Physica D 65 (1993) 163–171
North-Holland

SDI: 0167-2789(92)00018-J

PHYSICA D

On intermittency in a cascade model for turbulence

R. Benzi, L. Biferale and G. Parisi
Dipartimento di Fisica, Università "Tor Vergata", Via della Ricerca Scientifica, I-00133 Rome, Italy

Received 28 April 1992
Revised manuscript received 8 September 1992
Accepted 16 October 1992
Communicated by U. Frisch

In this note we study the possibility of performing analytic computations of the exponents characterizing the multifractal behaviour of turbulence. A simple analytic computation is presented in the framework of the cascade model (or shell model).

1. Introduction

One of the most striking properties of the Navier–Stokes equation in the fully developed limit is the scaling behaviour of the increments of the velocity field $u(x)$:

$$\langle \delta_l u^n \rangle \propto l^{\zeta(n)}$$
$$\text{with } \delta_l u = |u(x+l) - u(x)| , \tag{1}$$

where $\langle \ldots \rangle$ is a spatial average and $\zeta(n)$ is a nonlinear function of n. Following the previous works [1–3] the nonlinearity of $\zeta(n)$ has been interpreted as an indication of the existence of many different kind of singularities of gradients of the velocity field, each one characterized by an exponent α [4,5]. Near one of these singularities the velocity field behaves as $\delta_l u \propto l^\alpha$ and the support of these singularities has a Hausdorff dimension which depends on α (let us call it $D(\alpha)$). The name multifractal was invented to describe this situation [4]. The concept of multifractal has many applications in different fields of physics and they have been reviewed in [6]. A simple computation [4,5] shows that the two functions $\zeta(n)$ and $D(\alpha)$ are simply related by a Legendre transform:

$$\zeta(n) = \min_\alpha [n\alpha + 3 - D(\alpha)] . \tag{2}$$

Intermittency and multifractality are obviously related. The kurtosis of $\delta_l u$ increases for small l as $l^{\zeta(4)-2\zeta(2)}$: the rare events leading to strong turbulence are due to the effects of singularities with a small value of α, which are assumed to be concentrated on a set with relative small Hausdorff dimension. Unfortunately up to now there is no first principle deviation of multifractality in three dimensional turbulence and we are very far from a computation of the critical exponents $\zeta(n)$. It may be wise to study similar phenomena in simplified models where a more detailed analysis may be performed. In this note we consider an energy cascade model [7] which recently has been intensively studied numerically [8,9]. One defines a scale k_j in a momentum space ($k_j = \lambda^j$ for $j = 1, \ldots, N$, N eventually goes to infinity); one also introduces the complex variables $u_j(t)$, which should characterize the collective behaviour of the velocity field of real turbulence in a shell of momenta $k = k_j$. Very often the simplifying choice $\lambda = 2$ is done. The total energy is given by $E = \Sigma_j |u_j|^2$ and the evolution equations are

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu k_j^2\right) u_j = F_j + \mathrm{i}\left(\frac{\lambda - 1}{2}\right) \times (2k_{j+1} u_{j+1}^* u_{j+2}^* - k_j u_{j-1}^* u_{j+1}^* - k_{j-1} u_{j-2}^* u_{j-1}^*) , \tag{3}$$

where ν is the viscosity and the forcing F_j is concentrated at small j. In the zero viscosity limit and in absence of forcing the energy is conserved. Numerical experiments [9] show that in the small viscosity limit $\langle u(k_j)^n \rangle \propto k_j^{-\zeta(n)}$ where $\langle \ldots \rangle$ denotes the time average. The shell model described in eq. (3) should be regarded as simple model which has some mathematical characteristics, among them the multifractal behaviour, which are in common with turbulence. We are convinced that the correct understanding of the origin of the multifractality in this model should be a crucial step for obtaining similar results in fully developed turbulence. The aim of this note is to present a possible approach to this problem and to do some simple computations, whose result is of the correct order of magnitude. We will concentrate our attention on the stationary probability $P[u]$, where we denote by $[u]$ the set of all u's. The knowledge of $P[u]$ is enough to obtain all the relevant information. We are interested in finding the form of $P[u]$ in the region of large j and in the zero viscosity limit (i.e. fully developed turbulence). In this region we may suppose that the results are universal in the sense that they do not depend on the detailed form of the forcing. Fortunately the form of $P[u]$ is not arbitrary: it is strongly constrained by the so called closure equations [10] which can be written as $\langle \mathrm{d}A[u]/\mathrm{d}t \rangle = 0$ where we use the equation of motion (3) to compute the derivative of A, A being any arbitrary functional of u's. Different choices of the function A lead to different closure equations. For instance, a possible complete set of equation of motion to be closed could be given by $\langle \mathrm{d}u^p/\mathrm{d}t \rangle$, for any p. For the $P[u]$ one can make, in the fully turbulent regime, the following ansatz:

$$P[u] \propto \exp\left(-\sum_j H_j\right) , \tag{4}$$

where H_j is given by $H(u_j, u_{j-1}, u_{j+1}, u_{j-2}, u_{j+2}, \ldots)$. It is easy to verify that the above expression is compatible with the closure equations. In other words we assume that the Hamiltonian, $H = \Sigma_j H_j$, corresponding to the stationary distribution is invariant under scale transformations (i.e. translations with respect to j). It is natural to suppose, in agreement to what happens in usual phase transitions, that H is essentially short range, i.e. the dependence of H_j on u_{j+m} can be neglected for m sufficiently large. In what follows we will assume for simplicity that H has a strictly finite range m. The other crucial assumption that we do consist in assuming that H_j is a homogeneous function of degree zero: it depends only on the ratio between the u's and their angles. This assumption automatically leads to a scaling law for u. In order to understand better the meaning of these hypotheses on $P[u]$, we recall a useful theorem which states that under some conditions the probability distribution (4) may be generated by a random (multiplicative) process. Let us consider the simple case where $H_j = H(u_j, u_{j+1})$. If the integral equation

$$\int \mathrm{d}y \exp[-H(x, y)]\, \psi(y) = \lambda \psi(x) \tag{5}$$

has a solution with positive λ, than the function

$$P(x, y) = \exp[-H(x, y)]\, \psi(y)/\lambda\psi(x) \tag{6}$$

is well normalized (i.e. $\int \mathrm{d}y P(x, y) = 1$). Thus we can construct the following Markov chain in which the conditional probability of having u_{j+1}, for given u_j, is just given by $P(u_j | u_{j+1})$. It is a very simple computation to verify that the probability distribution of the u's generated by this process is given by eq. (4).

The fact that the transition probability P is a function of degree zero in the u's tell us that this process is essentially a random multiplicative process. More generally we are supposing that there is a conditional probability $P(\ldots, u_{j-2}, u_{j-1}, u_j|u_{j+1})$ which is an homogeneous function of zero degree in the u's, the dependence by the far away u's may be neglected and the process generated by it produces the equilibrium distribution. We prefer to concentrate our attention on the process generating the probability distribution, more than on the probability distribution itself because in this case the computations, both analytic and numerical are much simpler. If we suppose that $P(\ldots, u_{j-2}, u_{j-1}, u_j|u_{j+1})$ depends only on m variables, we remain with a function of $m+1$ variables to be determined. Our proposal is to determine it by imposing that the closure equations are satisfied as much as possible. We remark that the idea that intermittency could be produced by random multiplicative processes goes back to the work of Novikov [11]. Recently Chabra and Sreenivasan [12] have found some experimental evidences that a random multiplicative process could exist in fully developed turbulence. This paper suggests why such a process might, in some sense, be consistent with the dynamical equation of a model of three dimensional Navier–Stokes equations.

In section 2 we write the energy equation of the system and we introduce the notation; in section 3 we discuss our main ansatz on the multiplicative process and we write the closure equation whose solution are given in section 4, conclusions follow in section 5.

2. The energy equation

In order to find a possible way to follow the ideas discussed in the introduction, we consider a slight different form of the model equations (3). Let us take the variables u_j defined as $u_j = k_j^{-1/3}\phi_j$.

The equations for ϕ_j are (in absence of forcing):

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu k_j^2\right)\phi_j = \mathrm{i}k_j^{2/3}(\phi^*_{j+1}\phi^*_{j+2} - \tfrac{1}{2}\phi^*_{j-1}\phi^*_{j+1} - \tfrac{1}{2}\phi^*_{j-2}\phi^*_{j-1})\,. \tag{7}$$

