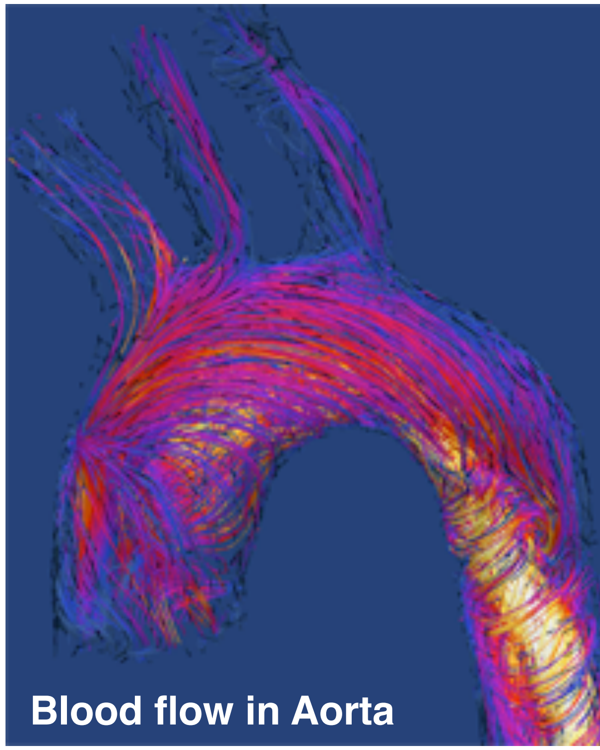
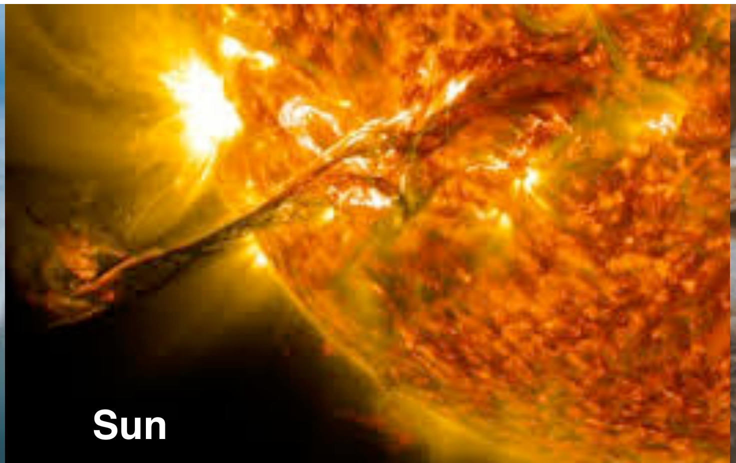
Outline

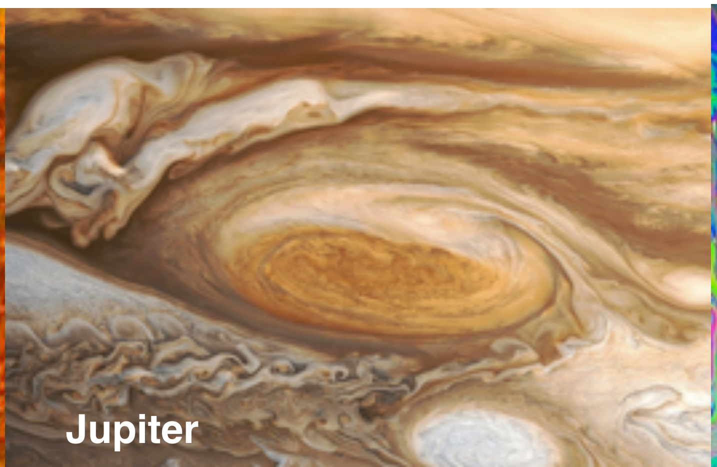
- Introduction
- Helically Decimated Navier-Stoke's equation
- Energy transfer and helicity
- Large and small scale structures

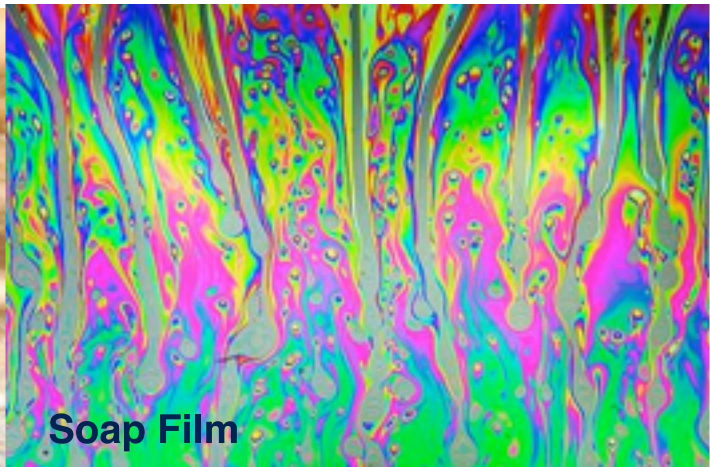
Turbulence everywhere

- All environmental flows are turbulent,
- Atmospheric boundary layer, Ocean Currents, interstellar clouds, flow of gas and oil in pipe lines, combustion in engines,









Conserved quantities

- Navier-Stoke's equations for incompressible flow

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u} + \mathbf{f},$$

$$\nabla \cdot \mathbf{u} = 0.$$

- Energy

$$E = \int \mathbf{u}(\mathbf{x}) \cdot \mathbf{u}(\mathbf{x}) d^3 x$$

- Helicity

$$H = \int \mathbf{u}(\mathbf{x}) \cdot \omega(\mathbf{x}) d^3 x$$

Betchov 1961

- are conserved in un-forced and non-dissipative flows.
- Helicity is a pseudoscalar: changes sign under parity.
- Unlike energy, helicity is not positive definite.

Kolmogorov theory (1941)

- For very high *Re*, the statistical properties of eddies of sizes in the inertial range of scales are
  - independent of the forced and dissipative scales, and are locally homogeneous and isotropic.
  - universally and uniquely determined by the length scale l, viscosity ν, and the rate of energy dissipation ε.
- Characteristic velocity of an eddy of size l scales as $u_l \sim (l\varepsilon)^{-1/3}$.
- Energy spectrum in the inertial range $E(k) \sim \varepsilon^{2/3} k^{-5/3}$,

$$\text{for} \quad L^{-1} << k << \eta^{-1}; \quad \eta = \left(\frac{\nu^3}{\varepsilon}\right)^{1/4}$$

- Self-Similarity hypothesis: Structure functions of p-th order scales as

$$S_p(l) = \langle \delta u_l^p \rangle \sim (\varepsilon l)^{p/3},$$
$$\delta u_l = [\mathbf{u}(\mathbf{r}+\mathbf{l}) - \mathbf{u}(\mathbf{r})] \cdot \frac{\mathbf{l}}{l}.$$

Effects of helicity

- Nonlinearity: Since u.w is nonzero, there could be decrease in the nonlinearity u x w. e.g. linear Bertram flows with maximal helicity.
- Nonlocality: Nonzero u.w also implies stronger coupling between large and small scales, i.e. increasing non-locality. e.g. production of large scale magnetic fields in conductive fluids.
- Self-production: At a very high *Re*, there is a growth of helicity at the small scales, even though total helicity remains finite, because of the symmetry.

Effects of helicity

- Energy gets distributed among scales by the nonlinear term in Navier-Stoke's equation and assuming a constant energy flux we observe the scaling behaviour

$$\delta u_l = [\mathbf{u}(\mathbf{r}+\mathbf{l}) - \mathbf{u}(\mathbf{r})] \cdot \frac{\mathbf{l}}{l} \sim \varepsilon^{1/3} l^{1/3}$$

$$\varepsilon = 2\nu \langle \partial_j u_i \partial_i u_j \rangle$$

- By similar dimensionality argument and assuming a constant helicity flux *h* [LT^{-3}], we obtain

$$\delta u_l = [\mathbf{u}(\mathbf{r}+\mathbf{l}) - \mathbf{u}(\mathbf{r})] \cdot \frac{\mathbf{l}}{l} \sim h^{1/3} l^{2/3}$$

$$h = 2\nu \langle \partial_j u_i \partial_i \omega_j \rangle$$

- But such a scaling is not observed. Why?

Effects of helicity

- There is no purely helicity dominated turbulence since both energy and helicity cascade to the small scales.

- For the joint cascade of energy and helicity we expect

$$\delta u_l = [\mathbf{u}(\mathbf{r}+\mathbf{l}) - \mathbf{u}(\mathbf{r})] \cdot \frac{\mathbf{l}}{l} \sim \varepsilon^{\beta} h^{\gamma} l^{\delta}$$

- But then, we can not determine the exponents, uniquely, from dimensionality argument.

- Presence of helicity changes the geometrical structure in a subtle way, which could not be captured by simple dimensional analysis.

Exact results

For pure energy cascade

$$\langle \delta u_L^3(\mathbf{r}) \rangle = -\frac{4}{5}\varepsilon r$$

For pure helicity cascade

$$\langle \delta u_L(\mathbf{r}) \delta \mathbf{u}(\mathbf{r}) \cdot \delta \omega(\mathbf{r}) \rangle - \frac{1}{2} \langle \delta \omega_L(\mathbf{r}) \delta \mathbf{u}(\mathbf{r}) \cdot \delta \mathbf{u}(\mathbf{r}) \rangle = -\frac{4}{3} h r$$

Where

$$\delta \mathbf{u}(\mathbf{r}) \equiv \mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x}); \delta u_L(\mathbf{r}) \equiv \delta \mathbf{u}(\mathbf{r}) \cdot \frac{\mathbf{r}}{r}$$

$$\delta \omega(\mathbf{r}) \equiv \omega(\mathbf{x}+\mathbf{r}) - \omega(\mathbf{x}); \delta \omega_L(\mathbf{r}) \equiv \delta \omega(\mathbf{r}) \cdot \frac{\mathbf{r}}{r}$$

Dimensionality

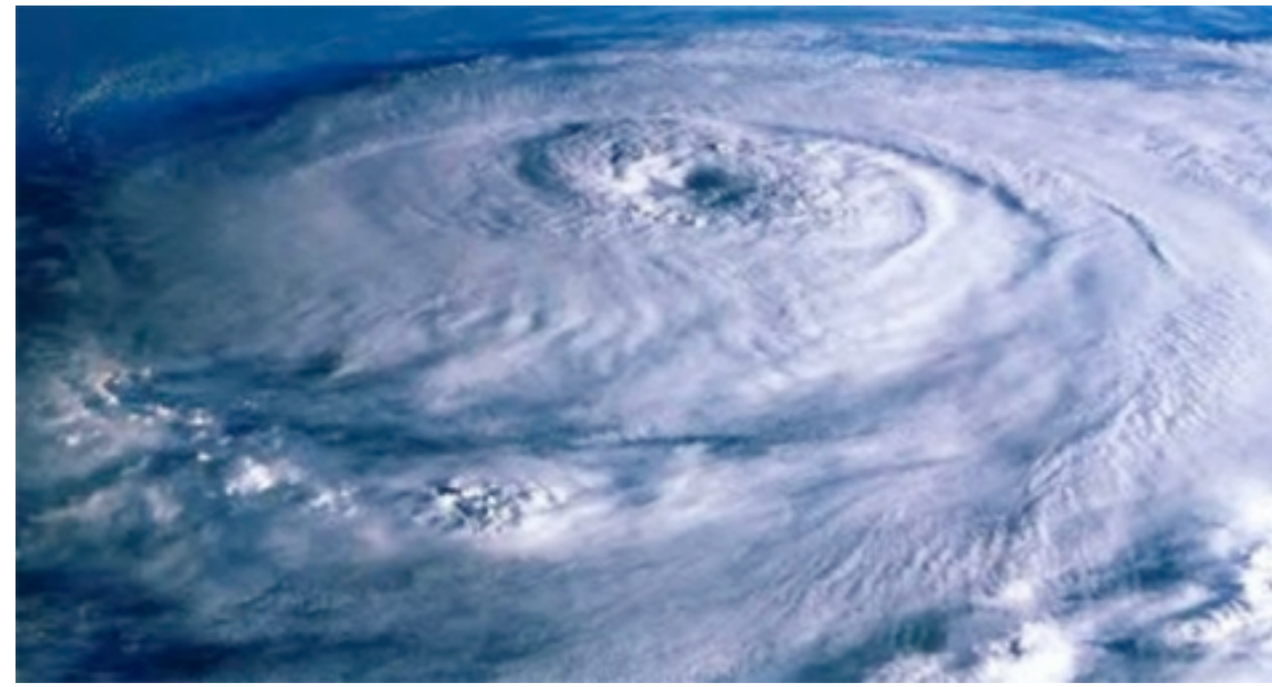
- The direction of cascade is determined by positive-definite inviscid invariants.
- In 2D: energy and enstrophy are conserved; both positive-definite.
- In 3D: energy and helicity are conserved; helicity is not positive-definite.

