Energy cascades in two-dimensional three-component flows

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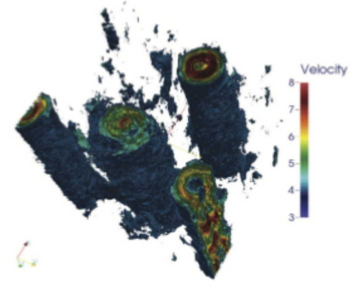
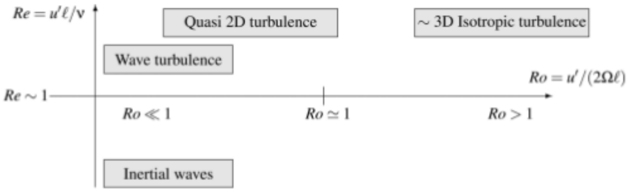
Naturally occurring 2D3C flows



$$\partial_t \boldsymbol{u}^{\mathrm{2D}} = -(\boldsymbol{u}^{\mathrm{2D}} \cdot \nabla)\boldsymbol{u}^{\mathrm{2D}} - \nabla P + \nu \Delta \boldsymbol{u}^{\mathrm{2D}} \ ,$$
$$\partial_t \theta = -(\boldsymbol{u}^{\mathrm{2D}} \cdot \nabla)\theta + \nu \Delta \theta$$

Biferale et al. PRX 2016



Godeferd & Moisy App. Mech. Rev. 2015

2D3C building blocks of the Navier-Stokes equations

Consider the Navier-Stokes equations on a periodic domain:

$$\partial_t \boldsymbol{u} = -(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla P + \nu \Delta \boldsymbol{u} \ ,$$

$$\partial_t \hat{\mathbf{u}}_{\boldsymbol{k}} = - \sum_{\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=0} (i\boldsymbol{k} \cdot \hat{\mathbf{u}}^*_{\boldsymbol{p}})\hat{\mathbf{u}}^*_{\boldsymbol{q}} - i\boldsymbol{k}\hat{\mathbf{P}} - \nu k^2 \hat{\mathbf{u}}_{\boldsymbol{k}} \ ,$$

Each triad of wavevectors defines an interaction of Fourier modes.

The wavevectors in each triad are linearly dependent
$\Longrightarrow$ Each triad defines a plane Fourier space, e.g., the (k_x, k_y)-plane.
$\Longrightarrow$ $k_z = 0$ and the corresponding flow has no variation along z.
(Moffatt, JFM **741** R3, (2014))

The Navier-Stokes equations consist of a sum over triads, each of which defines a 2D3C flow.

Outline

1. 2D3C and helical decomposition
2. DNS
   - helical vs nonhelical forcing
   - transition 2D3C $\longrightarrow$ 3D

Main points:

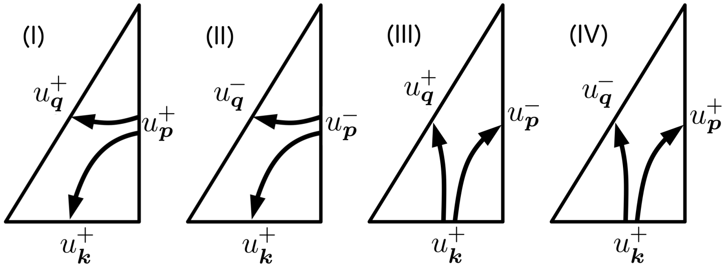
1. 2D3C flows can be described by two stream functions
2. Projection onto homochiral subspace $\Longrightarrow$ θ is no longer passive
3. zero-flux nonequilibrium dynamics
   - helical vs nonhelical forcing
   - transition 2D3C $\longrightarrow$ 3D

Helical Fourier decomposition

Helicity $\quad H = \langle \boldsymbol{u} \cdot \boldsymbol{\omega} \rangle = \sum_{\boldsymbol{k} \in \mathbb{Z}^3} \hat{\mathbf{u}}_{-\boldsymbol{k}} \cdot i\boldsymbol{k} \times \hat{\mathbf{u}}_{\boldsymbol{k}} \leqslant \sum_{\boldsymbol{k} \in \mathbb{Z}^3} k |\hat{\mathbf{u}}_{\boldsymbol{k}}|^2 \ .$

$$\hat{\mathbf{u}}_{\boldsymbol{k}}(t) = \hat{\mathbf{u}}^+_{\boldsymbol{k}}(t) + \hat{\mathbf{u}}^-_{\boldsymbol{k}}(t) = \sum_{s_k \in \{+,-\}} u^{s_k}_{\boldsymbol{k}}(t) \boldsymbol{h}^{s_k}_{\boldsymbol{k}} \ ,$$

where $i\boldsymbol{k} \times \boldsymbol{h}^{s_k}_{\boldsymbol{k}} = s_k k \boldsymbol{h}^{s_k}_{\boldsymbol{k}}$ and $i\boldsymbol{k} \times \hat{\mathbf{u}}^{s_k}_{\boldsymbol{k}}(t) = s_k k \hat{\mathbf{u}}^{s_k}_{\boldsymbol{k}}(t)$.