Next we perform a polar decomposition of the variables ϕ (we define $\phi_j = \rho_j \exp(\mathrm{i}\theta_j)$) and look for the equations of the moduli ρ_j:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu k_j^2\right)\rho_j = k_j^{2/3}[\rho_{j+1}\rho_{j+2}\sin(\theta_j + \theta_{j+1} + \theta_{j+2}) - \tfrac{1}{2}\rho_{j-1}\rho_{j+1}\sin(\theta_j + \theta_{j+1} + \theta_{j-1}) \tfrac{1}{2}\rho_{j-2}\rho_{j-1}\sin(\theta_j + \theta_{j-1} + \theta_{j-2})]\,. \tag{8}$$

In the following we will use the variables $\Delta_j = \theta_{j-2} + \theta_{j-1} + \theta_j$ in order to simplify the notation. With this choice equation (8) becomes:

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \nu k_j^2\right)\rho_j = k_j^{2/3}[\rho_{j+1}\rho_{j+2}\sin(\Delta_{j+2}) - \tfrac{1}{2}\rho_{j-1}\rho_{j+1}\sin(\Delta_{j+1}) - \tfrac{1}{2}\rho_{j-2}\rho_{j-1}\sin(\Delta_j)]\,. \tag{9}$$

In the inertial range we will set $\nu = 0$. In this case the Kolmogorov solution correspond to $\rho_j =$ constant. The reader should notice that the first term in the square bracket is the transfer of energy from small scales, the last term is the transfer of energy from large scales, while the second term could be a transfer of energy either from the large or from the small scales depending on the sign of $\sin(\Delta_{j+1})$. We argue that in order to have a cascade of energy from large to small scale, $\sin(\Delta_{j+1})$ should be negative, at least in the average.

It is interesting to note that one can prove that the probability distribution of θ_j is uniform in the interval $[0, 2\pi]$. Indeed, by simply taking:

$$\theta_{3j} \to \theta_{3j} + 2\epsilon\,,$$
$$\theta_{3j-1} \to \theta_{3j-1} - \epsilon\,,$$
$$\theta_{3j+1} \to \theta_{3j+1} - \epsilon\,, \tag{10}$$

for any j, we transform a solution of the equation of motion in another solution. Because of the existence of this $U(1)$ symmetry, we can choose ϵ in a random uniform way and therefore the variable θ_j must be uniformly distributed (another symmetry of the equation consists in changing the sign of the real part of the u's). The fact the phases θ_j are uniformly distributed between $[0, 2\pi]$ does not imply that the Δ_j are uniformly distributed. Thus even in the Kolmogorov picture we should introduce some phase coherency in order to satisfy the requirement of an energy cascade. Next we shall consider the time average $\langle \ldots \rangle$ for the moment of order p of ρ_j. In the inertial range ($\nu = 0$), we obtain:

$$\begin{aligned} 0 &= \left\langle \rho_j^p \frac{\mathrm{d}}{\mathrm{d}t} \rho_j \right\rangle \\ &= \langle \rho_j^p \rho_{j+1} \rho_{j+2} S_{j+2} \rangle - \tfrac{1}{2} \langle \rho_j^p \rho_{j+1} \rho_{j-1} S_{j+1} \rangle \\ &\quad - \tfrac{1}{2} \langle \rho_j^p \rho_{j-1} \rho_{j-2} S_j \rangle\,, \end{aligned} \tag{11}$$

where we have introduced the variables $S_j = \sin(\Delta_j)$. Our aim is to solve equations (11) for all p by using the idea that a multiplicative process, in the sense discussed in the introduction, could represent a reasonable approximation of the equal time probability distribution of the real dynamical system.

3. The choice of the multiplicative process

Our starting point is the hypothesis that

$$\rho_{j+1} = a_{j+1} \rho_j\,, \tag{12}$$

where a_j is a random variable to be specified. By substituting eq. (12) into (11) we obtain:

$$\begin{aligned} &\langle a_{j+2} a_{j+1}^2 a_j^{p+2} a_{j-1}^{p+2} S_{j+2} \rangle - \tfrac{1}{2} \langle a_{j+1} a_j^{p+1} a_{j-1}^{p+2} S_{j+1} \rangle \\ &\quad - \tfrac{1}{2} \langle a_j^p a_{j-1}^{p+1} S_j \rangle = 0\,. \end{aligned} \tag{13}$$

