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**Measuring anisotropy and universality in turbulence using
SO(3) decomposition**

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Measuring anisotropy and universality in turbulence using $SO(3)$ decomposition

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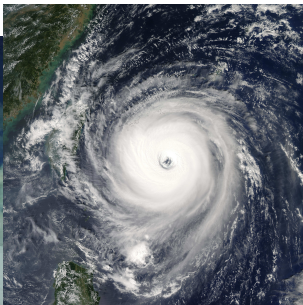
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Supercomputing resources at CINECA

source: Wikipedia



- ▶ Incompressible NSE invariant under Rotation + Translation

At large scales:

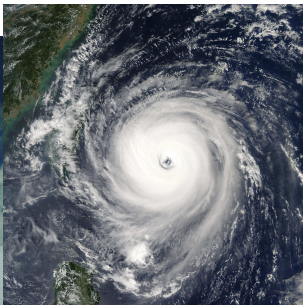
forcing, B.C

break rotation invariance

$$\left\{ \begin{array}{l} \partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} + \mathbf{f} \\ \nabla \cdot \mathbf{v} = 0 \\ + \text{boundary conditions} \end{array} \right.$$

- ▶ Is **breaking of rotational symmetry** passed down-scale ?

source: Wikipedia



- ▶ Incompressible NSE invariant under Rotation + Translation

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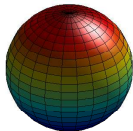
Universal signatures in small-scale fluctuations?

Longitudinal structure function: $S^{(n)}(\mathbf{r}) \equiv \langle [(\mathbf{v}(\mathbf{x} + \mathbf{r}) - \mathbf{v}(\mathbf{x})) \cdot \hat{\mathbf{r}}]^n \rangle$ (0-rank tensor)

$$S^{(n)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{m=j} S_{jm}^{(n)}(r) Y_{jm}(\hat{\mathbf{r}})$$

Arad et. al. PRL'98

$$Y_{00} = 1/\sqrt{4\pi}$$



$$Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$$



$$Y_{22} = \sqrt{\frac{15}{16\pi}} (\sin^2\theta \cos 2\phi)$$



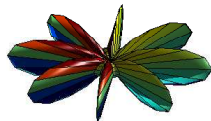
$$Y_{40}$$



$$Y_{42}$$



$$Y_{44}$$



rotational invariant operators

$$\partial_t S^{(2)}(\mathbf{r}) + \Gamma^{(3)} S^{(3)}(\mathbf{r}) - 2\nu \nabla^2 S^{(2)}(\mathbf{r}) = f^{(2)}(\mathbf{r})$$

$$r \ll L_f$$

Universality at small scales

$$\partial_t S^{(2)}(\mathbf{r}) + \Gamma^{(3)} S^{(3)}(\mathbf{r}) - 2\nu \nabla^2 S^{(2)}(\mathbf{r}) \sim 0$$

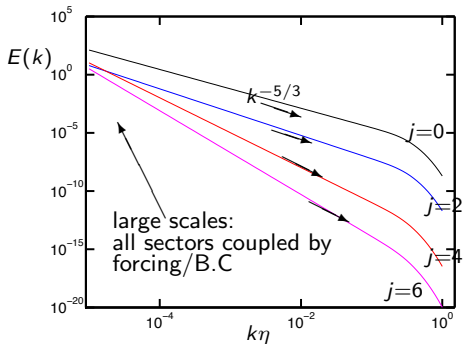
+ SO(3)

Weak Anisotropy

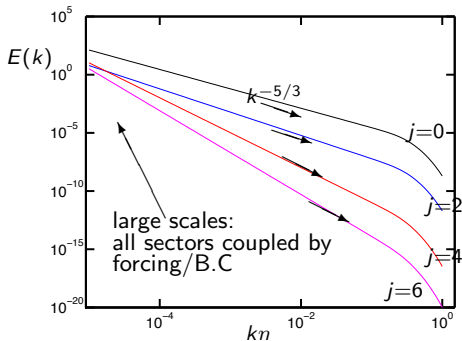
$$S_j^{(2)}(\mathbf{r}) = \sum_{m=-j}^{m=+j} Y_{jm}(\hat{\mathbf{r}}) \int S_{jm}^{(2)}(r) Y_{jm}(\hat{\mathbf{r}}) d\hat{\mathbf{r}}$$

$$\partial_t S_j^{(2)}(\mathbf{r}) + \Gamma_j^{(3)} S_j^{(3)}(\mathbf{r}) - 2\nu \nabla^2 S_j^{(2)}(\mathbf{r}) \sim 0 \quad j = 0, 1, 2, \dots$$

