Coherent structures and phases synchronization in non linear Burgers equation

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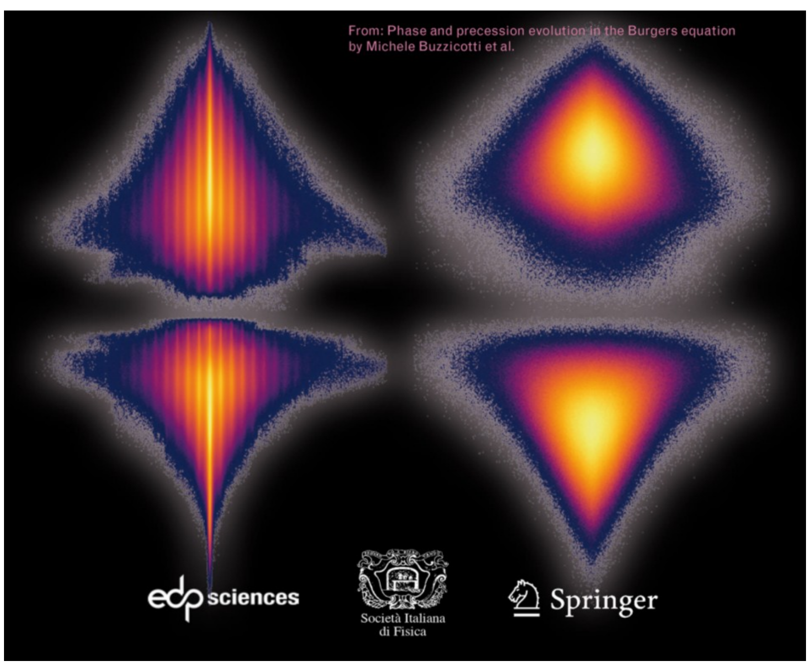

Motivation:

Understanding the robustness of the energy transfer mechanism driven by singular events.

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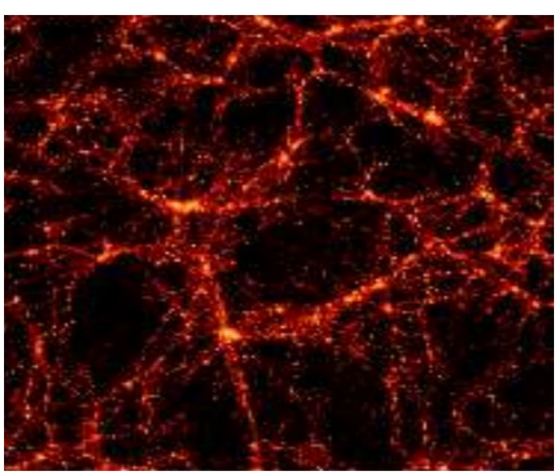
Burgers' equation

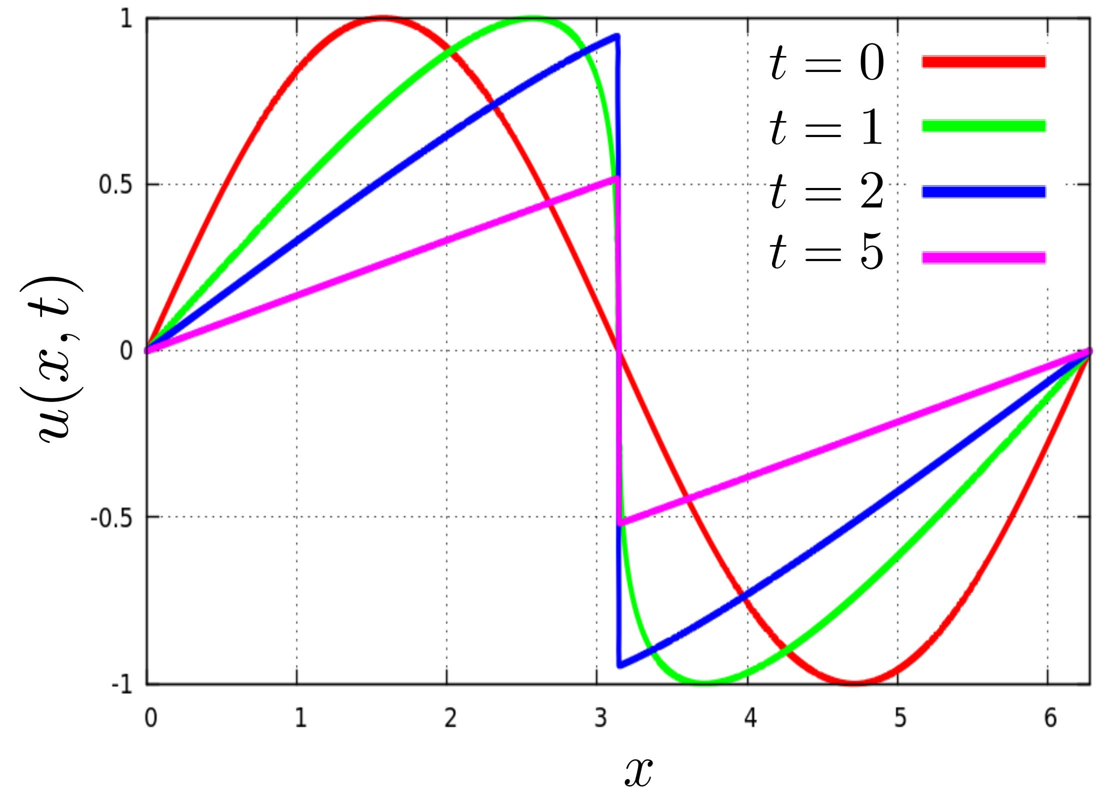
$$\frac{\partial u(x,t)}{\partial t} + u\frac{\partial u(x,t)}{\partial x} = \nu \frac{\partial^2 u(x,t)}{\partial x^2}$$