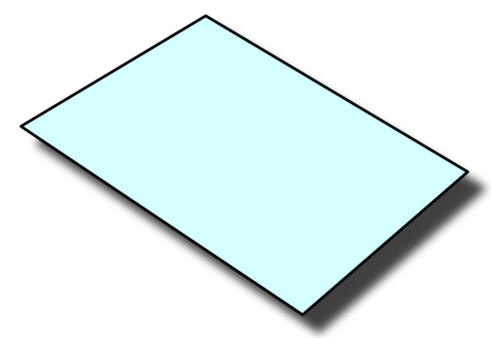
3D: Kinetic energy is transferred from large to small eddies

2D: Kinetic energy is transferred from small to large eddies

2D or 3D ?

- Many flows are quasi-2D, like thick films, geophysical flows like ocean and atmosphere.
- Physical phenomenas change the dimensionality of the system, like rotation.
- There have been evidence of inverse energy cascade in such systems.
- Also conducting fluids transfer energy to the large scales.





A4 paper ($80 gr/m^2$)
L_1= 210 mm
L_2 = 297 mm
h = 0.1 mm

Pacific Ocean
N-S = 15000 km
E-W = 19800 km
average depth = 4.28 km

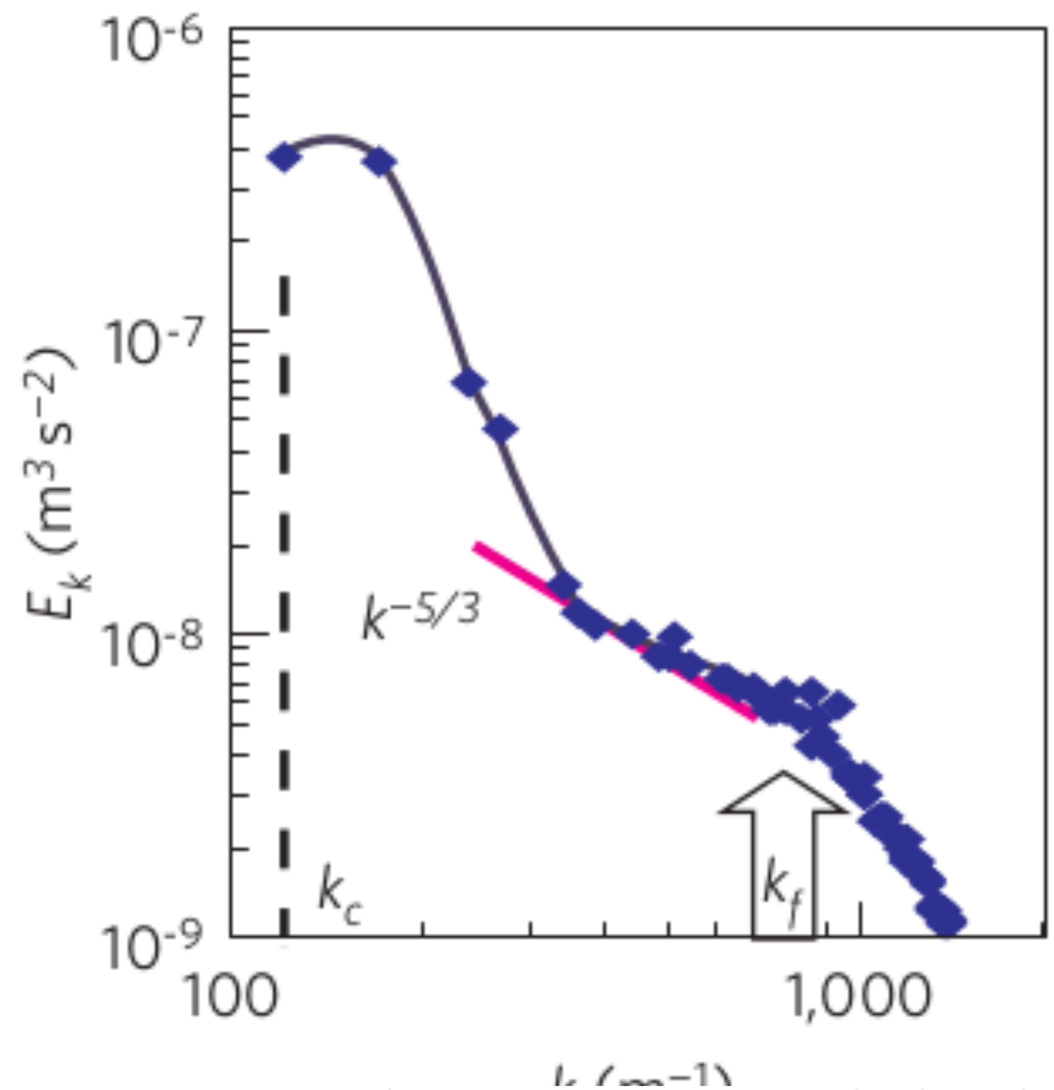
Transition from 3D to 2D

- Dimensional transition occurs in turbulent fluid layers from 3D direct energy cascade to 2D inverse energy cascade as we decrease the thickness of the layer.

- Depending upon the aspect ratio there is a coexistence of inverse and direct cascade.

- Enstrophy (w.w) becomes quasi-invariant as only conserved by large scale dynamics where the flow is two dimensional.

- Inverse cascade develops because of existence of another positive definite conserved quantity.



Upscale energy transfer in thick turbulent fluid layers

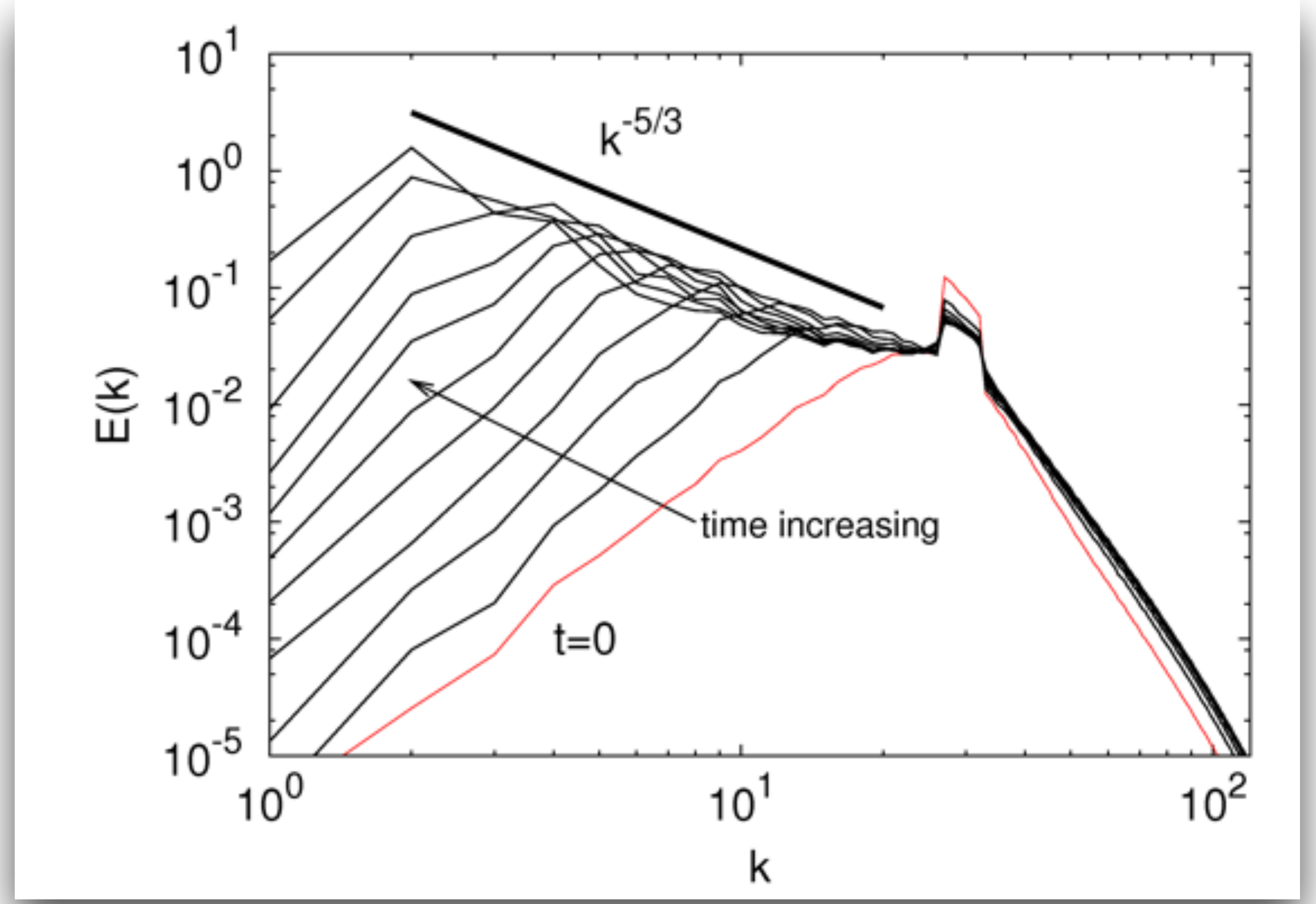
H. Xia[1], D. Byrne[1], G. Falkovich[2] and M. Shats[1]*

Nat. Phys. 7, 321 (2011)

- **If we make helicity positive definite, do we see inverse energy transfer in 3D?**

Inverse energy cascade in 3D

- Making the helicity sign-definite, we observe inverse cascade of energy.