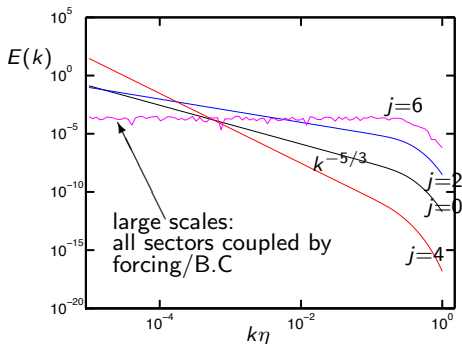
FOLIATION



- (1) **Universality**: leading isotropic sector
- (2) Foliation of j -sectors for $k \gg k_F$
- (3) different physics in different sectors
- (4) **return-to-isotropy**



- (1) **Universality**: leading isotropic sector
- (2) Foliation of j -sectors for $k \gg k_F$
- (3) different physics in different sectors
- (4) **return-to-isotropy**



- (1) **NO Universality**: sub-leading isotropic sector
- (2) **NO Foliation**: j -sectors coupled at $k \gg k_F$
- (3) **Isotropy not recovered at small-scales**

$$S^{(n)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{m=j} S_{jm}^{(n)}(r) Y_{jm}(\hat{\mathbf{r}})$$

Working Hypothesis

$$S_{jm}^{(n)}(r) = A_{jm}^{(n)} \left(\frac{r}{L}\right)^{\zeta_j^n}$$

- ▶ Projection on sector- j has universal scaling exponent ζ_j^n in inertial range depending on that sector **only**
- ▶ Power law behavior **only** in each separated sector
- ▶ Prefactors depend on large scale physics

$$S^{(n)}(\mathbf{r}) \sim A_0 \left(\frac{r}{L}\right)^{\zeta_0^n} + A_1 \left(\frac{r}{L}\right)^{\zeta_1^n} + A_2 \left(\frac{r}{L}\right)^{\zeta_2^n} + \dots$$

$$S^{(n)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{m=j} S_{jm}^{(n)}(r) Y_{jm}(\hat{\mathbf{r}})$$

$$S^{(n)}(\mathbf{r}) \sim A_0 \left(\frac{r}{L}\right)^{\zeta_0^n} + A_1 \left(\frac{r}{L}\right)^{\zeta_1^n} + A_2 \left(\frac{r}{L}\right)^{\zeta_2^n} + \dots$$

$$S^{(n)}(\mathbf{L}) \sim A_0 + A_1 + A_2 + \dots$$

Prefactors cannot be universal!

$$S^{(n)}(\mathbf{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{m=j} S_{jm}^{(n)}(r) Y_{jm}(\hat{\mathbf{r}})$$

$$S^{(n)}(\mathbf{r}) \sim A_0 \left(\frac{r}{L}\right)^{\zeta_0^n} + A_1 \left(\frac{r}{L}\right)^{\zeta_1^n} + A_2 \left(\frac{r}{L}\right)^{\zeta_2^n} + \dots$$

$$S^{(n)}(\mathbf{L}) \sim A_0 + A_1 + A_2 + \dots$$

Prefactors cannot be universal!

Open Questions

- ▶ Are scaling exponents ζ_j^n in j -sector m -independent ?
- ▶ Are scaling exponents ζ_j^n universal ?
- ▶ **Return-to-Isotropy** ?

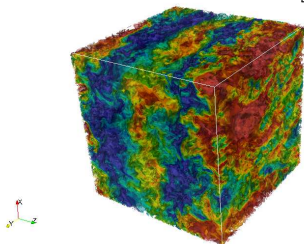
$$\zeta_0^n \leq \zeta_2^n \leq \zeta_4^n \leq \dots$$

- ▶ No rigorous inferences from NSE at least thus far ...