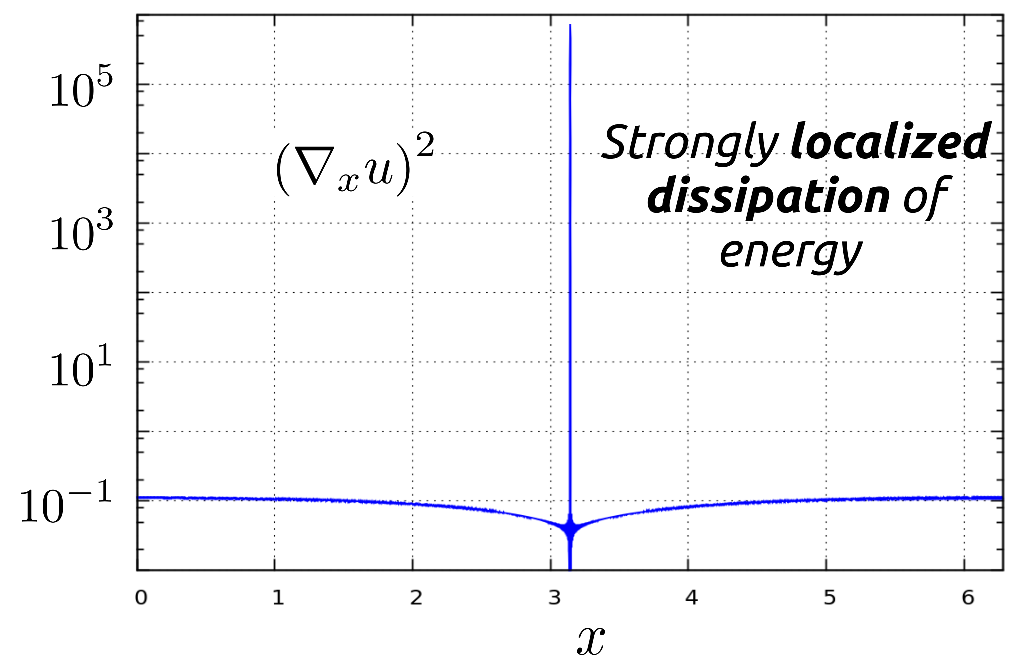
Kardar - Parisi - Zhang (KZP)

$$\frac{\partial h}{\partial t} - \frac{1}{2}|\nabla h|^2 = \nu \nabla^2 h + F$$

2D Burgers; Cosmological model







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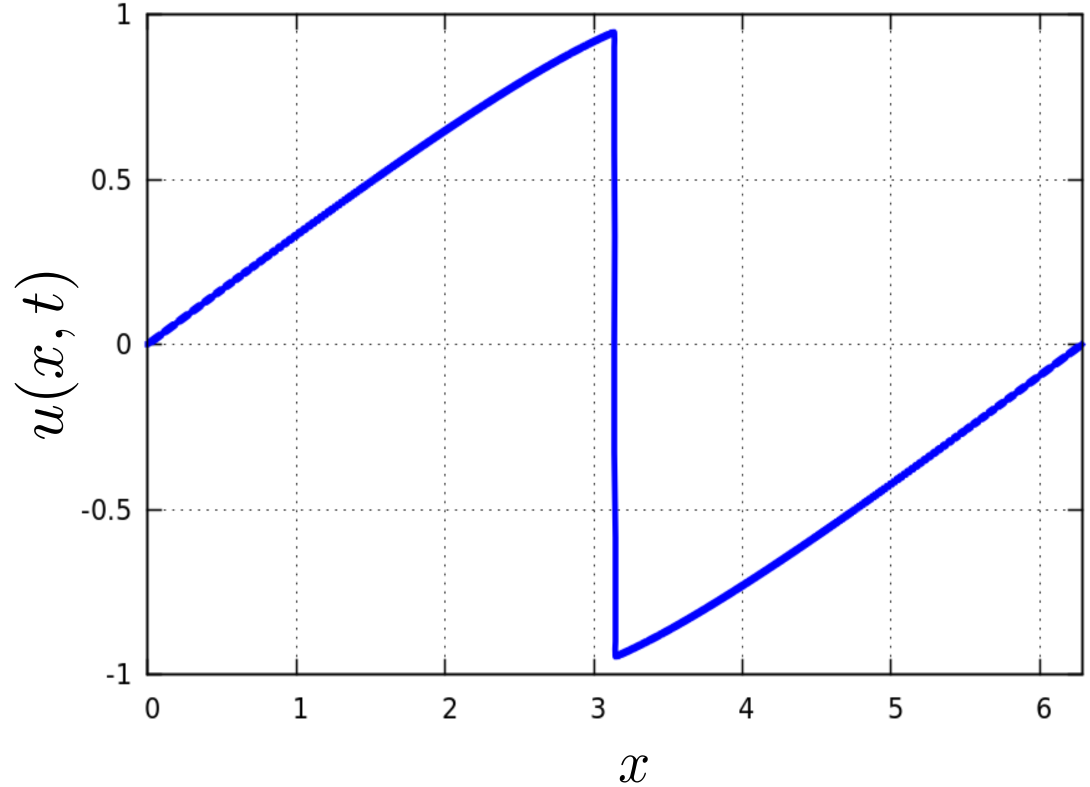
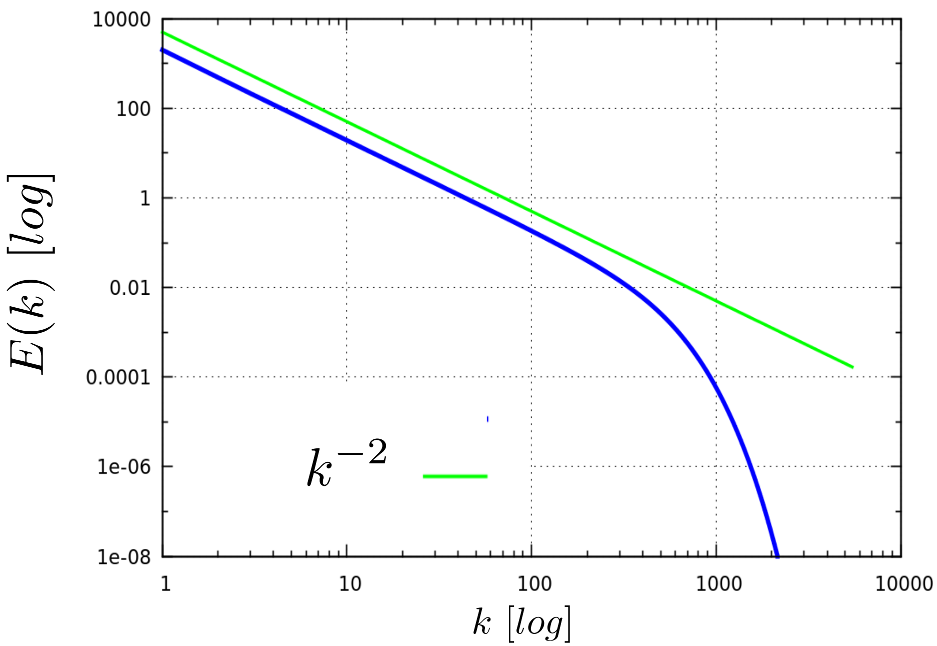
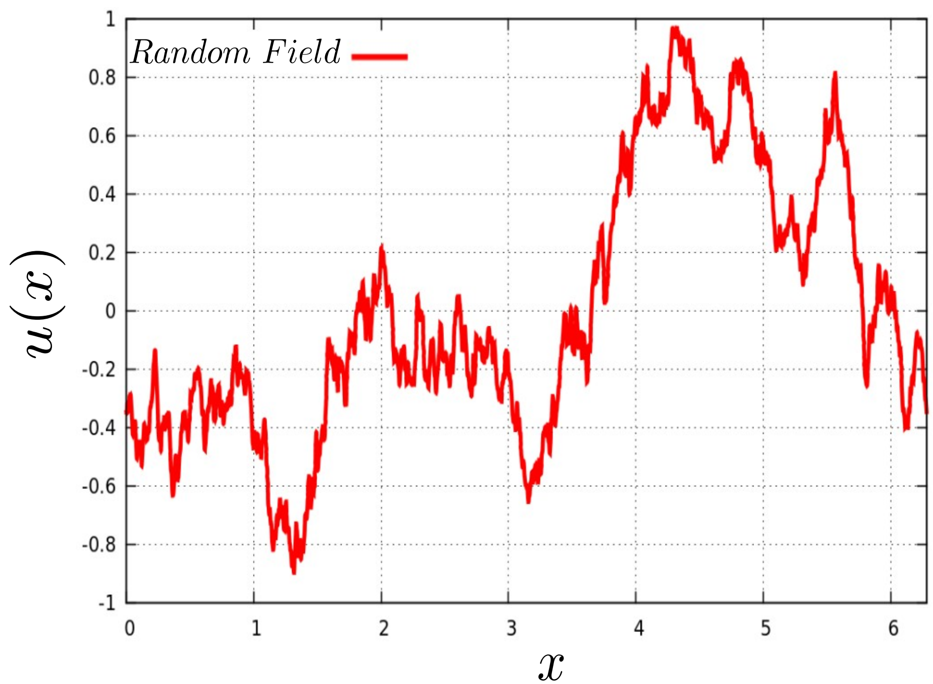