Inverse energy cascade in three-dimensional isotropic turbulence,

Biferale, L., Musacchio, S., Toschi, F., Phys. Rev. Lett. 108, 164501 (2012)

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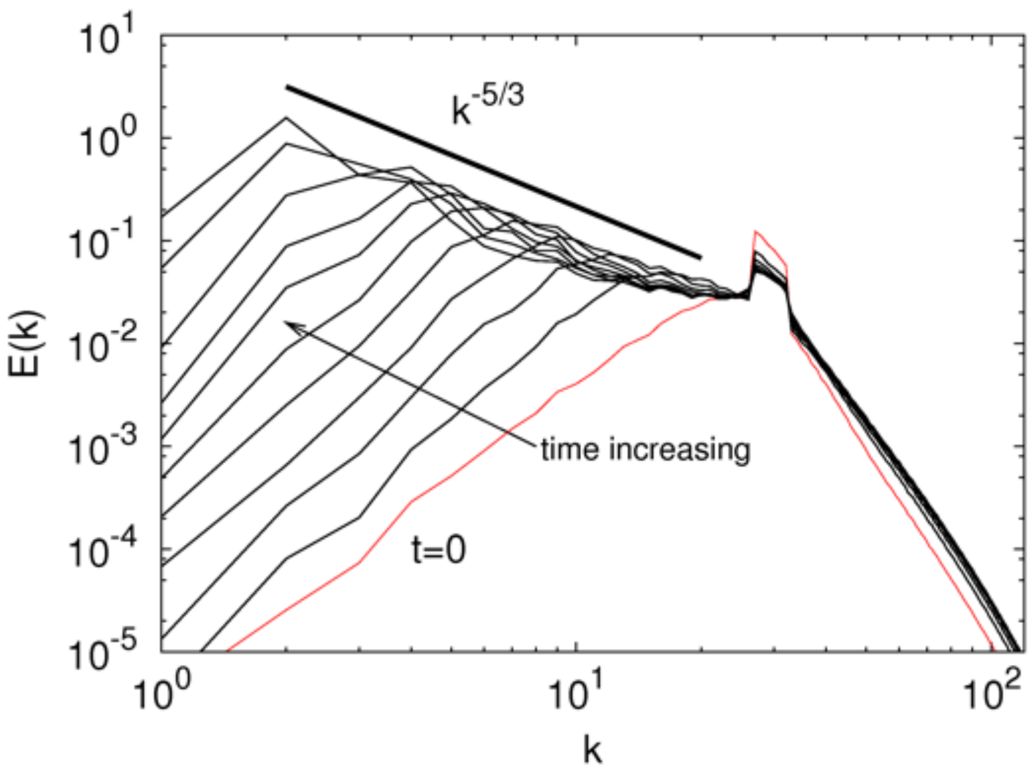

Inverse energy cascade in three-dimensional isotropic turbulence,

Biferale, L., Musacchio, S., Toschi, F., Phys. Rev. Lett. 108, 164501 (2012)

Can direct and inverse cascade of energy co-exist?

Direct Numerical Simulations

Pseudospectral method for DNS

- We solve the Navier-Stokes equations on a triply periodic box of size 2π.
- Initial velocity field is in Fourier space on a grid of size N^3.
- The nonlocal terms like $\vec{\nabla} \times \vec{u}$, $\nabla^2 \vec{u}$ are evaluated in in Fourier space.
- Terms like $\vec{u} \times \vec{\omega}$ are calculated in real space.
- Switch between real and Fourier space by using the FFT algorithm FFTW.
- For the first step of evolution a Runge-Kutta scheme is used.
- Then an Adams-Bashforth second-order scheme is used.

For an equation of the form

$$\frac{dq}{dt} = -\alpha q + f(t) \qquad (1)$$

A second-order Adams-Bashforth scheme

$$q(t+\delta t) = e^{-2\alpha\delta t} q(t-\delta t) + \frac{1 - e^{-2\alpha\delta t}}{2\alpha} \times [3f(t) - f(t-\delta t)]. \qquad (2)$$

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Navier-Stokes equations

- 3D Navier-Stokes equations in Fourier-space

$$\dot{u}_i(k) + \left(\delta_{ij} - \frac{k_i k_j}{k^2}\right) N_j(k) = -\nu k^2 u_i(k),$$

$$\text{where} \quad N_i(q) = \sum_{\mathbf{q}=\mathbf{k}+\mathbf{p}} i k_j u_i(k) u_j(p)$$

- In Fourier space, $\mathbf{u}(\mathbf{k}, t)$ has two degrees of freedom since $\mathbf{k} \cdot \mathbf{u}(\mathbf{k}, t) = 0$.
- We chose projection on orthonormal helical waves with definite sign of helicty.

Helical decomposition

- Following Waleffe Phys. Fluids (1992)

$$\mathbf{u}(\mathbf{k},t) = \underbrace{a^+(\mathbf{k},t)\mathbf{h}^+(\mathbf{k})}_{\mathbf{u}^+} + \underbrace{a^-(\mathbf{k},t)\mathbf{h}^-(\mathbf{k})}_{\mathbf{u}^-}$$

where $\mathbf{h}^{\pm}(\mathbf{k})$ are the complex eigenvectors of the curl operator $i\mathbf{k}\times\mathbf{h}^{\pm}(\mathbf{k}) = \pm k\mathbf{h}^{\pm}(\mathbf{k})$.

- $\mathbf{h}_s^* \cdot \mathbf{h}_t = 2\delta_{st}$; $\mathbf{h}_s^* = \mathbf{h}_{-s}$,
  where s and t could be $+1$ or -1
- Choose $\mathbf{h}^{\pm}(\mathbf{k}) = \hat{\mu}(\mathbf{k}) \times \hat{\mathbf{k}} \pm i\hat{\mu}$,
  where $\hat{\mu}$ is an arbitrary unit vector orthogonal to $\mathbf{k}$
- reality of the velocity field requires $\hat{\mu}(\mathbf{k}) = -\hat{\mu}(-\mathbf{k})$
- Such requirement is satisfied, e.g., by the choice $\hat{\mu}(\mathbf{k}) = \mathbf{z}\times\mathbf{k}/||\mathbf{z}\times\mathbf{k}||$, with $\mathbf{z}$ an arbitrary vector.

Helically decimated Navier-Stokes equations

- Decimated Navier-Stokes equations in Fourier space:

$$\partial_t \mathbf{u}^{\pm}(\mathbf{k},t) = \mathcal{P}^{\pm}(\mathbf{k})\mathbf{N}_{u^{\pm}}(\mathbf{k},t) + \nu k^2 \mathbf{u}^{\pm}(\mathbf{k},t) + \mathbf{f}^{\pm}(\mathbf{k},t)$$

where ν is kinematic viscosity and $\mathbf{f}$ is external forcing.

- The nonlinear term containing all triadic interactions

$$\mathbf{N}_{u^{\pm}}(\mathbf{k},t) = \mathcal{F}T(\mathbf{u}^{\pm}\cdot\nabla\mathbf{u}^{\pm} - \nabla p)$$

- Projection operator:

$$\mathcal{P}^{\pm}(\mathbf{k}) \equiv \frac{\mathbf{h}^{\pm}(\mathbf{k}) \otimes \mathbf{h}^{\pm}(\mathbf{k})^*}{\mathbf{h}^{\pm}(\mathbf{k})^* \cdot \mathbf{h}^{\pm}(\mathbf{k})}$$

$$\mathbf{u}^{\pm}(\mathbf{k},t) = \mathcal{P}^{\pm}(\mathbf{k})\mathbf{u}(\mathbf{k},t)$$

$$\mathbf{u}(\mathbf{k},t) = \mathbf{u}^{+}(\mathbf{k},t) + \mathbf{u}^{-}(\mathbf{k},t)$$

- Eneregy $E(t) = \sum_{\mathbf{k}} |\mathbf{u}^{+}(\mathbf{k},t)|^2 + |\mathbf{u}^{-}(\mathbf{k},t)|^2$.
- Helicity $\mathcal{H}(t) = \sum_{\mathbf{k}} k(|\mathbf{u}^{+}(\mathbf{k},t)|^2 - |\mathbf{u}^{-}(\mathbf{k},t)|^2)$.

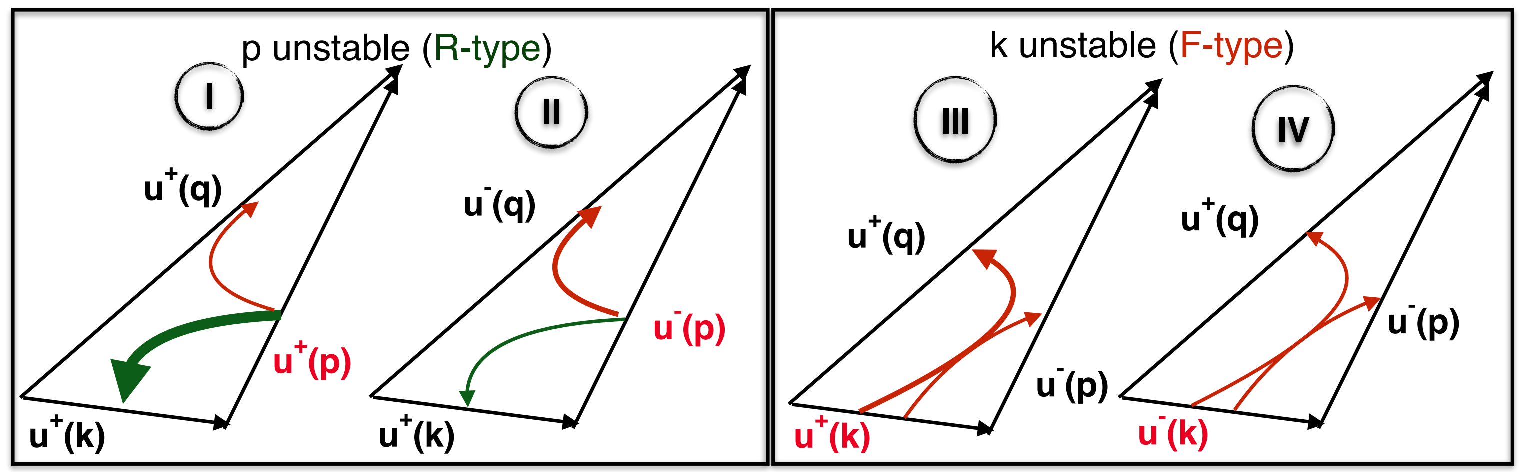
Classes of triadic interactions in NS equations

R-type: When large wavenumbers have same sign, middle one is unstable and could transfer energy to both small and large wavenumbers;

