Entropic Lattice Boltzmann Method

An implicit Large-Eddy Simulation?

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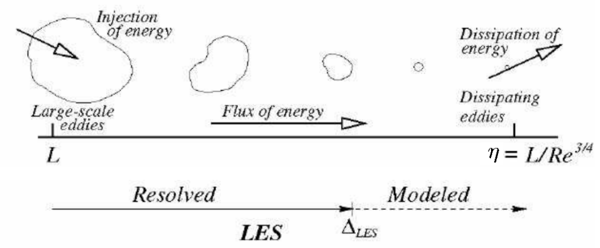
Introduction: Large Eddy Simulation (LES)

- Reduces the number of degrees of freedom by resolving scales only up to a cutoff scale and modeling the remaining smaller scales
- Enables cost-effective high Reynolds turbulent flow simulations

LES equation: Filtered Navier-Stokes + SGS model

$$\partial_t \overline{u}_i + \partial_j (\overline{u}_i \, \overline{u}_j) = -\frac{1}{\rho} \partial_i p + \nu \, \partial_j \overline{S}_{ij} - \partial_j \tau_{ij}, \text{ where } \overline{S}_{ij} = (\partial_j \overline{u}_i + \partial_i \overline{u}_j)$$

$\tau_{ij} = \overline{u_i u_j} - \overline{u}_i \, \overline{u}_j$ must be modeled using a Sub-Grid Scale (SGS) Model



Introduction: Lattice Boltzmann Method (LBM)

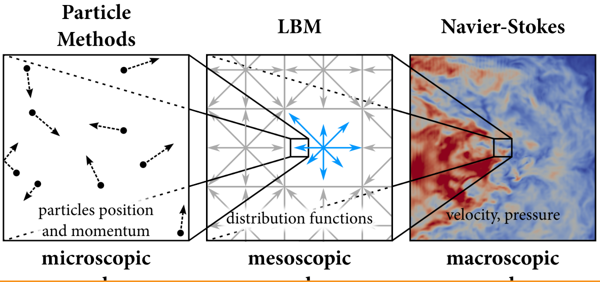
LBGK Equation **[Succi, 2001]**

$$f_i(\vec{x} + \vec{c}_i, t+1) - f_i(\vec{x}, t) = -\frac{1}{\tau}\left[\, f_i(\vec{x}, t) - f_i^{eq}(\vec{x}, t)\right]$$

which is a relaxation of typical time τ to the local equilibirum distribution:

$$f_i^{eq}(\vec{x}, t) = w_i\, \rho(\vec{x}, t)\left[1 + \frac{\vec{c}_i \cdot \vec{u}(\vec{x}, t)}{c_s^2} + \frac{\left(\vec{c}_i \cdot \vec{u}(\vec{x}, t)\right)^2}{2\, c_s^4} - \frac{|\vec{u}(\vec{x}, t)|^2}{2\, c_s^2}\right]$$

a 2^{nd} order expansion in $\frac{\vec{u}}{c_s}$ of the Maxwell-Boltzmann distribution



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- Chapman-Enskog expansion shows the relation between viscosity ν and the relaxation time τ

$$\nu = c_s^2\,(\tau - 0.5) \quad \text{where } c_s \text{ is the speed of sound in the lattice}$$

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Can we have an equivalent LES turbulence model for LBM?

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ELBM: A search for LBM stabilization

Can we use LBM to study turbulent flows?
Instabilities arise when $\tau \to 0.5$ ($\nu \to 0$) making standard LBGK unadapted to the study of turbulent flows

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ELBM principle is to equip LBM with an in-built H-theorem

ELBM: A LBM with an in-built H-theorem

- $\mathbf{f^{eq}}$ is defined as the maxima of a convex H-function under the constraints of mass and momentum conservation:

$$H(\mathbf{f}) = \sum_{0}^{q-1} f_i \log\left(\frac{f_i}{\omega_i}\right), \qquad \rho = \sum_i f_i^{eq}, \quad \rho\vec{u} = \sum_i \vec{c}_i\, f_i^{eq}$$

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- The ELBM eq. introduces a fixed parameter β and a local one α

ELBM Equation [Karlin *et al.*, 1999]

$$f_i(x + c_i, t + 1) = f_i(x, t) + \alpha\beta\left[f_i^{eq}(x, t) - f_i(x, t)\right]$$

where $\beta = \frac{1}{2\tau}$

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- Setting $\mathbf{f^{mirror}} = \mathbf{f} - \alpha\,(\mathbf{f} - \mathbf{f^{eq}})$, we can rewrite the ELBM eq.