F. Waleffe PoF A, **4**, 350-363 (1992)

Decompositions of a 2D3C flow

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, \boldsymbol{y}, t) \; ; \quad \nabla \cdot \boldsymbol{u} = 0 \; ;$$

Stream function and passive scalar

$$\boldsymbol{u}^{\mathrm{2D}} = \begin{pmatrix} u_x \\ u_y \\ 0 \end{pmatrix} \equiv \begin{pmatrix} \partial_y \psi^{\mathrm{2D}} \\ -\partial_x \psi^{\mathrm{2D}} \\ 0 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{\theta} = \begin{pmatrix} 0 \\ 0 \\ \theta \end{pmatrix} \; .$$

Helical stream functions

$$\boldsymbol{u}^{\mathrm{2D}} = \begin{pmatrix} \partial_y(\psi^+ + \psi^-) \\ -\partial_x(\psi^+ + \psi^-) \\ 0 \end{pmatrix} \quad \text{and} \quad \theta = (-\Delta)^{1/2}(\psi^+ - \psi^-) \; .$$

Quadratic inviscid invariants:

$$E^{2D} = \frac{1}{2} \sum_{\boldsymbol{k} \in \mathbb{Z}^3} k^2 |\hat{\psi}^+_{\boldsymbol{k}} + \hat{\psi}^-_{\boldsymbol{k}}|^2 \ ,$$

$$E^{\theta} = \frac{1}{2} \sum_{\boldsymbol{k} \in \mathbb{Z}^3} k^2 |\hat{\psi}^+_{\boldsymbol{k}} - \hat{\psi}^-_{\boldsymbol{k}}|^2 \ ,$$

$$\Omega = \sum_{\boldsymbol{k} \in \mathbb{Z}^3} k^4 |\hat{\psi}^+_{\boldsymbol{k}} + \hat{\psi}^-_{\boldsymbol{k}}|^2 \ ,$$

$$H = 2 \sum_{\boldsymbol{k} \in \mathbb{Z}^3} k^3 (|\hat{\psi}^+_{\boldsymbol{k}}|^2 - |\hat{\psi}^-_{\boldsymbol{k}}|^2) = 2\langle \theta\omega \rangle \ ,$$

Enstrophy of the z-component:

$$\langle |\boldsymbol{\omega}^{\theta}|^2 \rangle = \sum_{\boldsymbol{k} \in \mathbb{Z}^3} k^4 |\hat{\psi}^+_{\boldsymbol{k}} - \hat{\psi}^-_{\boldsymbol{k}}|^2 \ .$$

If $\hat{\psi}^-_{\boldsymbol{k}} = 0$ or $\hat{\psi}^+_{\boldsymbol{k}} = 0$ then $\langle |\boldsymbol{\omega}^{\theta}|^2 \rangle = \Omega \implies$ Inverse energy cascade of θ.

Projection onto subspace $\psi^- = 0$

Removal of one helical degree of freedom

- enforced correlation between θ and ω: $\langle \theta\omega \rangle \neq 0$,
- θ being no longer passive,
- inverse energy cascade of θ.

2D3C flow described by a single stream function:

$$\partial_t \psi^+ = \frac{(-\Delta)^{-1/2}}{2}(\nabla\psi^+ \times \nabla(-\Delta)^{1/2}\psi^+)_z + \frac{(-\Delta)^{-1}}{2}(\nabla\psi^+ \times \nabla(-\Delta)\psi^+)_z \, .$$

Summary

2D3C flows can be described in two ways:

$$(\psi^{2\mathrm{D}}, \theta)$$

- decomposition into independent 2D and 3C dynamics
- rapidly rotating flows
- passive scalar advection in 2D turbulence

$$(\psi^{+}, \psi^{-})$$

- decomposition into interacting 2D3C structures
- 3D turbulent flows

Numerical simulations

Pseudospectral DNS of hyperviscous equations

$$\partial_t \boldsymbol{u} = -\nabla \cdot (\boldsymbol{u} \otimes \boldsymbol{u}) - \nabla P + \nu(-1)^{n+1}\Delta^n \boldsymbol{u} + \boldsymbol{f} \ ,$$
$$\nabla \cdot \boldsymbol{u} = 0 \ ,$$