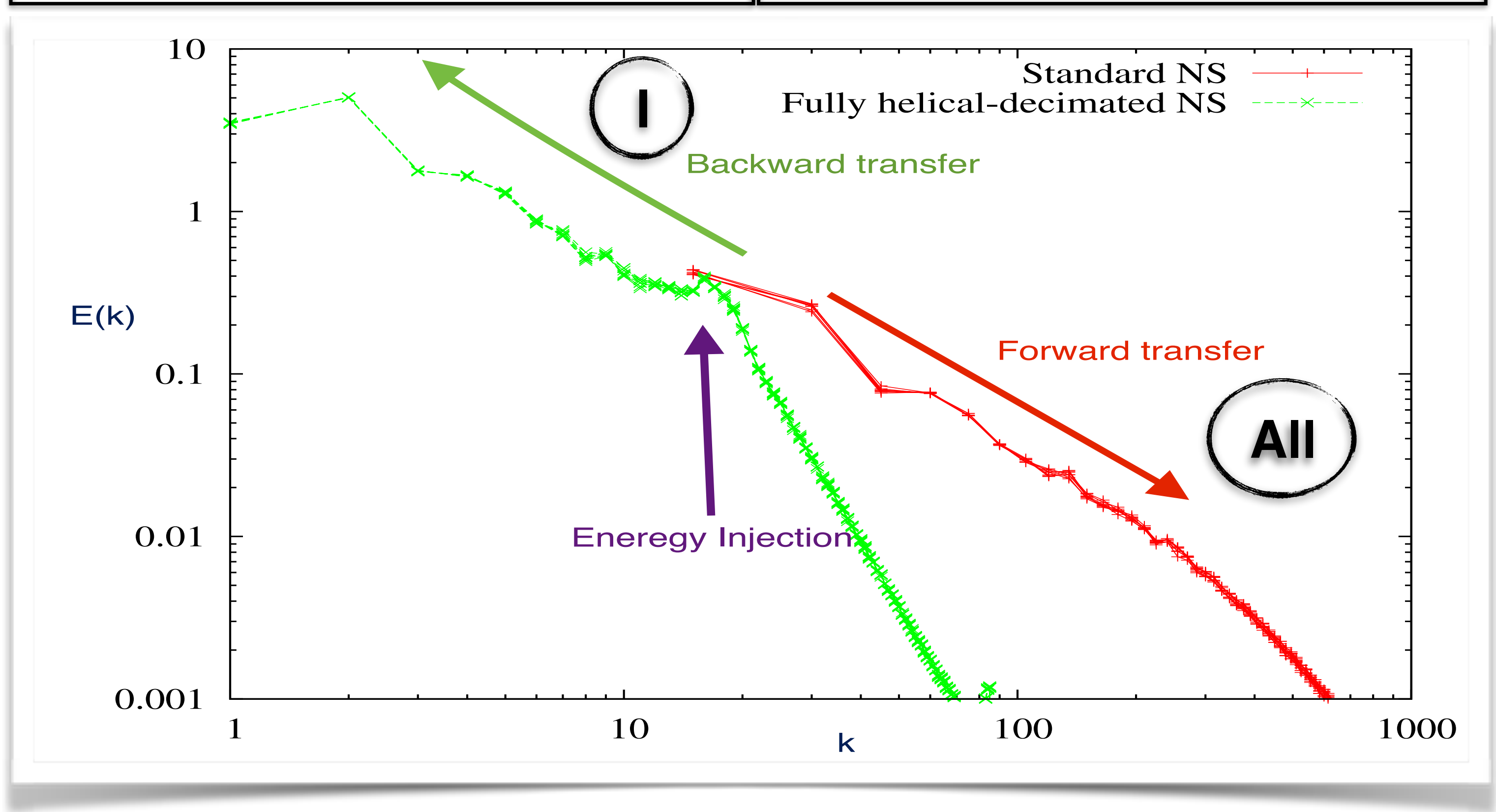
- predominantly to the smallest wavenumber if it has the same sign **[Class-I (+, +, +)]**.
- mixed transfer if smallest wavenumber has the opposite sign [Class-II (+, -, -)].

F-type: When large wavenumbers have opposite sign, smallest one is unstable and could transfer energy only to large wavenumbers, for both Class-III (+, -, +) and Class-IV (-, -, +).

$$\mathbf{N}_{\mathbf{u}^{\pm}}(\mathbf{q}) = \mathcal{FT}\left[\mathbf{u}^{\pm}(\mathbf{k})\cdot\nabla\mathbf{u}^{\pm}(\mathbf{p})\right];\ \mathbf{q}=\mathbf{k}+\mathbf{p};\ k\le p\le q$$



- Energy and helicity are conserved for each individual triad.
- Triads with only u^+, i.e. Class-I, lead to reversal of energy cascade.
- Energy spectra in the inverse cascade regime shows a $k^{-5/3}$ slope.



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Partial Helical-decimation

What happens in between??
when we give different weights to different class of triads…

- Modified projection operator:

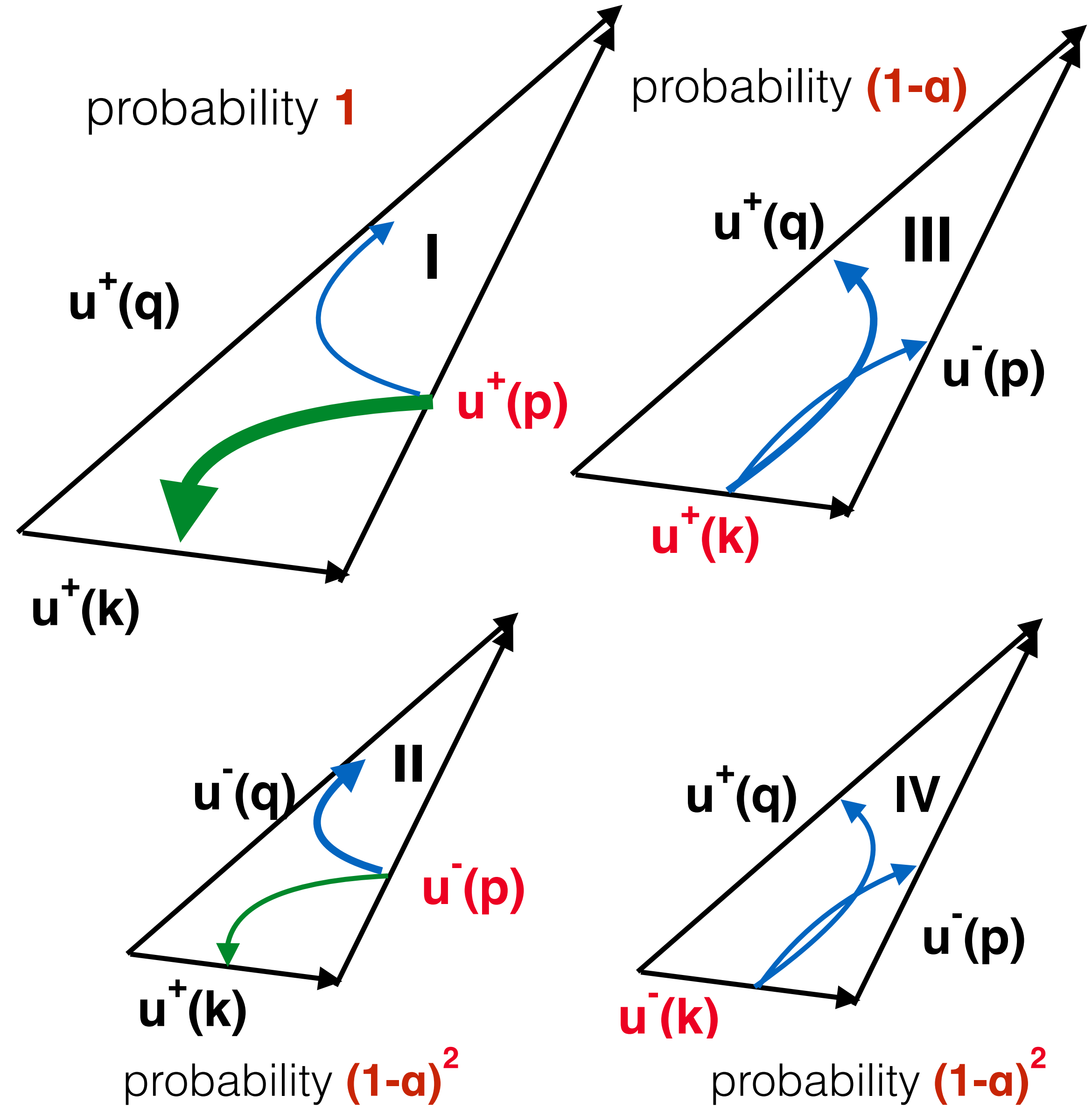
$$\mathcal{P}_\alpha^+(\mathbf{k})\mathbf{u}(\mathbf{k},t) = \mathbf{u}^+(\mathbf{k},t) + \theta_\alpha(\mathbf{k})\mathbf{u}^-(\mathbf{k},t)$$

  where $\theta_\alpha(\mathbf{k})$ is 0 with probability α and is 1 with probability $1-\alpha$.

- We consider triads of Class-I with probability 1, Class-III with probability $1-\alpha$ and Class-II and Class-IV with probability $(1-\alpha)^2$.

- $\alpha = 0 \rightarrow$ Standard Navier-Stokes.
  $\alpha = 1 \rightarrow$ Fully helical-decimated NS.

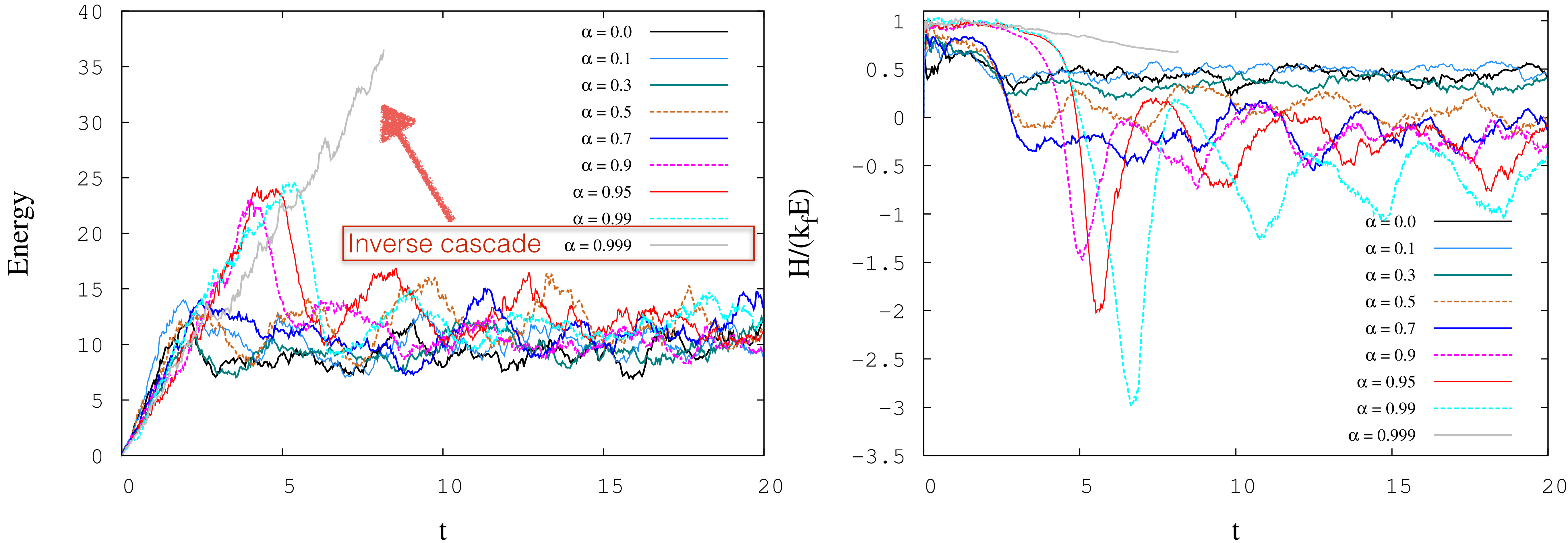
- Critical value of α at which forward cascade of energy stops?
  alternatively, inverse cascade of energy stops if forced at small scales.



$$\mathbf{N}_{\mathbf{u}^\pm}(\mathbf{q}) = \mathcal{FT}\left[\mathbf{u}^\pm(\mathbf{k})\cdot\boldsymbol{\nabla}\mathbf{u}^\pm(\mathbf{p})\right];\ \mathbf{q}=\mathbf{k}+\mathbf{p};\ k\le p\le q$$