Fourier Space Energy spectrum:

$$E(k) = u(k)u^*(k)$$





Phases must be the responsible of the singular energy focusing (shock formation).

Question: How many degrees of freedom do we need?

Reduce to learn!

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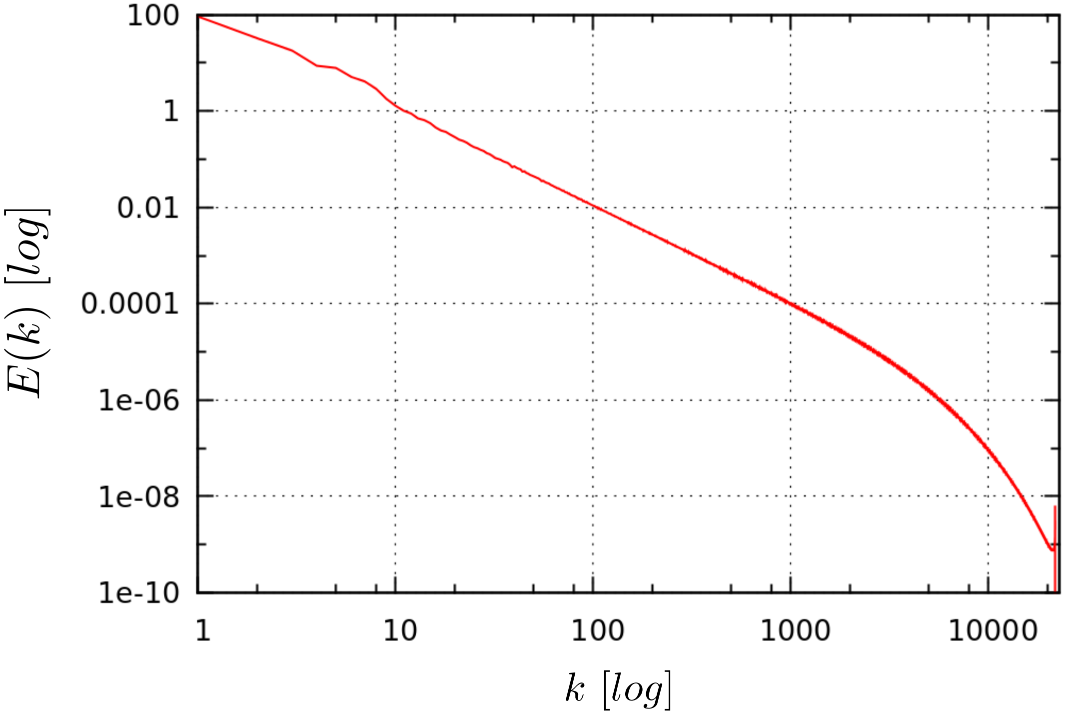