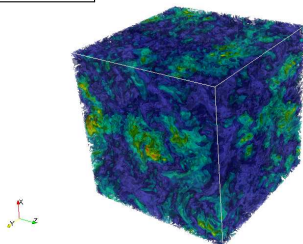
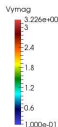
Random Kolmogorov Flow (*Biferale, Toschi PRL '01*)

- ▶ RKF is stationary, **homogeneous on average** and **anisotropic**

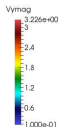
$$1024^3, R_\lambda = 280$$



$t/T_E=1$



$t/T_E=3$



- ▶ Anisotropic forcing:

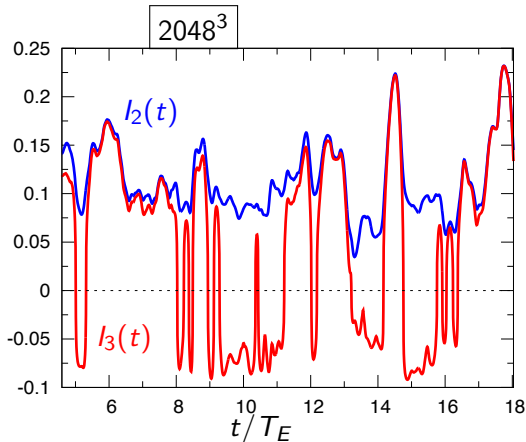
$$f_i(\mathbf{k}_{1,2}) = \delta_{i,2} f_{1,2}(t) e^{i\theta_{1,2}(t)}, \mathbf{k}_1 = (1, 0, 0), \mathbf{k}_2 = (2, 0, 0)$$

- ▶ $f_i(t)$ random time-varying amplitudes
- ▶ $\theta_i(t)$ random time-varying phases

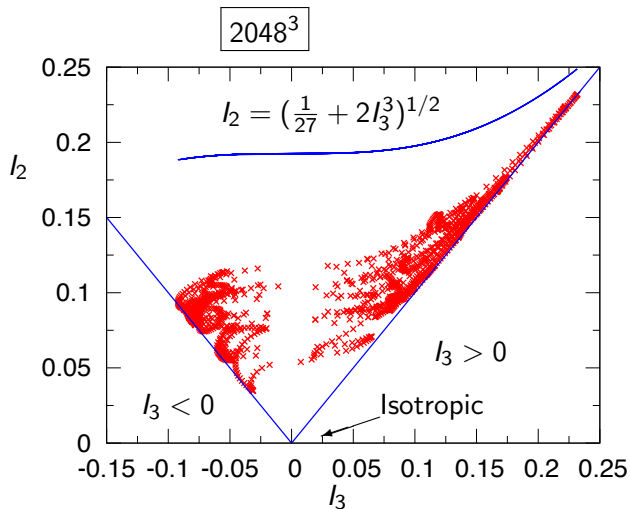
RKF: test-bed for anisotropic turbulence

► Reynolds Stress Tensor $b_{ij} \equiv \langle u_i u_j \rangle / \langle u_k u_k \rangle - \delta_{ij}/3$

► $I_2 \equiv b_{ii}^2/6$ $I_3 \equiv (b_{ij} b_{jk} b_{ki})^{1/3}/6$



Lumley Triangle



- Different large scale configurations (l_2 - l_3) possible in RKF

Isotropic sector vs undecomposed structure function

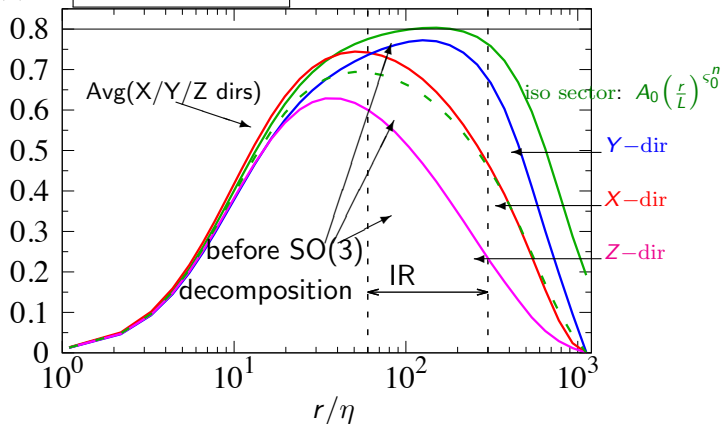
$$S^{(n)}(\mathbf{r}) = A_0 \left(\frac{r}{L}\right) s_0^n + A_1 \left(\frac{r}{L}\right) s_1^n + \dots$$