Evolution of Energy and helicity

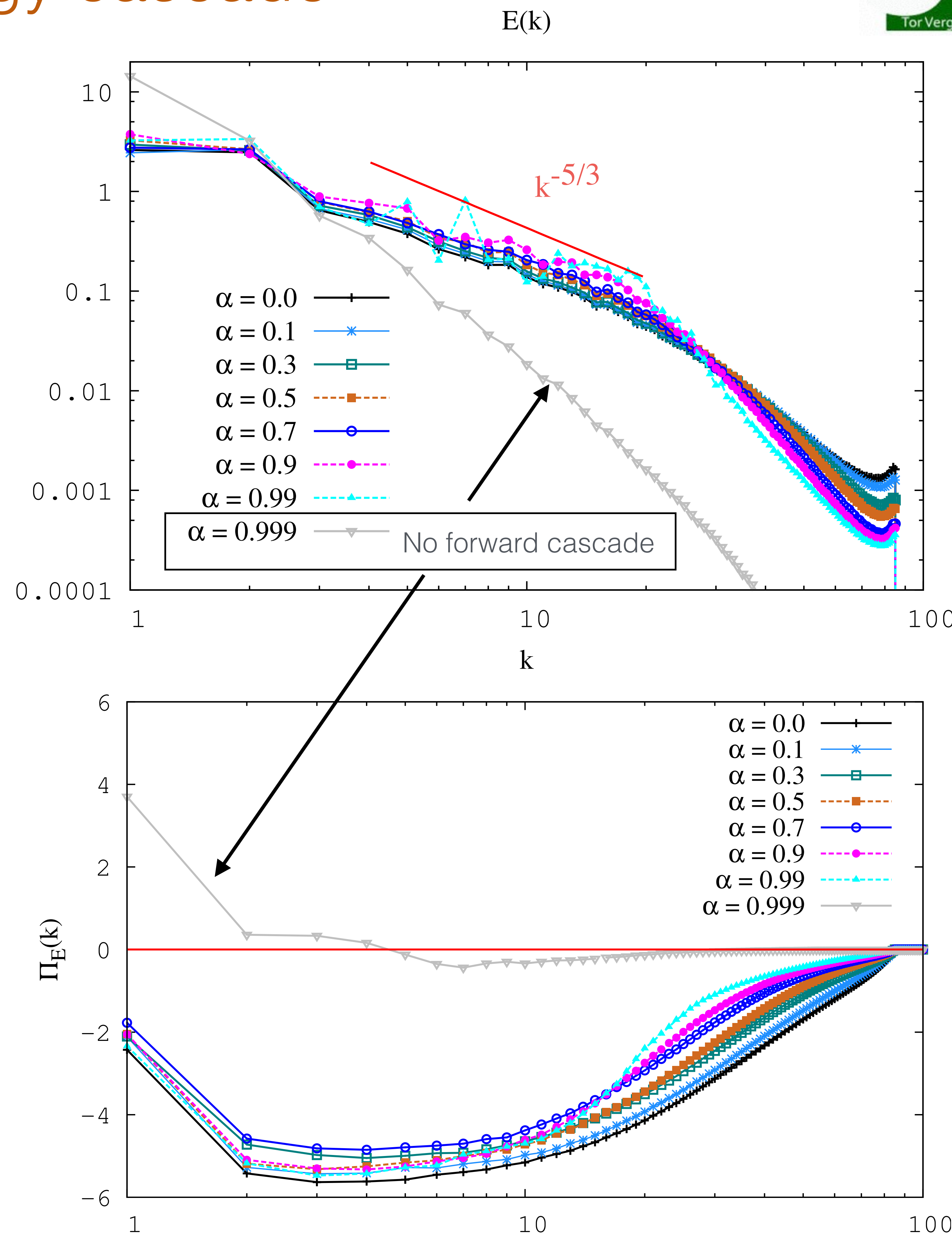
- Pseudo-spectral DNS on a triply periodic cubic domain of size $L = 2\pi$ with resolutions up to 512^3 collocation points.



- The peaks suggest the building up of the energy at forced large scales before being able to transfer to the small scales.
- The cascade of energy starts only when helicity becomes active, i.e., modes with negative helicity becomes energetic.
- With increase in α the peak grows, a signature of inverse cascade.

Robustness of energy cascade

- Spectra for all values of α showing $k^{-5/3}$ suggest the forward cascade of to be strongly robust.
- Unless we kill almost all the modes of one helicity-type energy always finds a way to reach small scales.
- The energy flux also remains unaffected by the decimation until α is very close to 1.
- **Critical value of α is ~ 1 !**

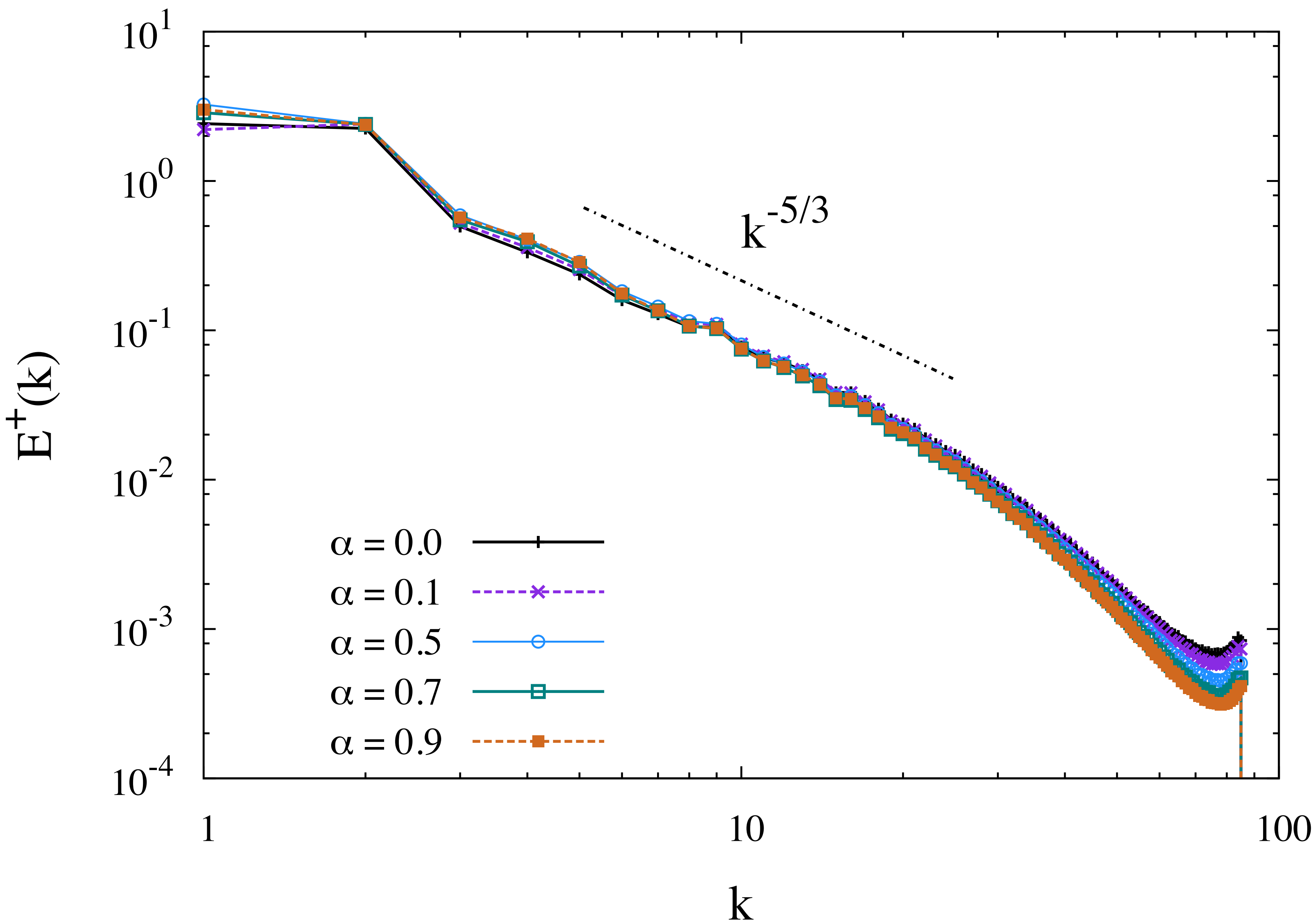


Energy in the positive helical modes

Chen, Phys. Fluids 2003

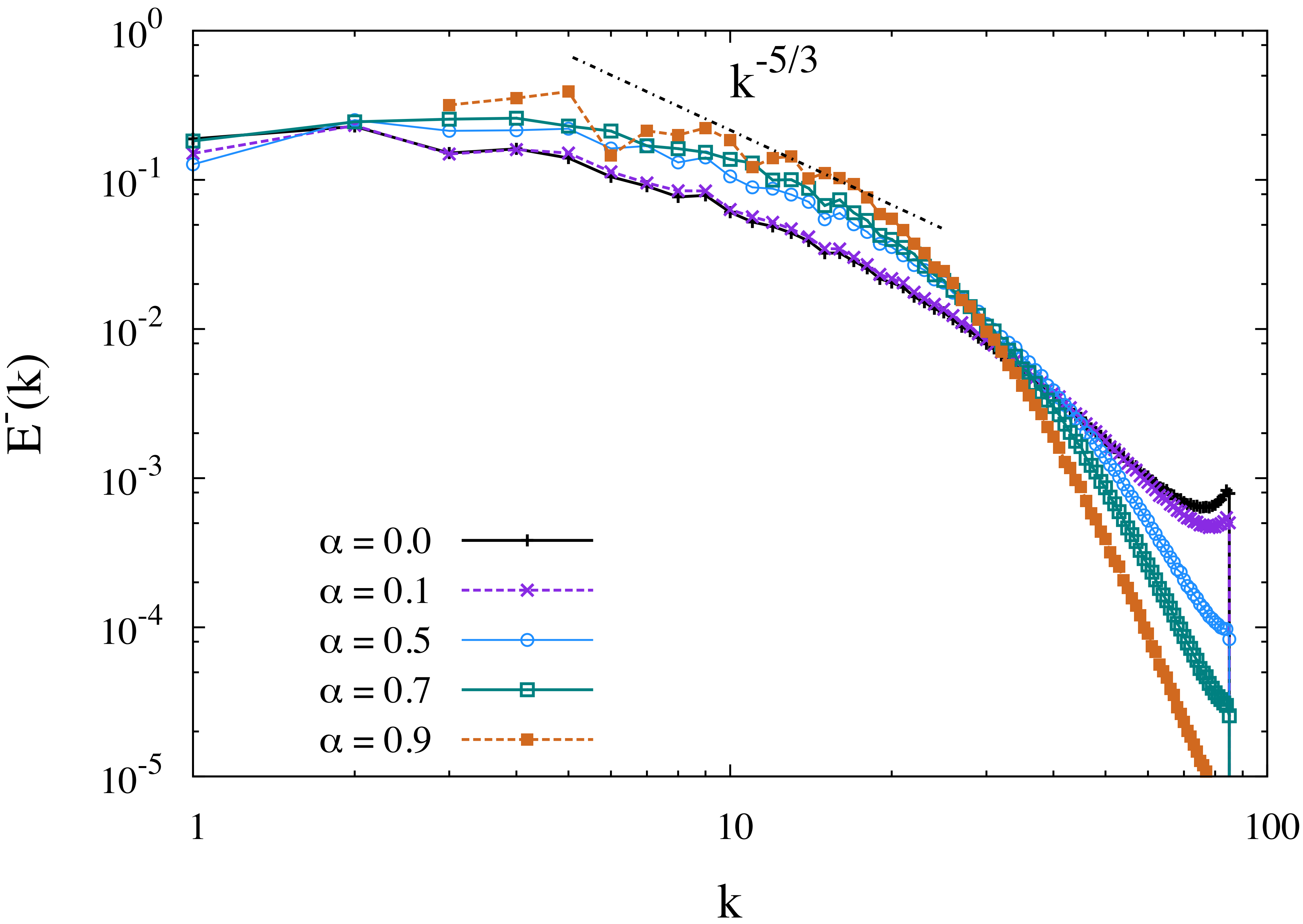
$$E^{\pm}(k) \sim C_1 \epsilon_E^{2/3} k^{-5/3} \left[1 \pm C_2 \left(\frac{\epsilon_H}{\epsilon_E}\right) k^{-1}\right],$$

where ϵ_E is the mean energy dissipation rate
and ϵ_H is the mean helicity dissipation rate.