In order to solve these equations we have to specify the correlation among the a_j and the S_j. We first assume that a_j are uncorrelated variables (among themselves). This is a quite strong assumption which should be considered to be a first order approximation to the real solution. At any rate this assumption has been numerically tested. It turns out that the normalized correlation:

$$Q_j(m) = \frac{\langle a(j+m)\, a(j) \rangle - \langle a(j) \rangle \langle a(m) \rangle}{\langle [a(j)]^2 \rangle - \langle a(j) \rangle^2} \tag{14}$$

is nearly zero already at $m = 1$ (see fig. 1), for m and j in the inertial range. Next we assume that

$$a_j = C(1 - \beta S_j)\,. \tag{15}$$

As a consequence of these two assumptions the S_j are uncorrelated variables. Although the assumptions we have done could be considered too much strong, they are consistent one with the other. For instance, if we set $a_j = f(S_j, S_{j+1})$,

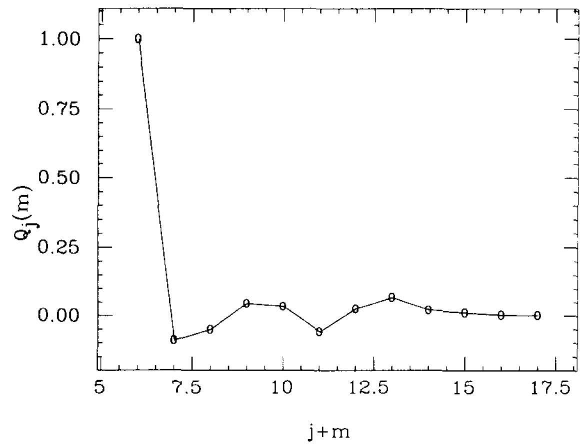

Fig. 1. Circles are the results from a direct integration of shell equations for the correlation $Q_j(m)$ versus $j + m$ with $j = 6$ and $m = 0, \ldots, 10$, both j and m are in the inertial range.

then the variables a_j would not be independent. Thus a_j could depend only on S_j or on S_{j+1}, if we still want to maintain statistically independence of a_j. The assumption (15) gives the Kolmogorov scaling law for $C = 1$ and $\beta = 0$. In order to find a different solution it is convenient to introduce the moments:

$$\Pi_p = \langle (1 - \beta S_j)^p \rangle , \tag{16}$$

where $\langle \ldots \rangle$ should be considered the average on the stochastic process βS_j. Using eq. (16) in (13), we obtain:

$$\begin{aligned} &2C^6 \Pi_{p+2}^2 \Pi_2 \langle (1 - \beta S) S \rangle \\ &\quad - C^3 \Pi_{p+1} \Pi_{p+2} \langle (1 - \beta S) S \rangle \\ &\quad - \Pi_{p+1} \langle (1 - \beta S)^p S \rangle = 0 . \end{aligned} \tag{17}$$

Given the probability distribution of S we can consider (17) as a set of equations $F_p(\beta) = 0$. It is not clear at this stage whether this infinite set of equations can be simultaneously satisfied for the same values of β and C. Let us now consider the equation for $p = 1$. We obtain

$$2C^6 \Pi_3^2 - C^3 \Pi_3 - 1 = 0 . \tag{18}$$

The solutions are $C^3 \Pi_3 = 1$ and $C^3 \Pi_3 = -\frac{1}{2}$. Because the a_j are positive definite, the only physical solution is $C^3 \Pi_3 = 1$ corresponding to $\zeta(3) = 1$. Eq. (18) is equivalent to the Kolmogorov equation and it is a consequence of the assumption that we have done on the independence of the a_j among themselves.

Eq. (18) also tells us that C is not an independent quantity and it is fixed by

$$C^3 = 1 / \Pi_3 . \tag{19}$$

We can get a better insight into eq. (17) by using the following trick. Let us assume that $\beta \neq 0$ and let use define $X = 1 - \beta S$, so that $S = (1 - X)/\beta$. Eq. (17) can be rewritten as:

$$\begin{aligned} &2C^6 \Pi_{p+2}^2 \langle (X - X^2)/\beta \rangle - C^3 \Pi_{p+1} \Pi_{p+2} \\ &\quad \times \langle (X - X^2)/\beta \rangle - \Pi_{p+1} \langle (X^p - X^{p+1})/\beta \rangle \\ &\quad = 0 . \end{aligned} \tag{20}$$

By noticing that $\Pi_p = \langle X^p \rangle$, we obtain:

$$\begin{aligned} &2C^6 \Pi_{p+2}^2 \Pi_2 (\Pi_1 - \Pi_2) - C^3 \Pi_{p+1} \Pi_{p+2} (\Pi_1 - \Pi_2) \\ &\quad - \Pi_{p+1} (\Pi_p - \Pi_{p+1}) = 0 . \end{aligned} \tag{21}$$