Isotropy in Inertial Range ($\eta \ll r \ll L$)

$$S^{(3)}(r) = -\frac{4}{5} \langle \epsilon \rangle r$$

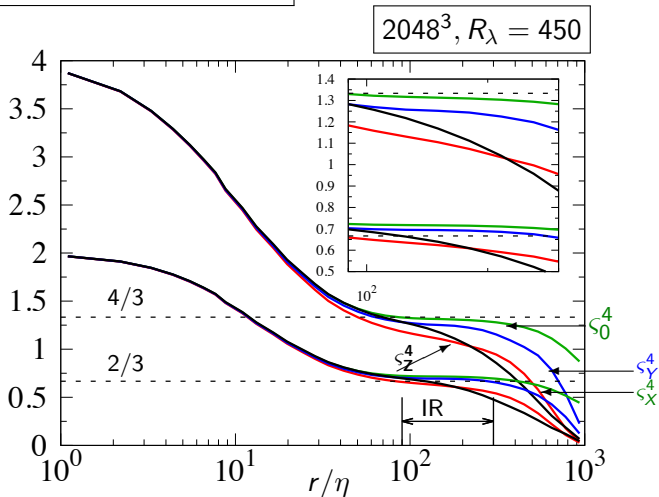
$-S^{(3)}(r)/r\langle\epsilon\rangle$

2048³, $R_\lambda = 450$



Local slopes: Isotropic sector vs Cartesian dirs

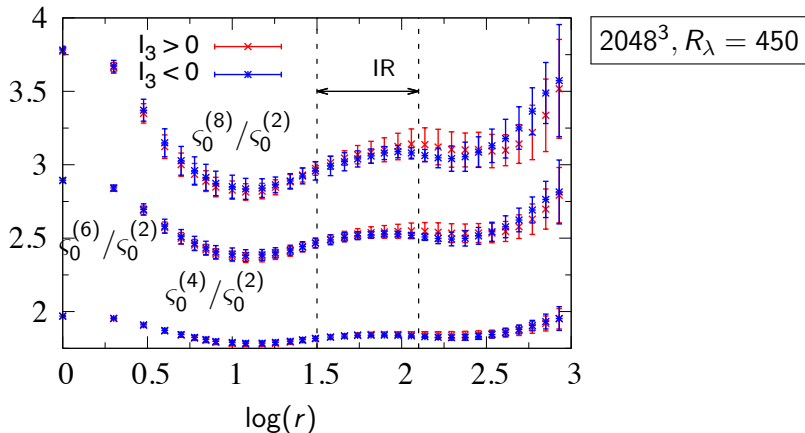
$$s_0^{(n)}(r) = d[\log(S_0^n(r))] / d[\log r]$$



- ▶ Power law scaling in isotropic sectors
- ▶ Intermittency exponent - contaminated by anisotropy

Universality of scaling exponent in isotropic sector

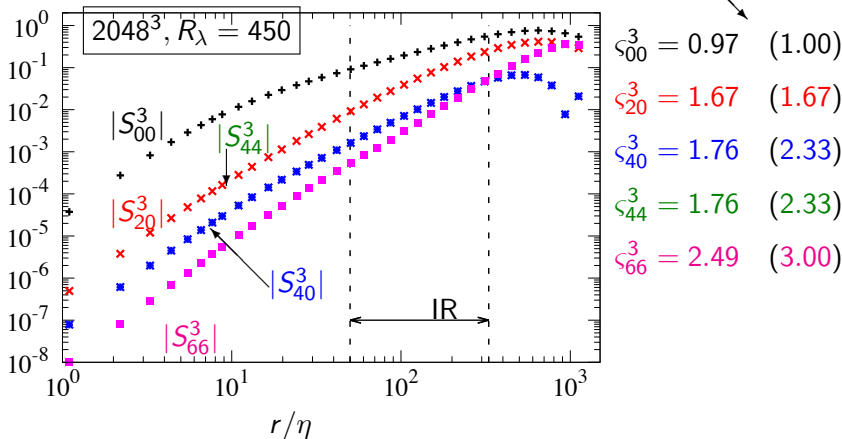
$$S^{(n)}(\mathbf{r}) = \boxed{A_0 \left(\frac{r}{L}\right)^{\zeta_0^{(n)}}} + A_1 \left(\frac{r}{L}\right)^{\zeta_n^{(1)}} + \dots \implies \text{Is } \zeta_0^{(n)} \text{ universal?}$$