- The $E^+(k)$ does not change with decimation.

Energy in the negative helical modes



- $E^-(k)$ shows that as we have fewer negative helical modes, they become more energetic in the inertial range of scales.

- The forward cascade of energy is though the triads of Class-III where two large wavenumber modes have opposite sign of helicity.
- The energy flux is carried by correlations of type

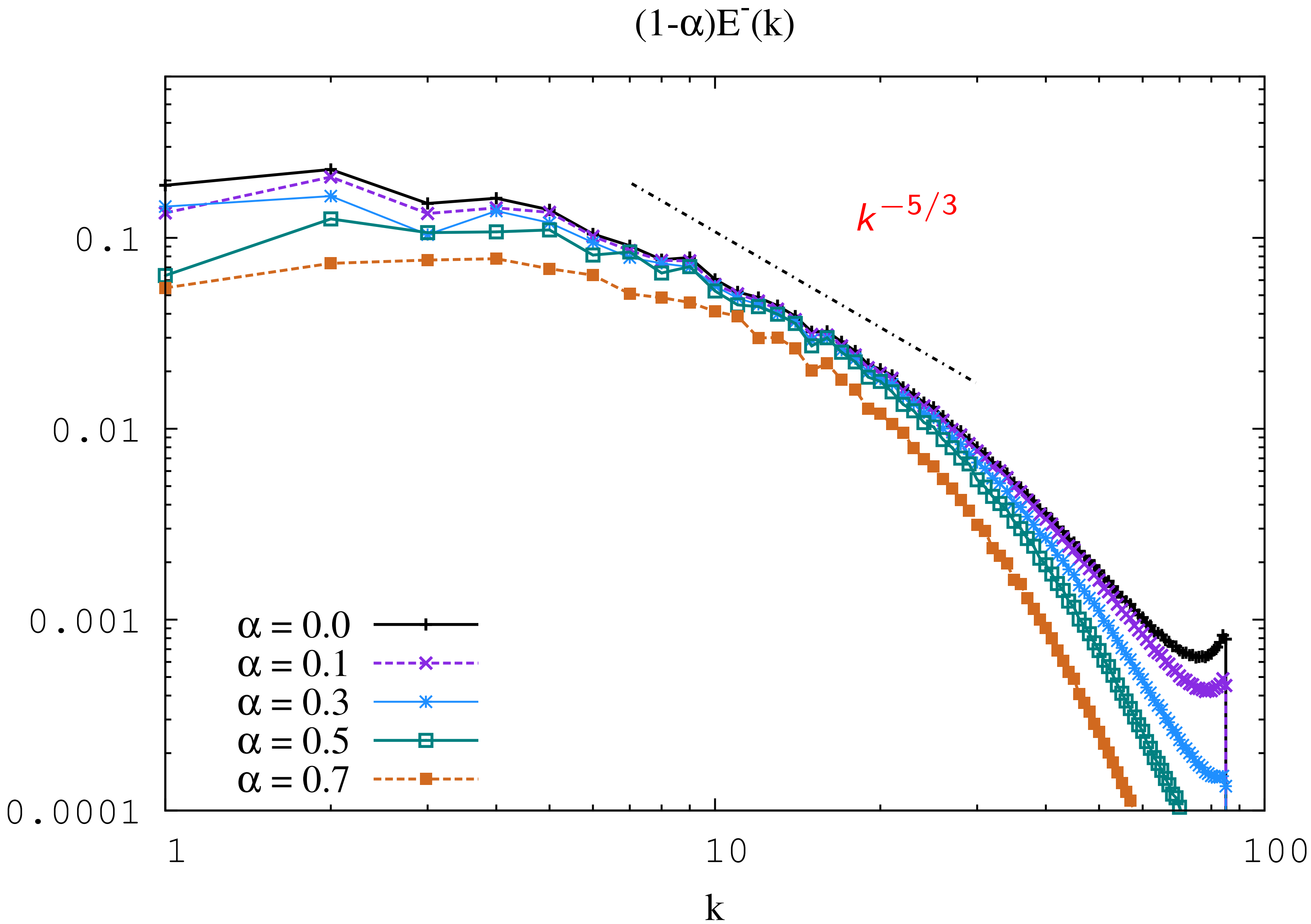
$$S(k|p,q) = \langle(\boldsymbol{k}\cdot u_{\boldsymbol{q}}^{-})(u_{\boldsymbol{k}}^{+}\cdot u_{\boldsymbol{p}}^{+})\rangle + \langle(\boldsymbol{k}\cdot u_{\boldsymbol{p}}^{+})(u_{\boldsymbol{k}}^{+}\cdot u_{\boldsymbol{q}}^{-})\rangle.$$

- This involves two positive helical modes and one negative helical modes.
- To maintain the constant flux, $\mathbf{u}^{-}$(k) must be rescaled by (1-α). since $\mathbf{u}^{-}$(k) exists with probability (1-α).

Reaction of negative helical modes

$$u^-_{\boldsymbol{k}} \to u^-_{\boldsymbol{k}}/(1-\alpha),$$

$$E^-(k) = \sum_{|\boldsymbol{k}|=k} (1-\gamma_{\boldsymbol{k}})|u^-_{\boldsymbol{k}}|^2 \to E^-(k)/(1-\alpha),$$



- Invariance of parity is restored through scaling of $E^-(k)$ by the factor $(1-\alpha)$.

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Conclusions

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- As we increase decimation of the modes with negative helicity (α), the contribution of triads leading to inverse energy cascade grows.

- The forward cascade of energy is very robust in 3D turbulence. It requires only a few negative modes to act as catalyst to transfer energy forward.

- Only when α is very close to 1, i.e., we decimate almost all modes of one helical sign, inverse energy cascade takes over the forward cascade.

- We observe a strong tendency to recover parity invariance even in the presence of an explicit parity-invariance symmetry breaking ($\alpha > 0$).

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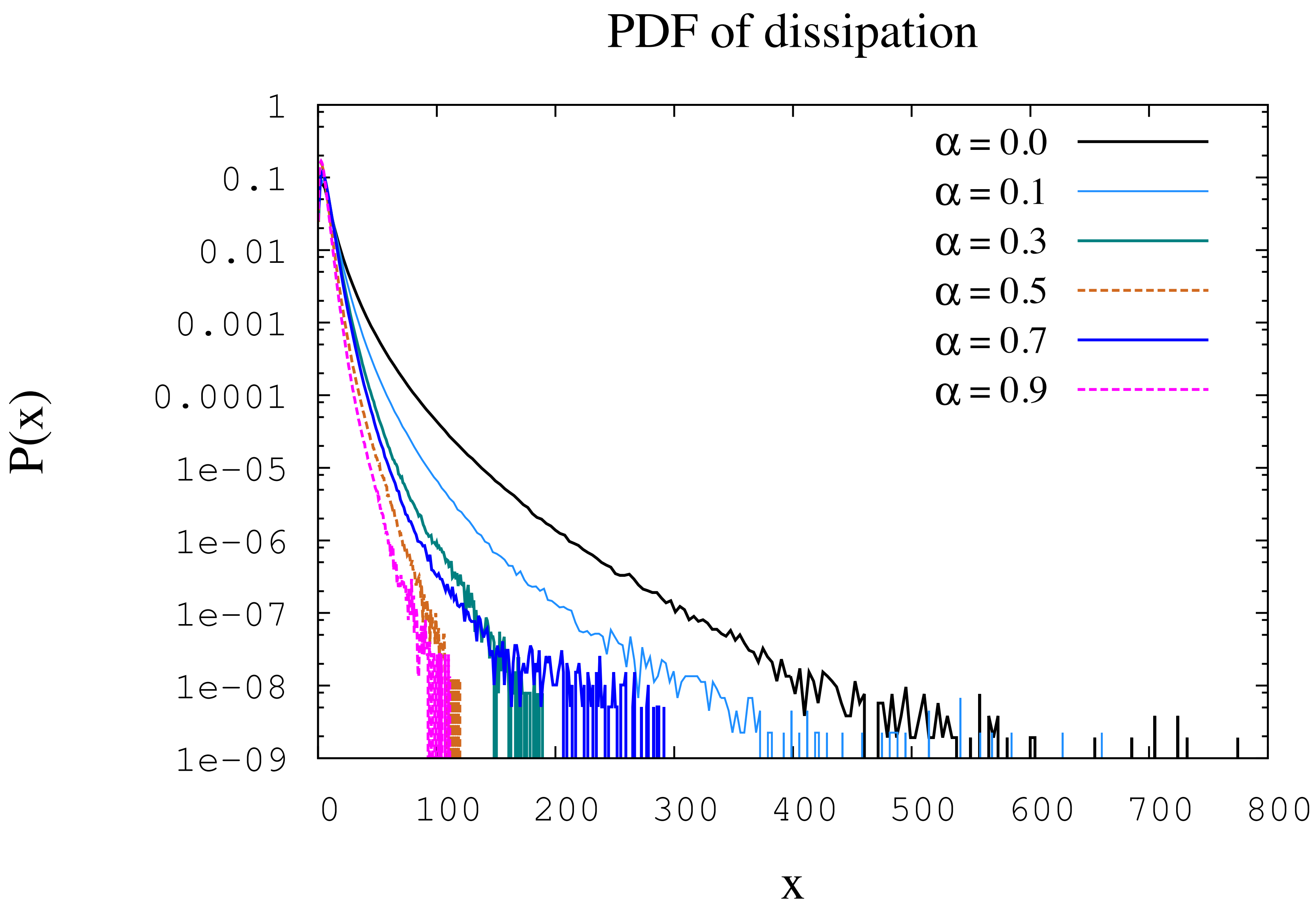
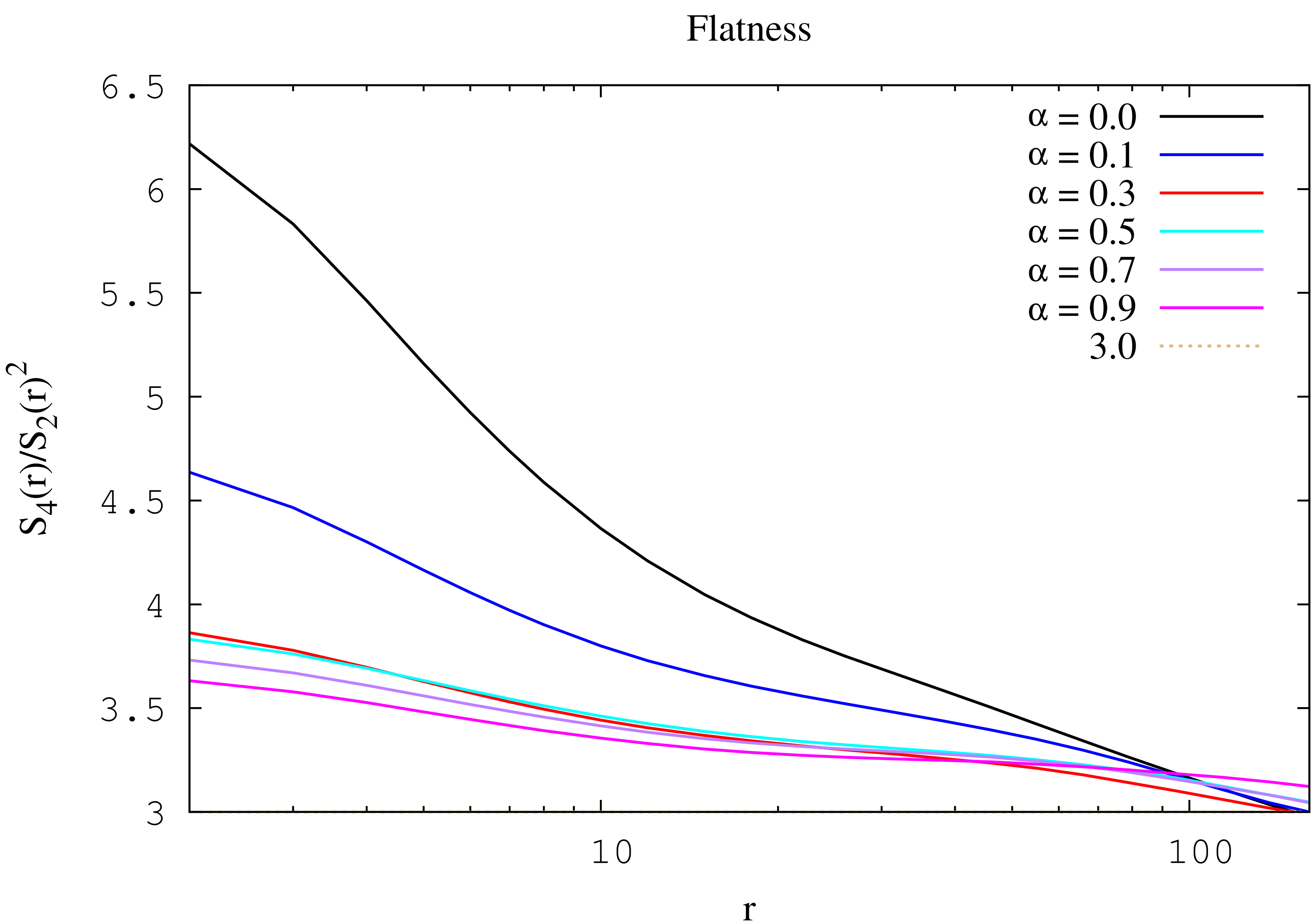
- What about abrupt symmetry breaking at some k_c?
    - can we stop the cascade by killing all negatives modes from $k>k_c$?
    - or can we start it at our wish (killing all modes up to k_c)?
- What about intermittency in the forward cascade regime at changing α?