These equations show the Kolmogorov solution $\Pi_p = 1 = C$. Also they can be used to compute Π_p as function of Π_1 and Π_2. Indeed from eq. (21) at $p = 1$ we have an equation for C^3. Next from $p = 2$ we can compute Π_4 as function of (Π_1, Π_2). From $p = 3$ we can compute Π_5 and so on. At this stage we can check whether or not the assumption we have done on the nature of the multiplicative process are at least consistent with the numerical results obtained by Jensen et al. [9]. Notice that this check is independent from the probability distribution of S. In order to perform this check we compute the exponent $\zeta(p)$, linked to Π_p in the following way:

$$\zeta(p) = \tfrac{1}{3} p - \log_2 (C^p \Pi_p) . \tag{22}$$

In fig. 2 we show $\zeta(p)$ computed from Π_1, Π_2 by

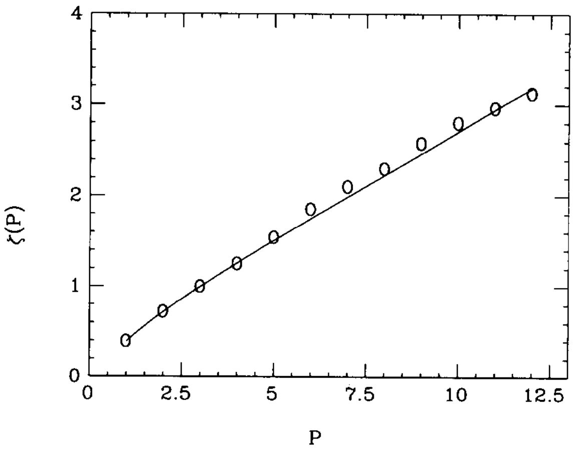

Fig. 2. Circles are the value of $\zeta(p)$ obtained from a numerical integration of eq. (3) [9]. The solid line is computed by inserting in eq. (21) the values of Π_1, Π_2 coincident with the correspondent numerical results [9].

eq. (21) and $\zeta(p)$ obtained numerically in [9]. Here we have chosen Π_1 and Π_2 by imposing that the values of $\zeta(1)$ and $\zeta(2)$ coincide with the numerical data of [9]. As we can see the numerical agreement is quite good. This tells us that our assumption on the multiplicative process could be considered to be a good first approximation.

4. The approximate solution to the closure equations

We next go back to eq. (17) and try to specify more about the probability distribution of S. Our starting point is to assume Δ_j to be uniformly distributed in the interval $[-\pi, 0]$. This assumption turns out to be reasonably good for j in the inertial range (see fig. 3).

Under this hypothesis we can compute the functions $F_p(\beta)$. In fig. 4 we plot them for $p = 0$, 2, 3, 4 ($F_1(\beta)$ is identically equal to zero for all β corresponding to the Kolmogorov equation). As we can see, the $F_p(\beta)$ functions cross the value 0 at $\beta = 0$ and at $\beta = \beta_p$; $\beta = 0$ corresponds to Kolmogorov scaling. The values β_p are not the same for all p, although they do not differ much

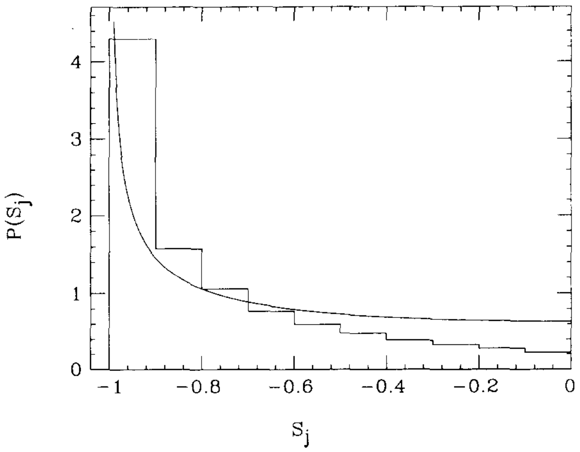

Fig. 3. Probability distribution function $P(S_j)$ of the variable S_j for $j = 9$ in the inertial range. The histogram has been carried out from a numerical integration of the shell model while the solid curve is our analytical ansatz. In the zone corresponding to positive value of S_j, $P(S_j)$ is nearly zero.