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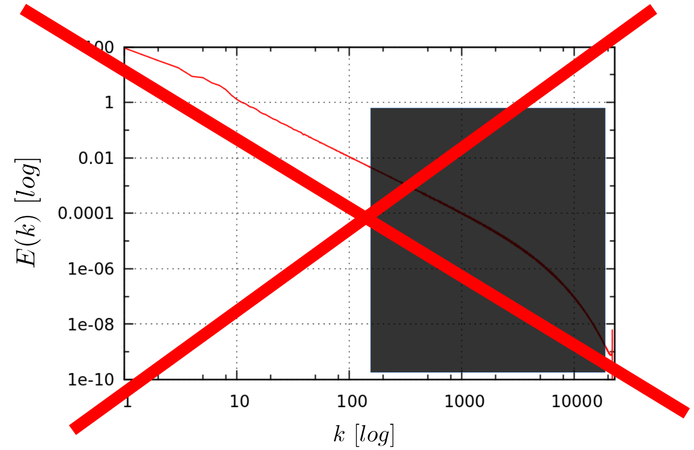
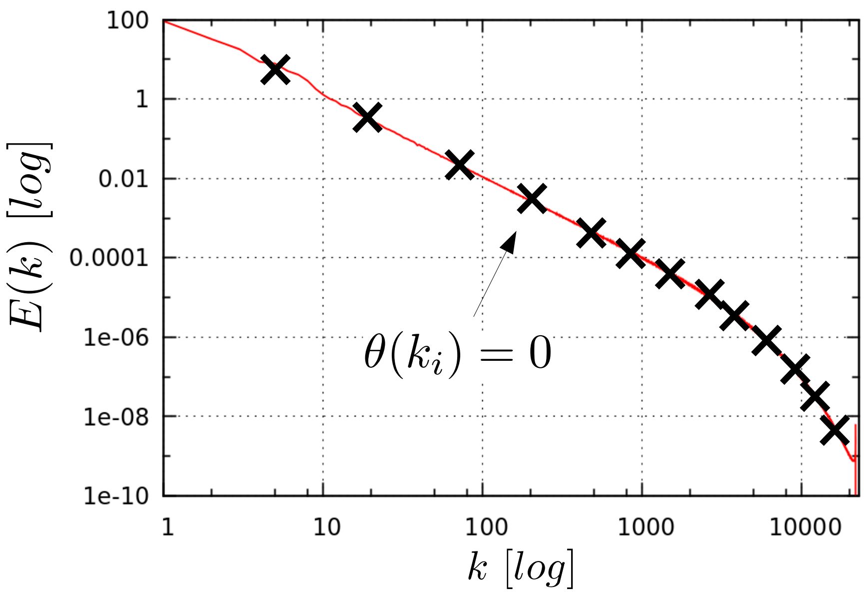




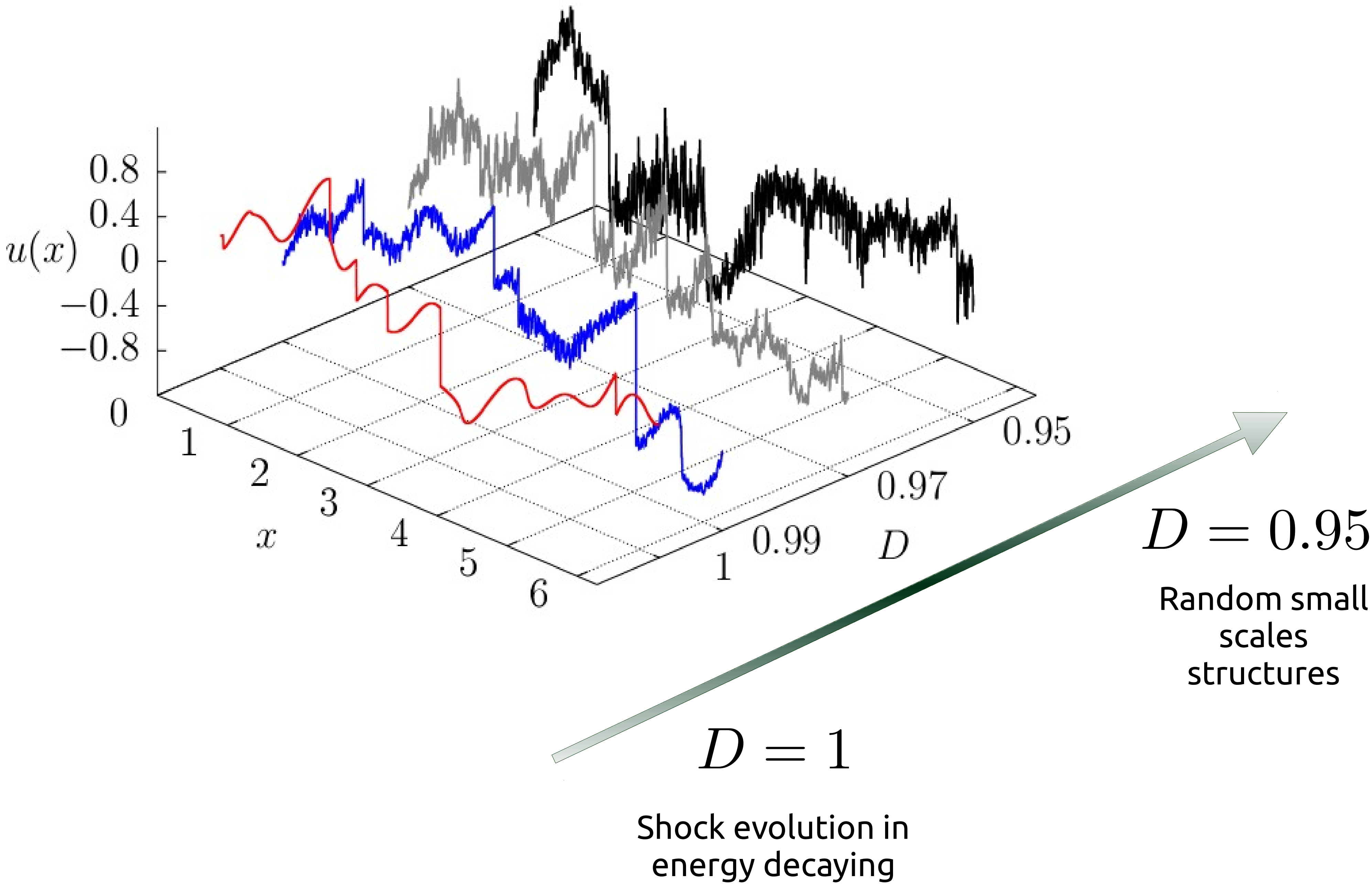
$$v(x,t) = \sum_{k \in \mathbb{Z}} e^{ikx} \theta(k) \hat{u}(k,t)$$