ELBM Equation **[Karlin *et al.*, 1999]**

$$f_i(x + c_i, t + 1) = (1 - \beta)\ f_i(x, t) + \beta\, f_i^{mirror}(x, t)$$

where $\beta = \frac{1}{2\tau}$, with $0 < \beta < 1$ as we have $0.5 < \tau < +\infty$

ELBM: A LBM with an in-built H-theorem

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- Setting $\mathbf{f^{mirror}} = \mathbf{f} - \alpha\,(\mathbf{f} - \mathbf{f^{eq}})$, we can rewrite the ELBM eq.
- α is calculated at each node and each time step as the solution of the following equation:

$$H(\mathbf{f}) = H\left(\mathbf{f^{mirror}}(\alpha)\right)$$

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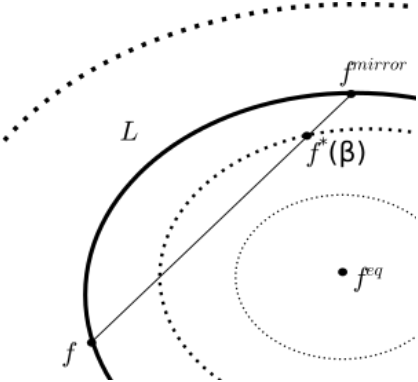
where $\beta = \frac{1}{2\tau}$, with $0 < \beta < 1$ as we have $0.5 < \tau < +\infty$

ELBM: Perspective from H-functional hypersurface

Calculation of α and convexity of H insure monotonic decreases of H



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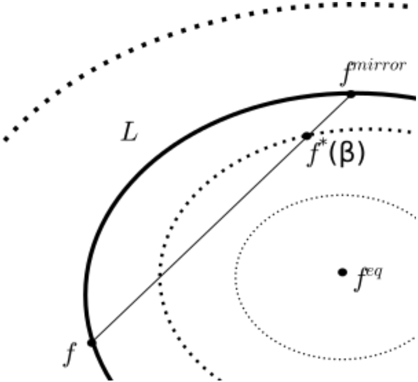

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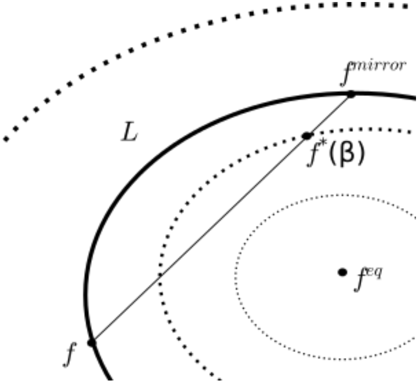

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... but we need to understand if the right physics is represented

Is ELBM a LBM with an implicit LES?

[Karlin *et al.*, 1999]

- The viscosity ν is allowed to fluctuate locally:

$$\nu(\alpha) = c_s^2 \left(\frac{1}{\alpha\beta} - 0.5 \right)$$

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[Malaspinas *et al.*, 2008]

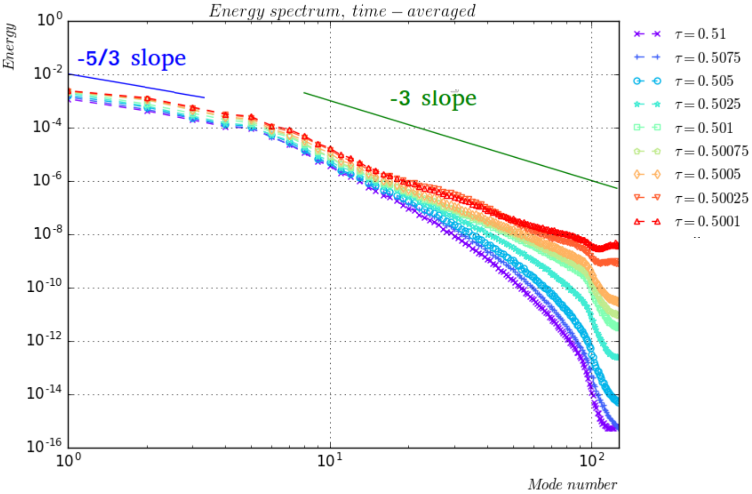
- Chapman-Enskog expansion was performed for $\alpha \approx 2$ and an additional term of the form $\nu_r S_{ij}$ appeared with:

$$\nu_r = -\frac{c_{s^2}\Delta t}{3(2\beta)^2} \frac{S_{\theta\kappa} S_{\kappa\gamma} S_{\gamma\theta}}{S_{\lambda\mu} S_{\lambda\mu}}$$

Very similar to a Smagorinsky subgrid scale model

Dissipative properties of 2D ELBM

D2Q9 forced 2D homogeneous turbulence simulations for different $\tau \to 0.5$



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Thank you for your attention!

Any questions?

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