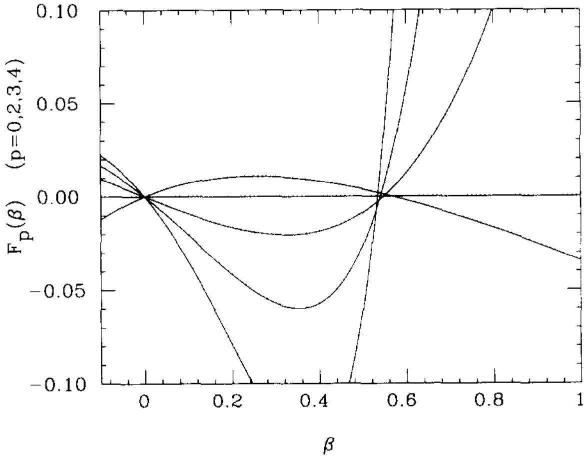

Fig. 4. The $F_p(\beta)$ curves for $p = 0$, 2, 3, 4 obtained with Δ uniformly distributed in the interval $[-\pi, 0]$, as a function of β; the exact zero's of the $F_p(\beta)$ function at $\beta = 0$ correspond to the Kolmogorov scaling while there is a weak dependence on p for the non trivial β_p values at which $F_p(\beta_p) = 0$.

one from the other. By considering $\beta = \beta_2$, we can compute the function $\zeta(p)$ for all p. Once again the $\zeta(p)$ so obtained are in quite good agreement with those computed in [9]. This result is rather impressive, because (contrary to the previous case) no numerical informations have been used here (apart from the distribution of Δ). It is also quite remarkable that although there is no small parameter (indeed $\beta_2 = 0.4 = \mathcal{O}(1)$), the deviations from Kolmogorov scaling are very small (less than 0.02 for $0 \le p \le 2$). Although this result should be considered quite good with respect to the approximation done so far, we want to understand whether it could be improved within similar approximations to those we have done. It turns out that this goal may be reached by a slightly more complex formulation of the stochastic process βS. We still consider Δ_j to be uniformly distributed. On the other hand we consider β not a constant, rather a stochastic variable independent of S with probability distribution:

$$P(\beta) = q\delta(\beta - \beta_0) + (1 - q)\delta(\beta) . \tag{23}$$

Using eq. (23) into (20), we obtain a set of

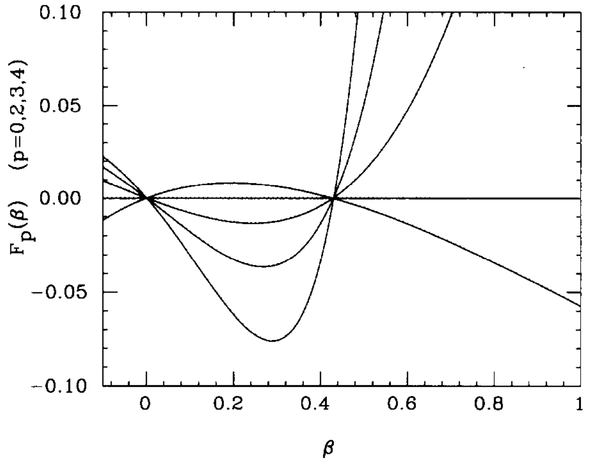

Fig. 5. The same as in fig. 4 but choosing the expression (23) for the probability distribution of β; it is evident that now also the non-trivial zero's ($\beta_p \neq 0$) collapse to the same value for every p's.

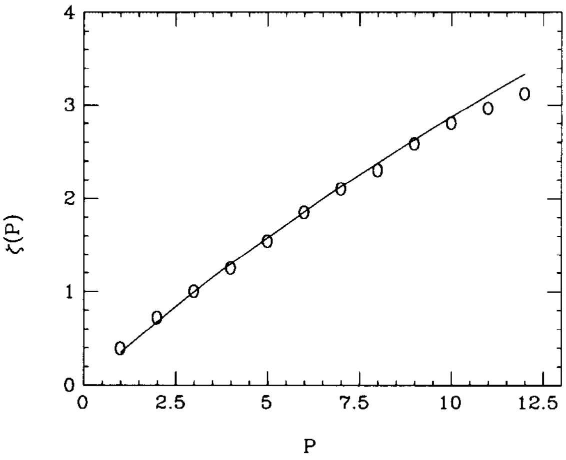

Fig. 7. The values of $\zeta(p)$ obtained from a numerical integration of eq. (3) (circles) from ref. [9]. The solid line is computed by using expression (23) for the probability distribution of β and the relation (22) in order to evaluate the $\zeta(p)$ function.