$$\theta(k) = \begin{cases} 1 & with \;\; probability \;\; h_k \sim (k/k_0)^{D-1} \\ 0 & with \;\; probability \;\; 1 - h_k \end{cases}$$

$$\#_{dof} = \int_0^k \theta(k') dk' \propto k^D$$

D is the system dimension; $0 < D \leq 1$

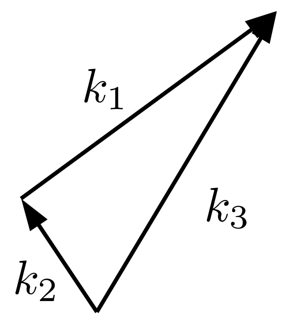
Real space evolution at changing of fractal dimension:



Fourier space Burgers' equation + Forcing

Triadic Interactions

$$\frac{d\hat{u}_k}{dt} = -\frac{ik}{2}\sum_{k_1,k_2\in\mathbb{Z}} \underline{\hat{u}_{k_1}\hat{u}_{k_2}\delta_{k1+k2,k}} - \nu k^2\hat{u}_k + \hat{F}_k$$



Amplitude – Phase representation:

$$\hat{u}_k = a_k(t)e^{i\phi_k(t)} \text{ where; } a_k(t) = |\hat{u}_k(t)|\,, \quad \phi_k(t) = \arg\hat{u}_k(t)$$

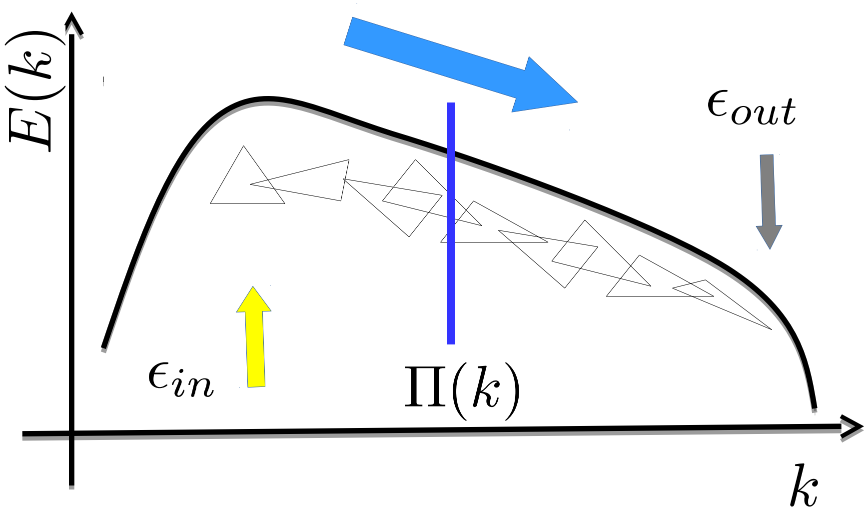
Energy flux towards small scales through k:

$$\Pi(k) = \sum_{k_1=1}^{k}\sum_{k_3+k_2=k_1}^{\infty} 2k_1\left\langle a_{k_1}a_{k_2}a_{k_3}\sin(\varphi_{k_1,k_2}^{k_3})\right\rangle$$

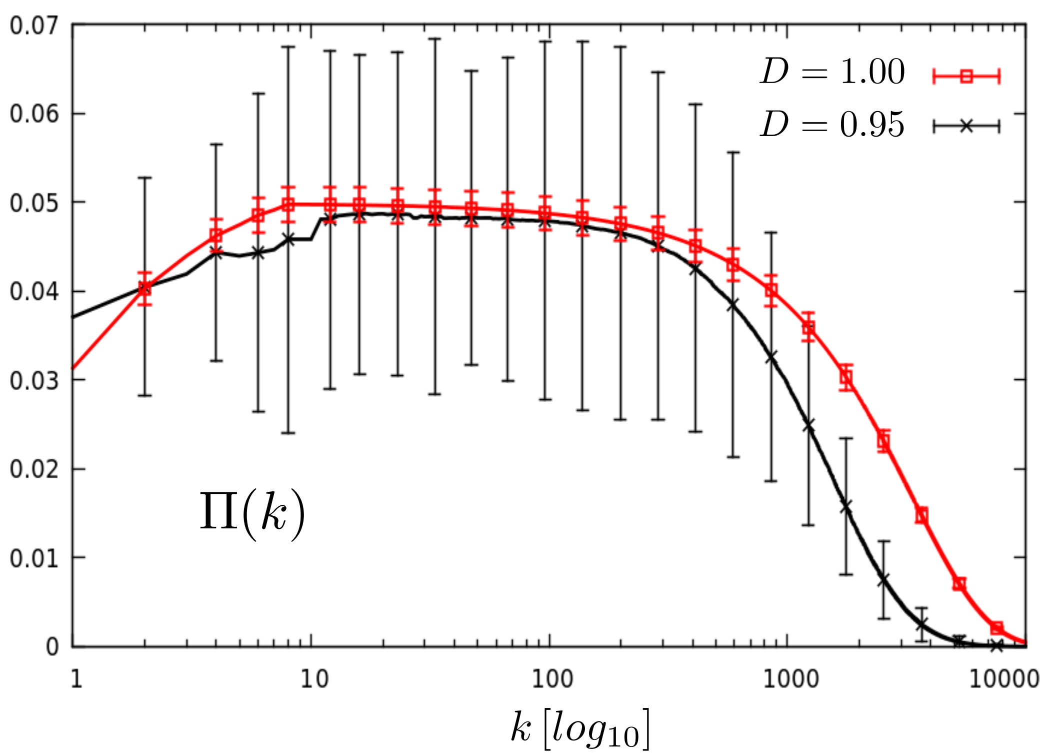
Key degrees of freedom: Triad dynamical phases

$$\varphi_{k_1\,k_2}^{k_3}(t) = \phi_{k_1}(t) + \phi_{k_2}(t) - \phi_{k_3}(t)\,, \qquad k_1 + k_2 - k_3 = 0$$