- ▶ Different I2-I3 confs in Lumley triangle have similar Inertial Range scaling at least up to order 8

Ansatz for projection coeffs: $S_{jm}^n(r) \sim A_{jm}^n r^{\zeta_j^n}$

Dimensional scaling exponent (*Biferale et. al. PRE'02*): $\zeta_j^n = (j + n)/3$

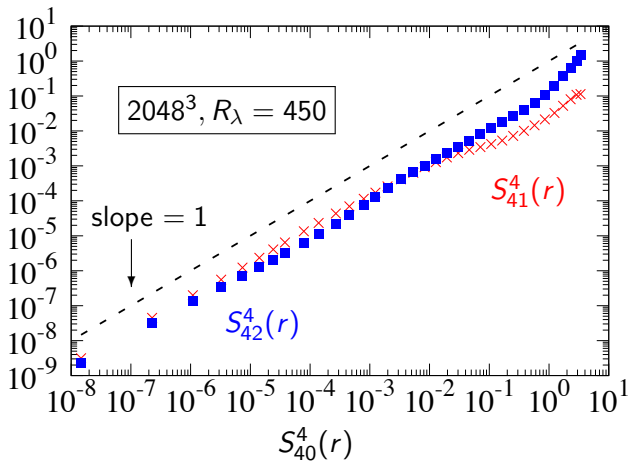


► Scaling exponents m -independent: $\zeta_{jm}^n = \zeta_j^n$

► $\zeta_0^n < \zeta_{20}^n < \zeta_{40}^n < \zeta_{66}^n \Rightarrow (r/L)^{\zeta_0^n} > (r/L)^{\zeta_{jm}^n}$ for $r/L \ll 1$

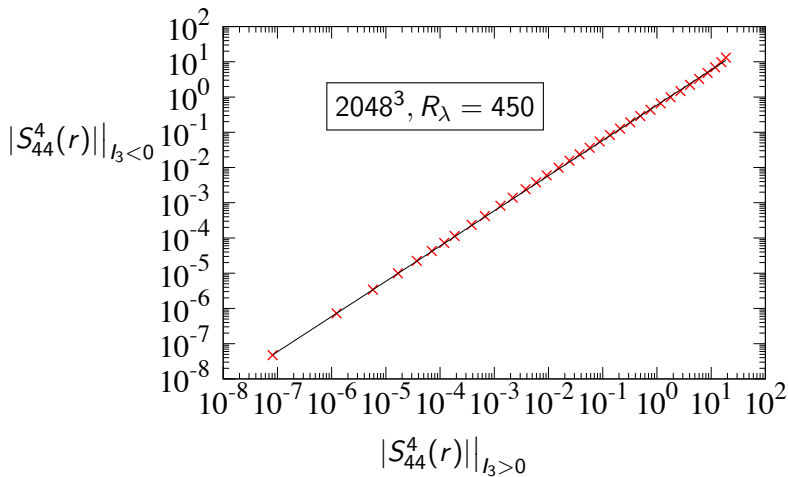
► **Isotropic sector dominant.** Anisotropic part: sub-leading

Scaling in anisotropic sectors



- Scaling of $S_{jm}^n(r)$ within same j -sector is m -independent

Universality of scaling exponents in anisotropic sectors



- ▶ j -sectors of different large-scale configurations scale similarly in inertial range

Conclusions

- ▶ $SO(3)$ decomposition- useful tool for decomposing different sectors, for a systematic investigation of anisotropic effects
- ▶ Isotropic sector: power law behavior
- ▶ Scaling in anisotropic sectors (j, m) is intermittent and m -independent
- ▶ Scaling exponents in given j -sector might be universal