equations

$$F_p(\beta_0, q) = 0 \,. \tag{24}$$

We have found that for $q = 0.95$ these equations (for $p = 0, \ldots, 4$) are nearly simultaneously satisfied as shown in figs. 5 and 6. It is quite interesting to look at the values of $\zeta(p)$ with the modified assumption (23). The agreement with the $\zeta(p)$ computed by Jensen et al. [9] is quite good, as one can see in fig. 7. In the case of assumption (23) the recursion equation (21) for the moment must be modified. Indeed we find:

$$\begin{aligned} &2C^6 \Pi_{p+2}^2 \Pi_2[(\Pi_1 - \Pi_2)A + B] \\ &\quad - C^3 \Pi_{p+1} \Pi_{p+2}[(\Pi_1 - \Pi_2)A + B] \\ &\quad - \Pi_{p+1}[(\Pi_p - \Pi_{p+1})A + B] \\ &\quad = 0 \,, \end{aligned} \tag{25}$$

where $A = q/\beta_0$ and $B = (1-q)\langle S \rangle = -2/\pi(1-q)$. We note that for $p = 1$ equation (24) reduces to $\Pi_3 C^3 = 1$. We can now use Π_1, Π_2 and C computed from the definition of the stochastic process and then we use (25) to compute all the other $\zeta(p)$. As we can see in fig. 8 the agreement is once again remarkable. In conclusion we can say that for the closure equation obtained from $\mathrm{d}/\mathrm{d}t\langle \rho_n^p \rangle = 0$, the assumptions made about the multiplicative process seem to satisfy all the functional constraints imposed by the equation of motion. The multiplicative process we propose here differs from that previously proposed by Benzi et al. [5]. Indeed in the present case the range of the exponents in the multifractal language is [0.2, 0.6]: regions with laminar velocities

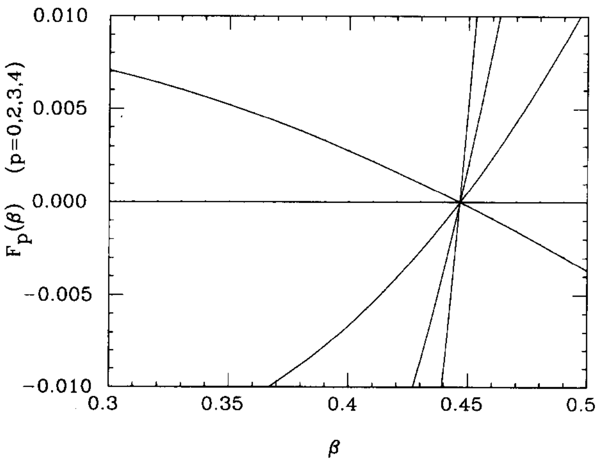

Fig. 6. An enlargement of the interval of fig. 5 where all the $F_p(\beta)$ curves for $p = 0, 2, 3, 4$ vanish.

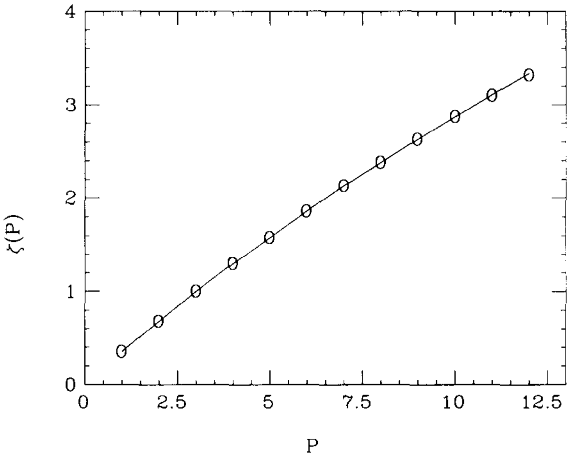

Fig. 8. The values of $\zeta(p)$ (circles) obtained using the closure relations (25) and the values of Π_1, Π_2 and C taken from the stochastic process (23). The solid line is the results from the naive relation (22) using the same stochastic process.

(i.e. less singular than the Kolmogorov scaling) are allowed. This is not possible in the β-random formalism where the less singular value is the one corresponding to Kolmogorov exponent $h=1/3$. The difference between the two cases can be better understood for the moment ($p=-1$). The β-random model predicts in this case $\zeta(-1)=-1/3$ while our multiplicative process yields $\zeta(-1)=-0.4$, in quite good agreement with the numerical results of Jensen et al. [9].