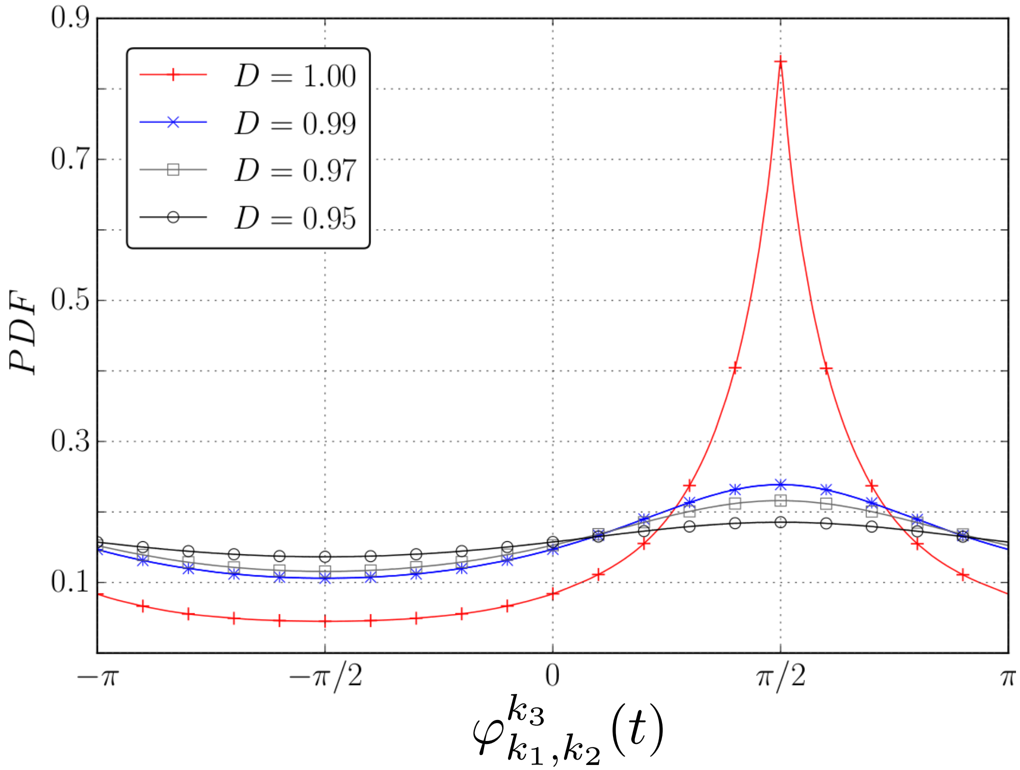
Energy flux towards small scales through k:



$$\Pi(k) =$$
$$= \sum_{k_1=1}^{k} \sum_{k_3=k+1}^{\infty} 2k_1 \left\langle a_{k_1} a_{k_2} a_{k_3} \sin(\varphi_{k_1,k_2}^{k_3}) \right\rangle$$

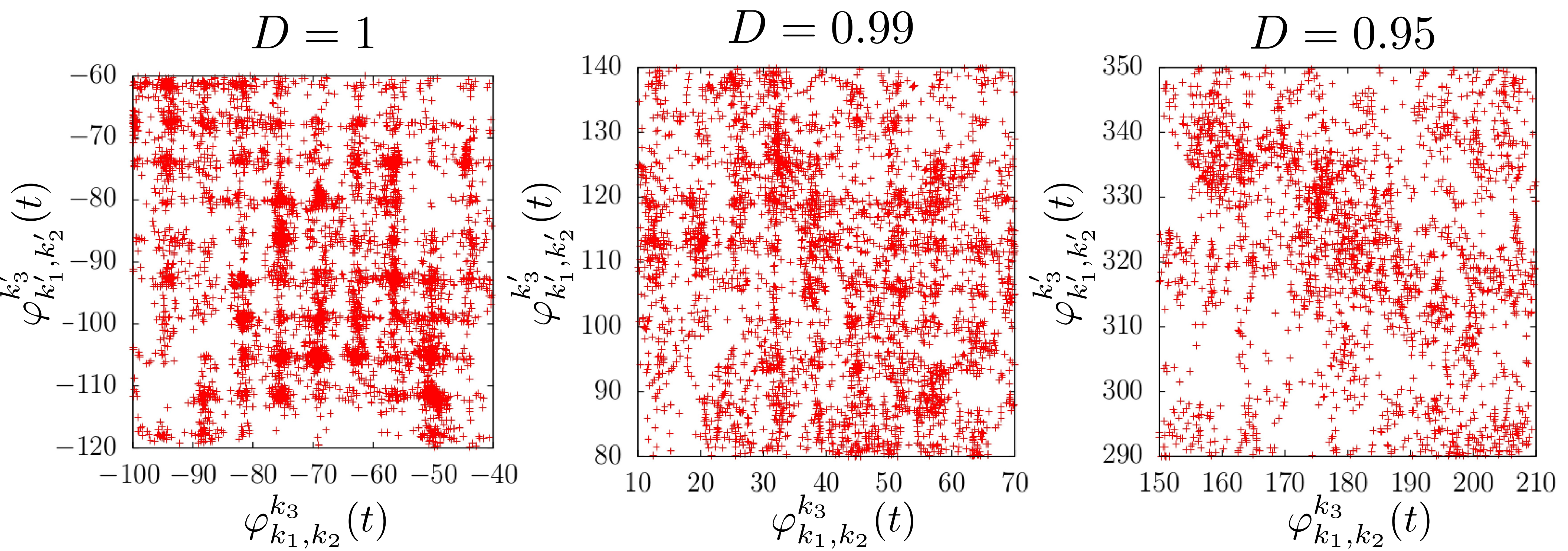


Triad **Phases Align** & **Synchronize** - Efficient Transfers



$$\Pi(k) = \\ = \sum_{k_1=1}^{k} \sum_{k_3=k+1}^{\infty} 2k_1 \left\langle a_{k_1} a_{k_2} a_{k_3} \sin(\varphi^{k_3}_{k_1,k_2}) \right\rangle$$