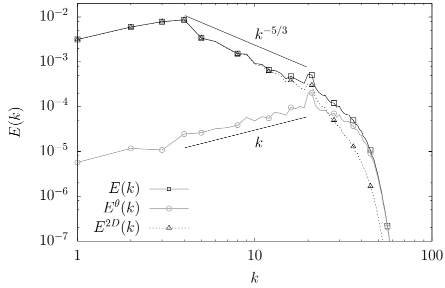
- $V = [0, 2\pi)^3$, periodic BC
- 256^3 grid points, dealiasing by 2/3-rule
- power of Laplacian $n = 4$
- no mean flow: $\langle \boldsymbol{u} \rangle = 0$

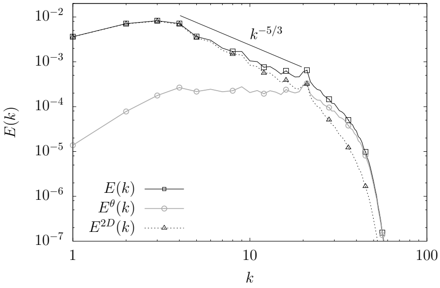
1. $\boldsymbol{f}$ random, $\delta(t)$-correlated forcing, $k_f \in [20, 21]$
   - helical forcing $\boldsymbol{f} = \boldsymbol{f}^+$
   - nonhelical forcing $\langle |\boldsymbol{f}^+|^2 \rangle = \langle |\boldsymbol{f}^-|^2 \rangle$
2. add percentage α of 3D Fourier modes randomly

Helical vs nonhelical forcing: energy spectra

nonhelical forcing

helical forcing





θ in equilibrium

θ out of equilibrium

Energy balance:

$$\sum_{k'=1}^{k} \partial_t E(k',t) = \underbrace{-\sum_{k'=1}^{k} \sum_{|\mathbf{k}|=k'} \hat{\mathbf{u}}_{\mathbf{k}} \cdot \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} (i\mathbf{k} \cdot \hat{\mathbf{u}}_{\mathbf{p}}) \hat{\mathbf{u}}_{\mathbf{q}}}_{\text{inertial flux across } k:\ \Pi(k)} + 2\nu \sum_{k'=1}^{k} k'^2 E(k',t) + F(k)$$

Energy cascade directions

$\Pi(k) > 0$ inverse cascade $\quad\quad$ $\Pi(k) < 0$ direct cascade

Subfluxes: 2D-component $\Pi^{\mathrm{2D}}(k)$, vertical component $\Pi^{\theta}(k)$,

$$\text{homochiral} \quad \Pi^{\mathrm{HO}}(k) = -\sum_{k'=1}^{k} \sum_{|\mathbf{k}|=k'} \sum_{s\in\{+,-\}} \hat{\mathbf{u}}^s_{\mathbf{k}} \cdot \sum_{\mathbf{k}+\mathbf{p}+\mathbf{q}=0} (i\mathbf{k} \cdot \hat{\mathbf{u}}^s_{\mathbf{p}}) \hat{\mathbf{u}}^s_{\mathbf{q}} \ ,$$

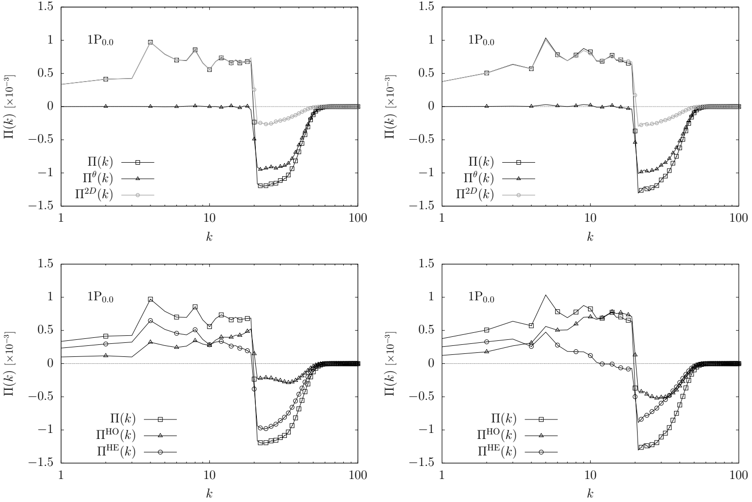
$$\text{heterochiral} \quad \Pi^{\mathrm{HE}}(k) = \Pi(k) - \Pi^{\mathrm{HO}}(k) \ .$$

nonhelical forcing

helical forcing

$(\psi^{2\mathrm{D}}, \theta)$

(ψ^{+}, ψ^{-})