5. Discussions and conclusions

We have seen that the idea of multiplicative process seems to play a key role in obtaining a reasonable accurate solution of the closure equations $\mathrm{d}/\mathrm{d}t\langle \rho_j^p\rangle = 0$. In the solution that we have found the moments of $|\phi_j|$ do not depend on λ, so that in the model for generic value of λ we find that

$$\langle |u_j|^p\rangle d \propto \lambda^{-p/3} 2^{-\zeta(p)+p/3} . \tag{26}$$

The λ independence of the corrections to pure Kolmogorov scaling is related to the assumption we have done of the absence of correlations in the multiplicative process. Certainly this is a very strong assumption and we do not expect that it will be exactly satisfied. In fact we can consider the closure equation $\mathrm{d}/\mathrm{d}t\langle \rho_n \rho_{n+1}\rangle = 0$. After some algebra we find that it implies:

$$\begin{aligned} G_2 = \lambda[&C^5\Pi_2^2\langle(1-\beta S)S\rangle \\ &- \tfrac{1}{2}C^2\Pi_1\langle(1-\beta S)S\rangle - \tfrac{1}{2}C^2\Pi_2^2\langle S\rangle] \\ &+ [\langle(1-\beta S)S\rangle C - \tfrac{1}{2}C^4\Pi_2\langle(1-\beta S)^2 S\rangle \\ &- \tfrac{1}{2}C^9\Pi_1\Pi_2\langle(1-\beta S)S\rangle] = 0 . \end{aligned} \tag{27}$$

This equation is not satisfied by the multiplicative process we have just introduced, at least for $\lambda = 2$, although the difference from zero is rather small. The same consideration apply to the closure equation $\mathrm{d}/\mathrm{d}t\langle \rho_n \rho_{n+2}\rangle = 0$. Actually this equation does not depend on λ:

$$\begin{aligned} G_3 = &C^6\langle(1-\beta S)S\rangle\Pi_2^3 - \tfrac{1}{2}C^3 \\ &\times\langle(1-\beta S)S\rangle\Pi_1\Pi_2 - \tfrac{1}{2}\langle S\rangle\Pi_1 = 0 . \end{aligned} \tag{28}$$

All the other closure equations of the kind $\mathrm{d}/\mathrm{d}t\langle \rho_n \rho_{n+k}\rangle = 0$ are automatically satisfied. These equations should actually be interpreted as equations for the correlations which we have neglected. The correlation will be thus λ-dependent and consequently the whole multiplicative process will eventually depend on λ. The good quality of the results obtained in the previous section suggest that this dependence on λ should be rather weak. It would be certainly interesting to verify numerically the approximate validity of eq. (26).

Acknowledgements

It is a pleasure for us to thank U. Frisch, G. Paladin and A. Vulpiani for useful discussions on the shell model. This work has been partially supported by contracts CEE SCI-0212 C, CEE EPOC CT-90-0012 and CEE SCI CT-91-0697.

References

[1] A.N. Kolmogorov, C.R. Acad. Sci. USSR 30 (1941) 301.
[2] B.B. Mandelbrot, J. Fluid. Mech. 62 (1974) 331.
[3] U. Frisch, P. Sulem and M. Nelkin, J. Fluid. Mech. 87 (1978) 719.
[4] U. Frisch and G. Parisi, in: Turbulence and Predictability of Geophysical Flows and Climate Dynamics, M. Ghil et al., eds. (North Holland, New York, 1987) p. 84.
[5] R. Benzi, G. Paladin, G. Parisi and A. Vulpiani, J. Phys. A 17 (1984) 3521.
[6] G. Paladin and A. Vulpiani, Phys. Rep. 156 (1987) 147.
[7] R. Grappin, J. Leorat and A. Pouquet, J. Phys. (Paris) 47 (1986) 1127, and references therein.
[8] M. Yamada and K. Ohkitani, Phys. Rev. Lett. 60 (1988) 983; Prog. Theor. Phys. 81 (1989) 329.
[9] M.H. Jensen, G. Paladin and A. Vulpiani, Phys. Rev. A 43 (1991) 798.
[10] A. Migdal, private communication.
[11] V.N. Desnyansky and E.A. Novikov, Prikl. Mat. Mekh. 38 (1974) 507.
[12] Chabra and Sreenivasan, Phys. Rev. Lett. 68 (1992) 2762.