Comparison of PDFs of local energy dissipation rates show reduction of longer tails with increase in fraction of decimation α. Less of extreme dissipation events show decrease in intermittency with increaseing α

Measure of intermittency: Flatness $F_4(r) = S_4(r)/[S_2(r)]^2$



- Measure of flatness shows the small scale intermittency reduces significantly when 10% of $\mathbf{u}^-$ modes are killed.
- It reduces further and seems saturated with increase in α

Small Scale structures

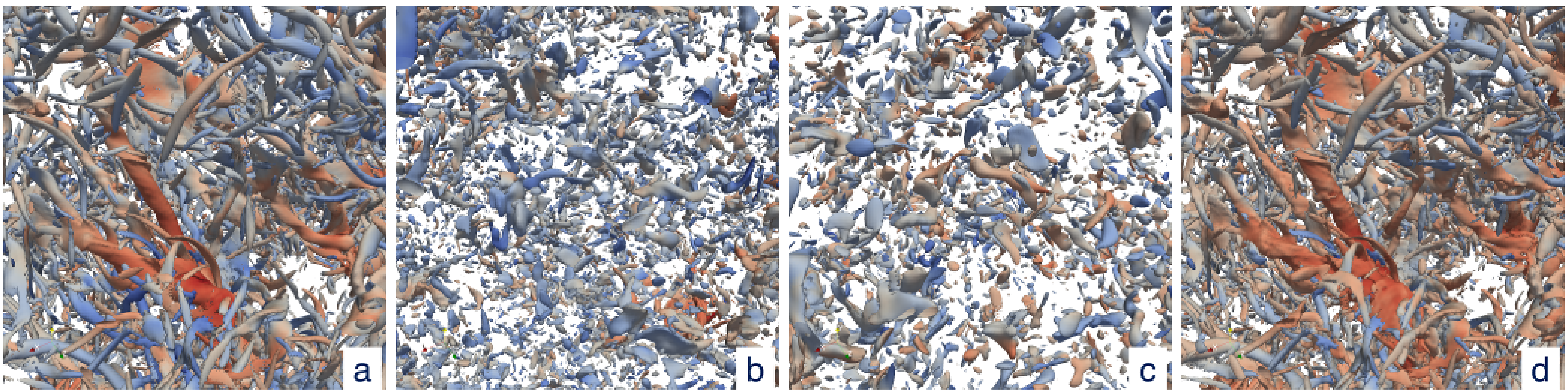


FIG. 3: (color online) iso-vorticity surfaces for: (a) $\alpha = 0$, (b) $\alpha = 0.5$, (c) $\alpha = 0.9$. Last plot (d) is obtained applying the projection with $\alpha = 0.5$ on the original NSE fields without any dynamical decimation. Color palette is proportional to the intensity of the helicity.

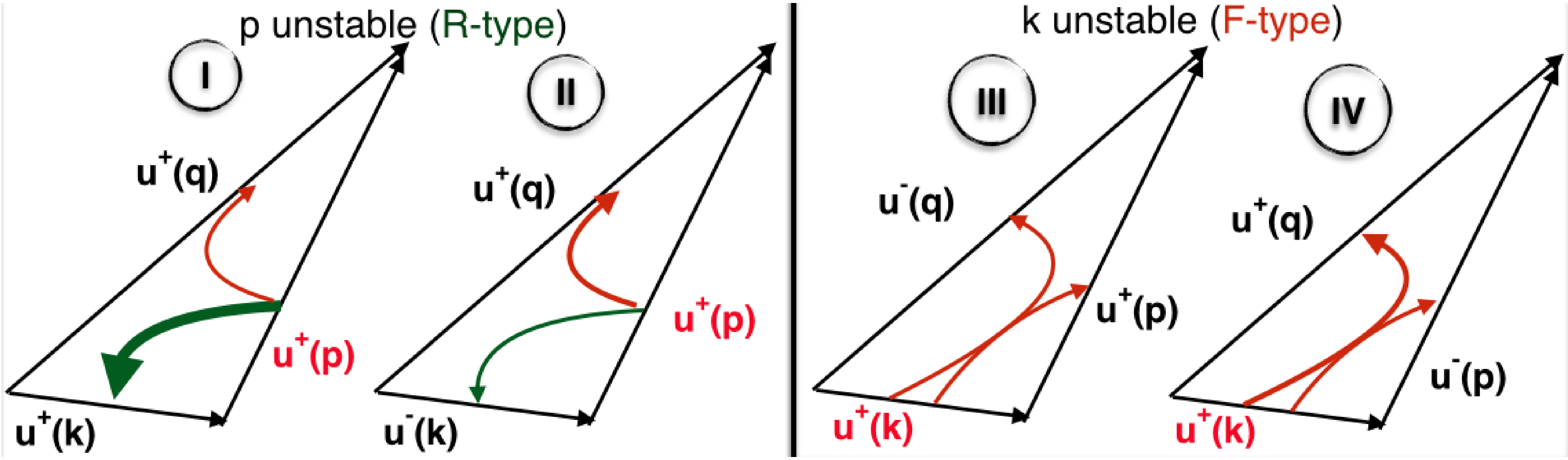
- There is a strong depletion of filament-like structures with dynamical decimation of negative helical modes.

- However, static decimation of negative helical modes preserves such structures.

Conclusion

- There is drastic reduction of intermittency with decimation.
- Vortex tubes usually associated with extreme events of energy dissipation disappear.
- Most importantly, only removal of helical modes dynamically, make this difference.
- Helicity surely plays a role in the direction of energy transfer and intermittency in the system.

Role of other triads of different classes

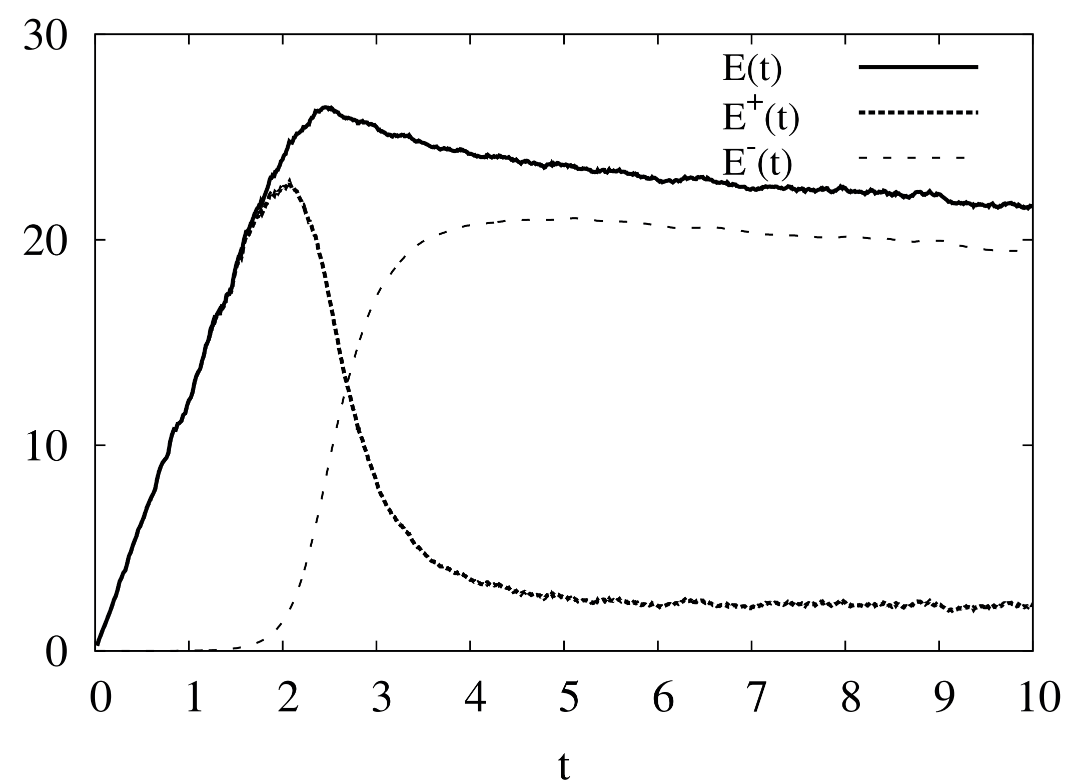


- We keep all triads of Class I by keeping positive helical modes at all wavenumbers.

- We add different class of triads to the dynamics by adding negative helical modes ONLY at $k = k_m$ and forcing $k = k_f$.