$\approx 160,000$ triads - inertial range



Triad dynamical **phase Precession**

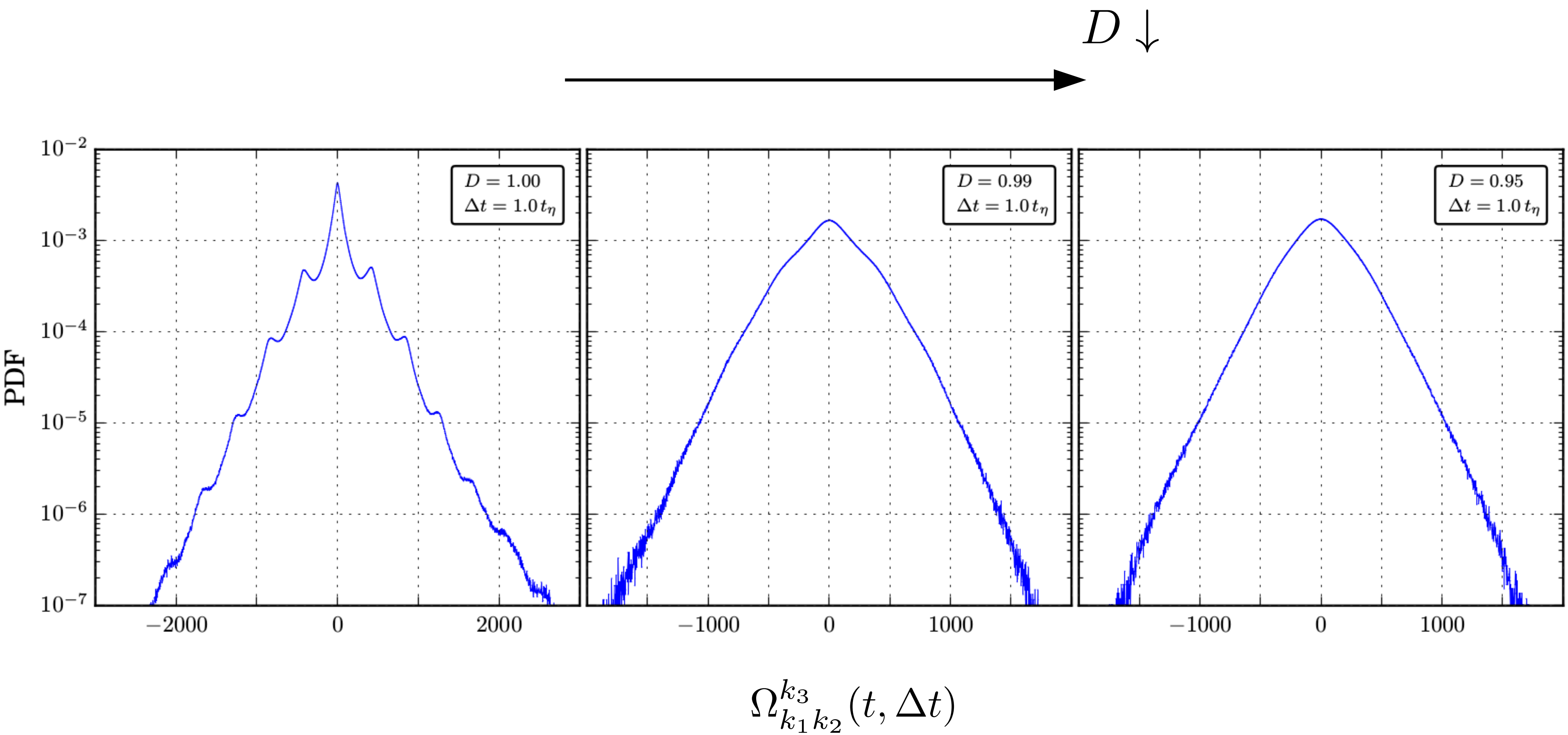
- Recall Triad dynamical Phase:

$$\varphi_{k_1\,k_2}^{k_3}(t) = \phi_{k_1}(t) + \phi_{k_2}(t) - \phi_{k_3}(t)$$

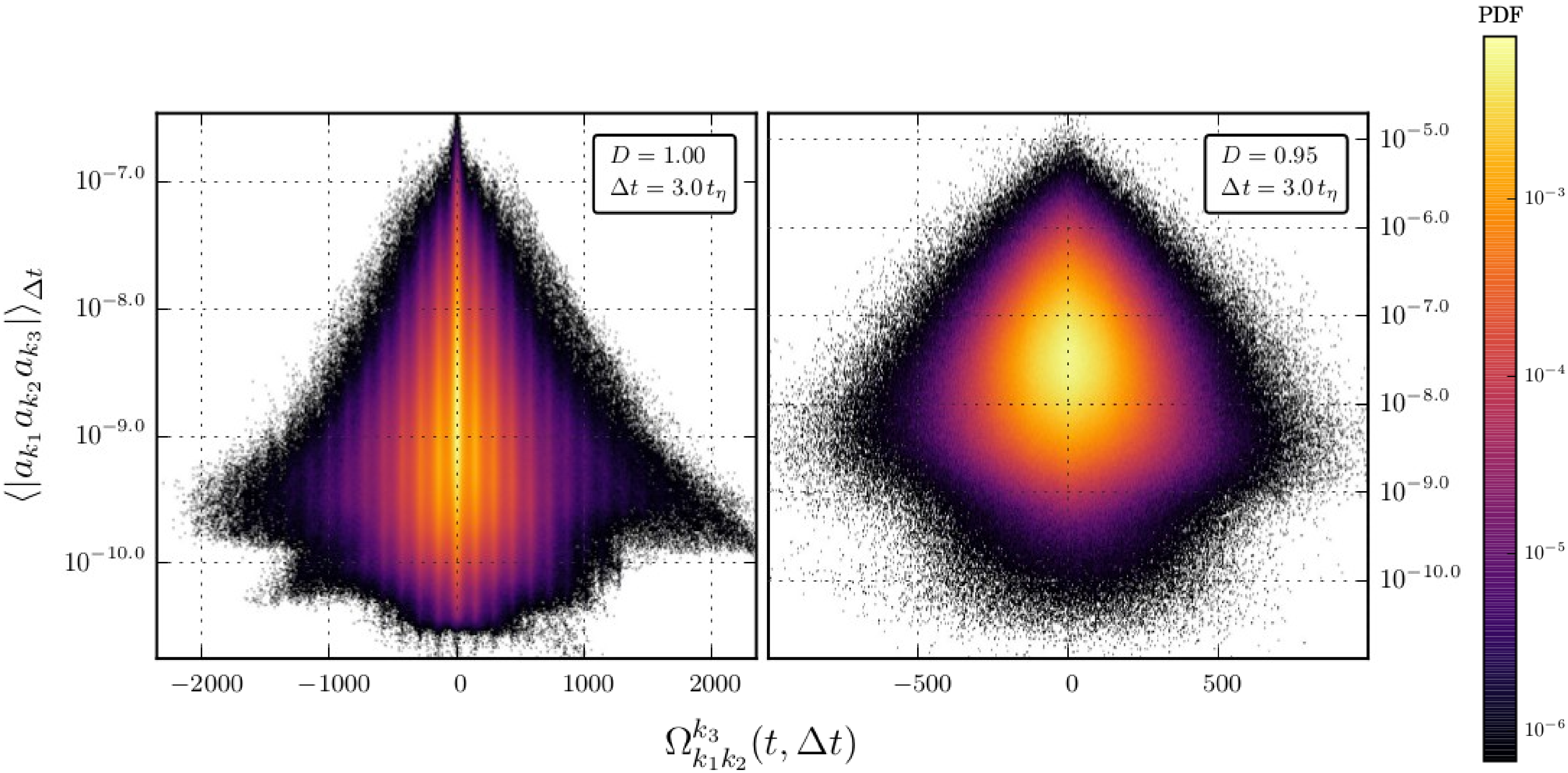
- Define the **Precession** of a triad's phase:

$$\Omega_{k_1 k_2}^{k_3} \equiv \lim_{t\to\infty} \frac{1}{t}\int_0^t \dot{\varphi}_{k_1 k_2}^{k_3}(t')dt' \equiv \left\langle \dot{\varphi}_{k_1 k_2}^{k_3}(t') \right\rangle_{t_\infty}$$

PDF triad phase' **Precession**



Joint PDF, Amplitude - Precession



Conclusions

We studied the evolution of forced **Burgers' equation under mode reduction in Fourier space**.

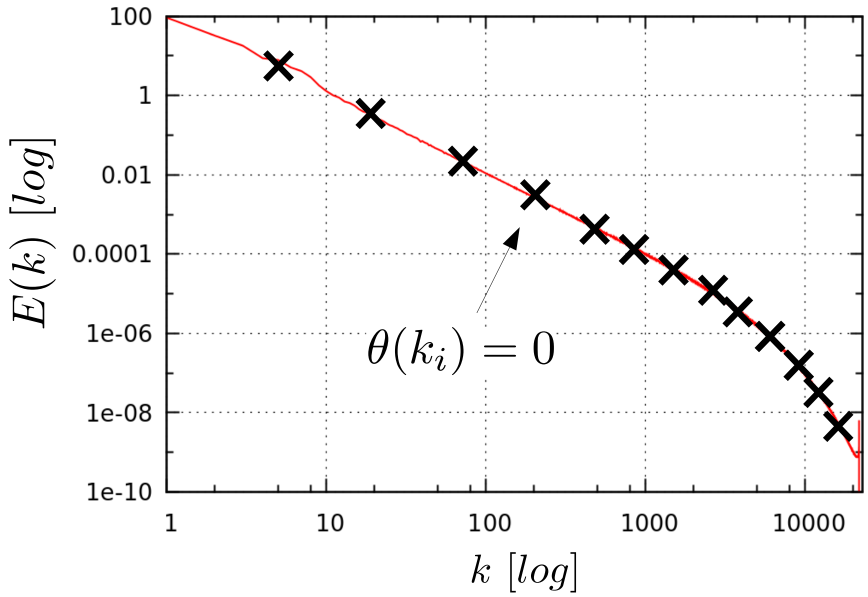
1) Energy transfer is strongly dependent on the mode reduction protocol.

2) Energy focusing is the result of a global phase correlation in Fourier space.

3) Bad news for modeling people.

4) Potential inspiration to search for similar phenomena in Navier-Stokes equations.

Modes Reduction, **Fractal Fourier decimation:**



Numerical approach:

$$v(x,t) = \sum_{k \in \mathbb{Z}} e^{ikx} \theta(k) \hat{u}(k,t)$$

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$$0 < D \leq 1$$

Decimated Burgers equation:

$$\partial_t v(x,t) + P_D \left[v \partial_x v(x,t) \right] =$$

$$= \nu \partial^2_{xx} v(x,t) + F_D$$

Decimation Main Properties:

1) The space **dimension** can **change continuously**

2) The original **symmetries** of the system are kept

3) It acts as a **Galerkin Truncation** without the introduction of any characteristic scale

4) The numerical evolution can be obtained via a **psedu-specral code**

Frisch, Pomyalov, Procaccia, and Ray. *Turbulence in non-integer dimensions by fractal Fourier decimation.* **Phys. Rev. Lett. 108, (2012)**.