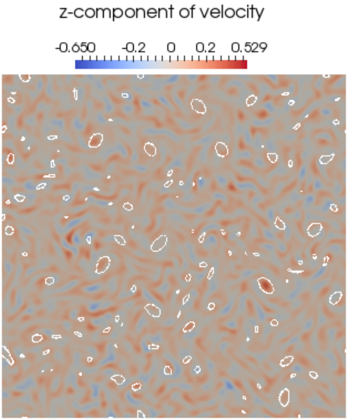


θ in equilibrium

θ out of equilibrium

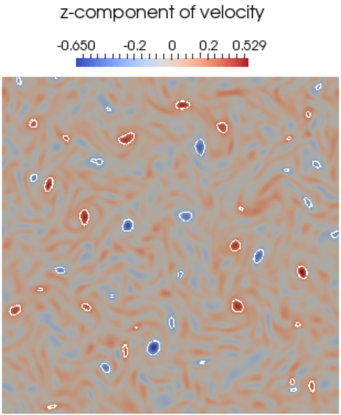
Visualisations θ, ω

nonhelical forcing



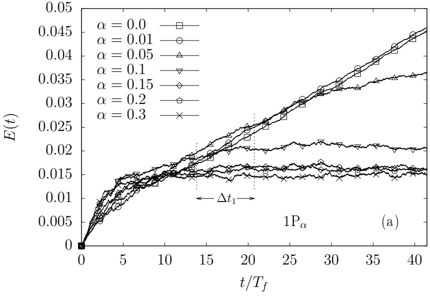
θ in equilibrium

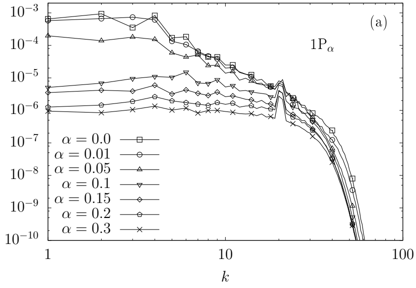
helical forcing

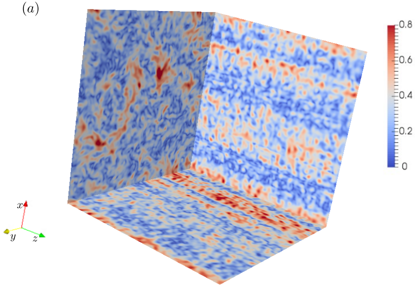


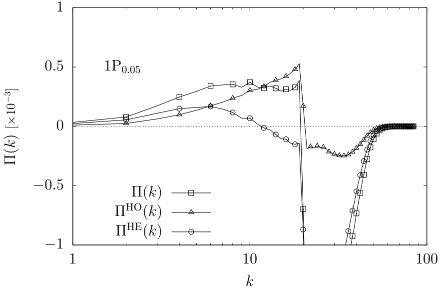
θ out of equilibrium

Transition 2D3C ⟶ 3D

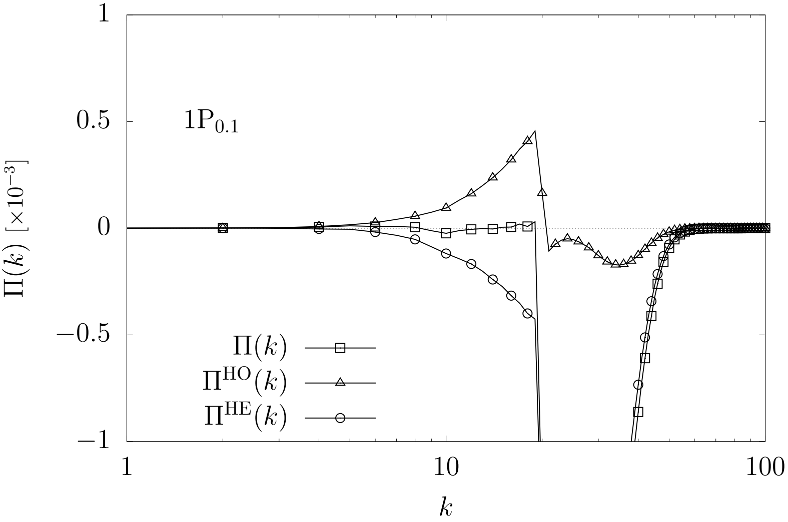





(a)
0.8
0.6
0.4
0.2
0
x
y
z


$1P_{0.05}$
$\Pi(k)$ $[\times 10^{-3}]$
k
1
0.5
0
−0.5
−1
1
10
100
$\Pi(k)$
$\Pi^{HO}(k)$
$\Pi^{HE}(k)$

Transition 2D3C $\longrightarrow$ 3D



Conclusions

1. 2D3C dynamics can be studied in two different ways: (ψ^{2D}, θ) or (ψ^+, ψ^-)
2. Projection onto ψ^+ leads to entanglement of all three components: θ is no longer a passive scalar
3. Transition from 2D3C $\longrightarrow$ 3D dynamics (inverse $\rightarrow$ direct cascade)
   - mainly 3D for $> 15\%$ of added 3D Fourier modes
   - cases where zero inverse flux is sustained by non-equilibrium dynamics

Thank you