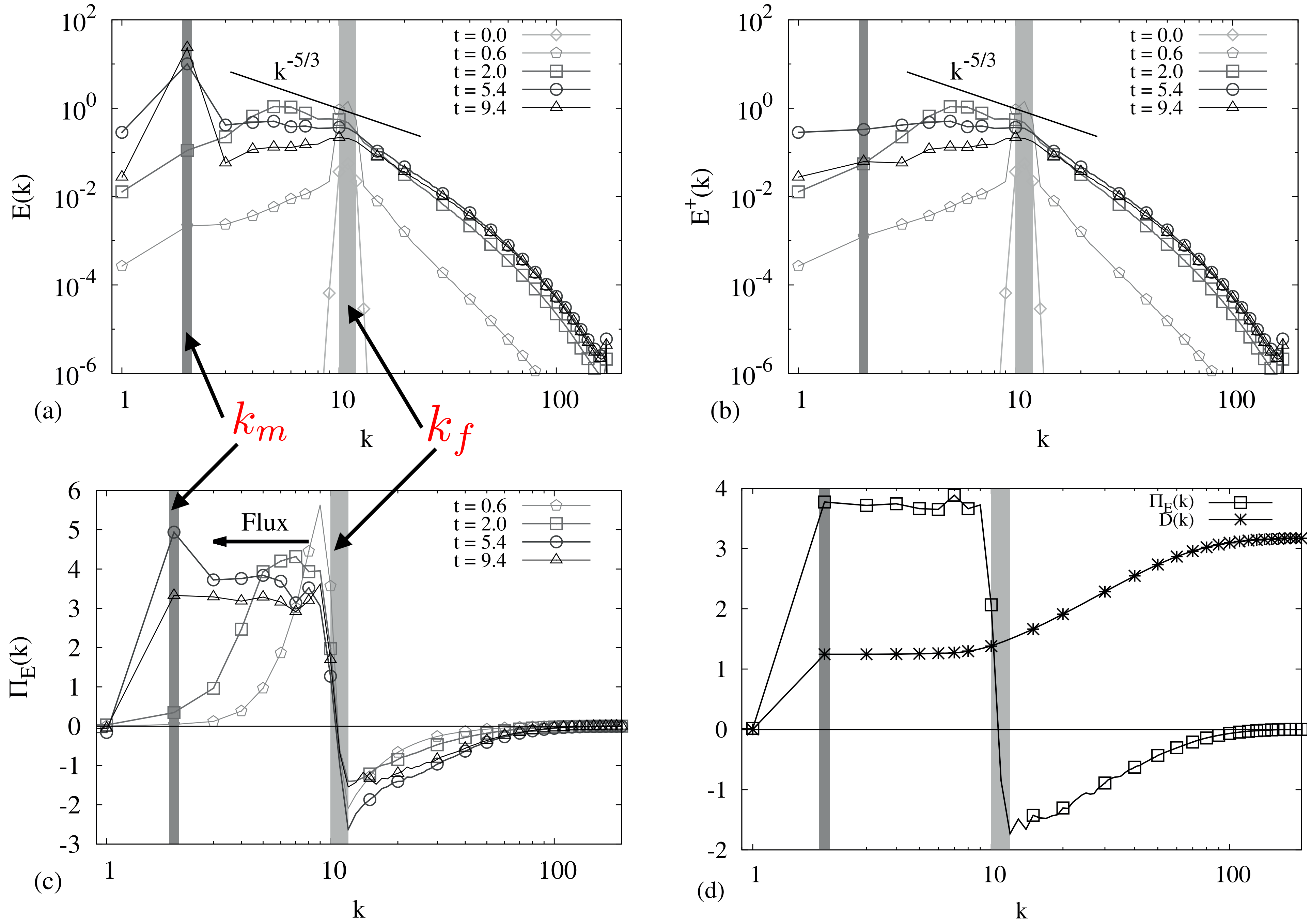
- We tried, two cases,

$$k_m < k_f \quad \text{and} \quad k_m > k_f$$



- In both cases, we reached a steady state as shown! But the dynamics were different.

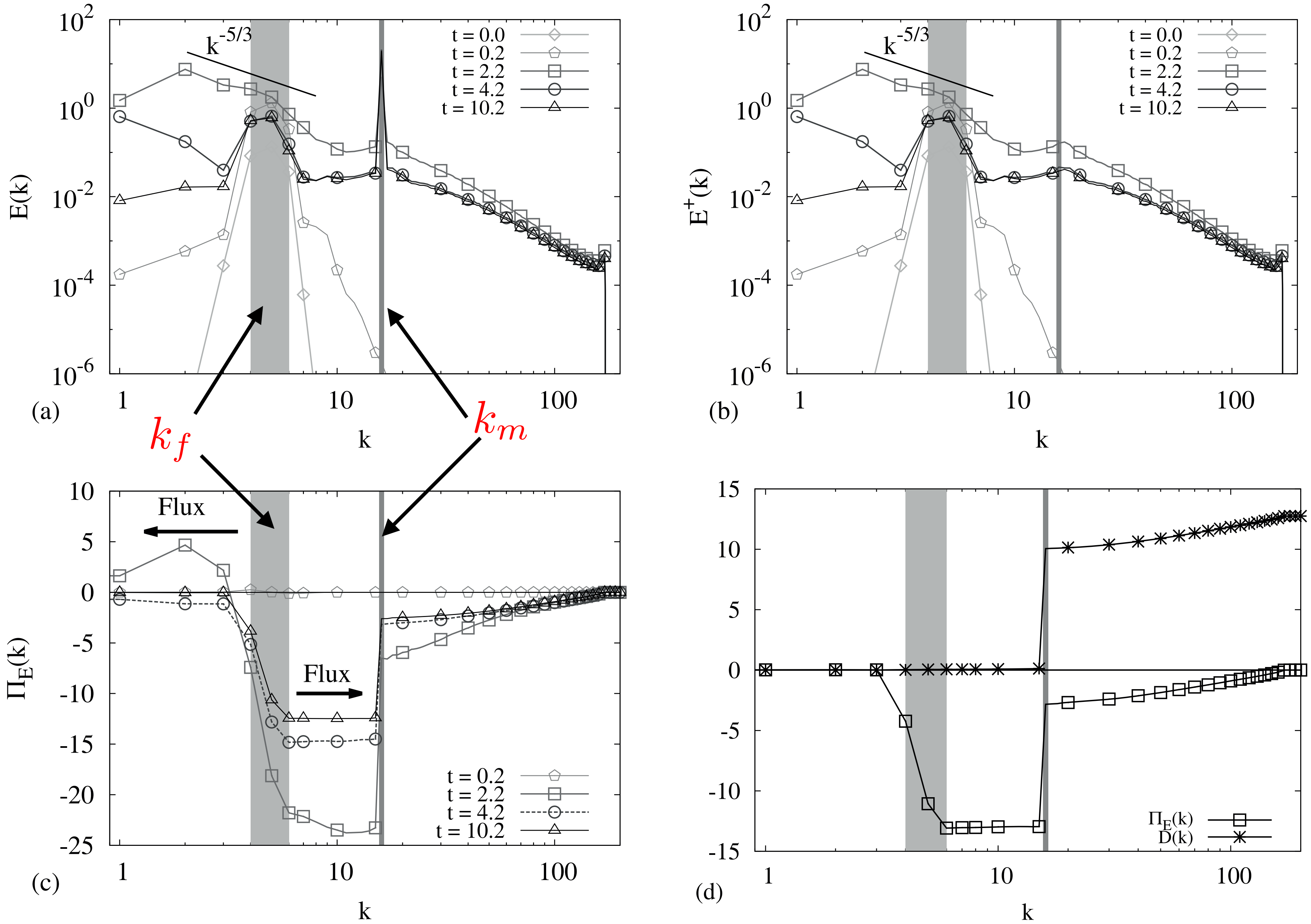
Added- Class II $k_m < k_f$



- Class-II is efficient in transferring energy from forced positive helical modes to negative helical modes at large scales.

- Inverse energy cascade does NOT need positive definite helicity!

Added- Class III and Class IV $k_m > k_f$



- Class-III and Class IV transfer energy from forced positive helical modes to negative helical modes at small scales.

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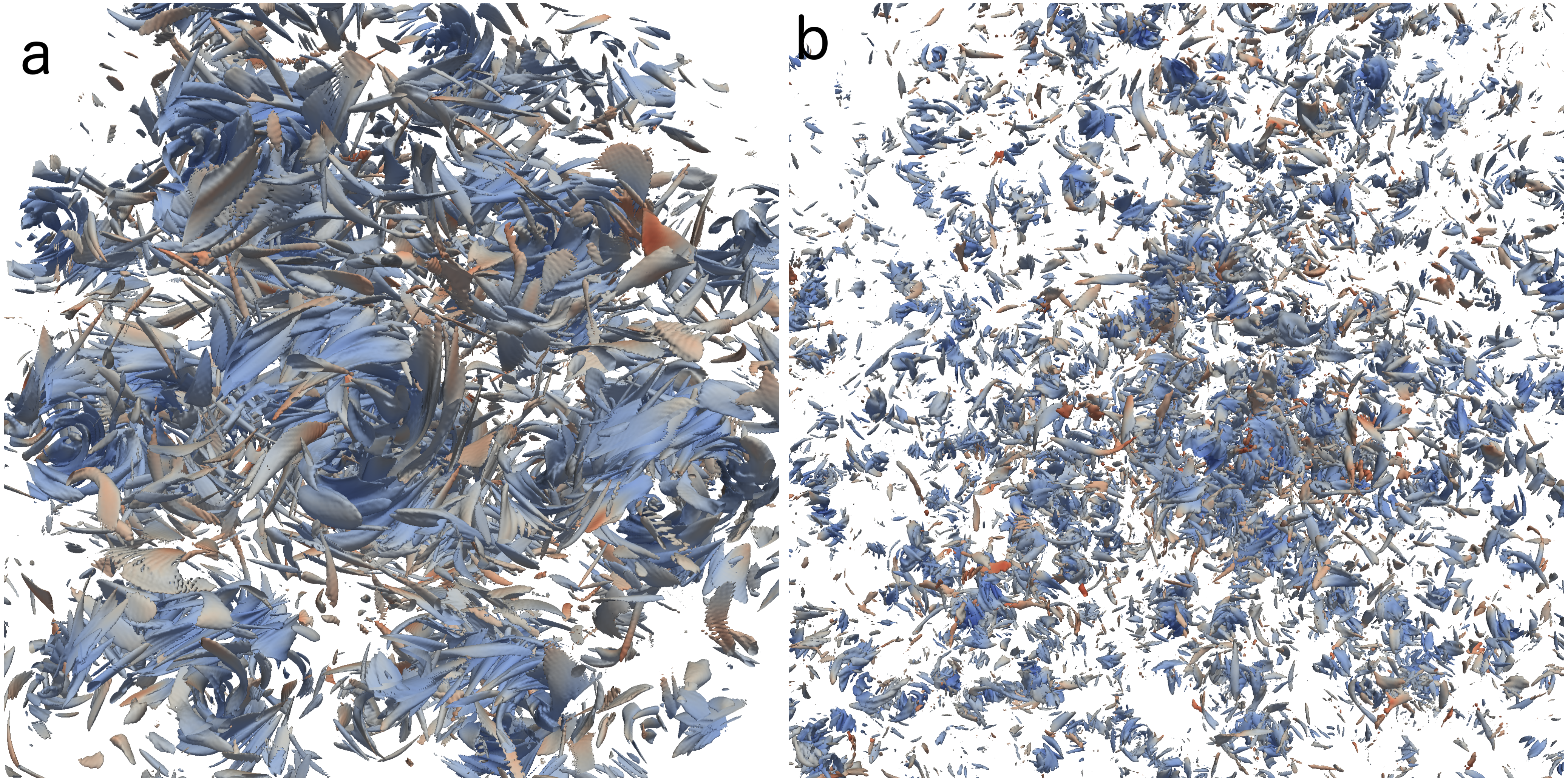

Class-II

Large scale condensates

Class-III and Class IV

small scale condensates

Conclusions

- Two classes of Triads (Class I and Class II) transfer energy to the large scales.
- It is possible to observe inverse energy cascade with out having a sign-definite helicity.
- When negative helical modes exist at only around one wavenumber, a large-scale or small-scale condensate is formed.

Thank you!

- R*ole of helicity for large-and small-scales turbulent fluctuations,* G Sahoo, F Bonaccorso, and L Biferale. Phys. Rev. E 92, 051002 (R) (2015).
- Disentangling the triadic interactions in Navier-Stokes equations, G Sahoo and L Biferale. Eur. Phys. J. E 38, 114 (2015).
- Inverse energy cascade in three-dimensional isotropic turbulence, L Biferale, S Musacchio, and F Toschi. Phys. Rev. Lett. 108, 164501 (2012).