Anomalous Scaling Exponents in Nonlinear Models of Turbulence

Luiza Angheluta,[1,2] Roberto Benzi,[3] Luca Biferale,[3] Itamar Procaccia,[1] and Federico Toschi[4,5]

[1]*The Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*
[2]*The Niels Bohr Institute, Blegdmasvej 17, Copenhagen, Demark*
[3]*Department of Physics and INFN, University of Rome "Roma Tor Vergata", Via della Ricerca Scientifica 1, 00133 Rome, Italy*
[4]*Istituto per le Applicazioni del Calcolo CNR, Viale del Policlinico 137, 00161 Roma, Italy*
[5]*INFN, Sezione di Ferrara, Via G. Saragat 1, I-44100 Ferrara, Italy*
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We propose a new approach to the old-standing problem of the anomaly of the scaling exponents of nonlinear models of turbulence. We construct, for any given nonlinear model, a *linear* model of passive advection of an auxiliary field whose anomalous scaling exponents are the same as the scaling exponents of the nonlinear problem. The statistics of the auxiliary linear model are dominated by "statistically preserved structures" which are associated with exact conservation laws. The latter can be used, for example, to determine the value of the anomalous scaling exponent of the second order structure function. The approach is equally applicable to shell models and to the Navier-Stokes equations.

The calculation of the scaling exponents of structure functions of nonlinear turbulent velocity fields remains one of the major open problems of statistical physics [1]. Dimensional considerations fail to provide the measured exponents, and present theory cannot even specify the mechanism for the so-called "anomaly," i.e., the deviation of the scaling exponents from their dimensional estimates. Theoretical attempts to calculate the exponents were mainly based on perturbative expansions [2] or on closures of the infinite correlation function hierarchy [3]. In this Letter we propose a new idea to ascertain the anomaly of the scaling exponents in turbulence. In addition, we propose an alternative way to determine the anomalous scaling exponent of the second order structure function. The proposed approach is equally applicable to Navier-Stokes turbulence and to simplified models of turbulence, like nonlinear shell models. The only distinction is in the ease of numerical demonstration.

The central idea is to construct a *linear* model whose scaling exponents are *the same* as those of the *nonlinear* problem. In this linear problem the exponents are universal to the forcing, and we understand the mechanism for the anomaly of the scaling exponents; we use this to show that also the *nonlinear* problem must have anomalous exponents. We exemplify the idea first in the context of the Navier-Stokes equations. Consider a model for two coupled vector fields $\boldsymbol{u}$ and $\boldsymbol{w}$,

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \lambda \boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{u} = -\boldsymbol{\nabla} p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}, \quad (1)$$

$$\frac{\partial \boldsymbol{w}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{w} + \lambda \boldsymbol{w} \cdot \boldsymbol{\nabla} \boldsymbol{w} = -\boldsymbol{\nabla} \tilde{p} + \nu \nabla^2 \boldsymbol{w} + \tilde{\boldsymbol{f}}. \quad (2)$$

Here ν is the kinematic viscosity, p and $\tilde{p}$ are pressure fields imposing $\boldsymbol{\nabla} \cdot \boldsymbol{u} = \boldsymbol{\nabla} \cdot \boldsymbol{w} = 0$, and $\boldsymbol{f}$ and $\tilde{\boldsymbol{f}}$ are two uncorrelated Gaussian random forcing. Finally, λ is a real number. Here and below we *assume* that the scaling exponents are universal to the forcing. We want to demonstrate their anomaly and to find their numerical values. For $\lambda = 0$ Eq. (1) reduces to the Navier-Stokes equations for $\boldsymbol{u}$, whereas Eq. (2) becomes a linear equation for $\boldsymbol{w}$, passively advected by $\boldsymbol{u}$. This linear problem was referred to before as a "passive vector with pressure" [4–6]. It exhibits anomalous exponents that are universal to the forcing. In addition, one understands the mechanism for the anomaly [7–10]; the linear model possesses "statistically preserved structures" (SPS) which are evident in the decaying problem. These are *left* eigenfunctions of eigenvalue 1 of the linear propagator for each order (decaying) correlation function; see below for more detail. Evidently, for any finite value of $0 < \lambda < \infty$ the scaling exponents of the two fields $\boldsymbol{u}$ and $\boldsymbol{w}$ must be the same, due to the symmetry $\lambda \boldsymbol{w} \leftrightarrow \boldsymbol{u}$ and the assumed universality to the forcing. Consider the two composite fields $\boldsymbol{u}_+ \equiv \boldsymbol{u} + \mu \boldsymbol{w}$ and $\boldsymbol{u}_- \equiv \boldsymbol{u} - \mu \boldsymbol{w}$. Choose the forcing terms in (1) and (2) such that $\langle (\boldsymbol{f} + \mu \tilde{\boldsymbol{f}})(\boldsymbol{f} - \mu \tilde{\boldsymbol{f}}) \rangle = 0$. Then $\boldsymbol{u}_+$ satisfies precisely the Navier-Stokes equations (with forcing $\boldsymbol{f} + \mu \tilde{\boldsymbol{f}}$) while $\boldsymbol{u}_-$ is a passive vector advected by $\boldsymbol{u}_+$ and forced by $\boldsymbol{f} - \mu \tilde{\boldsymbol{f}}$. Namely, $\boldsymbol{u}_+$ and $\boldsymbol{u}_-$ satisfy Eqs. (1) and (2) for $\boldsymbol{u}$ and $\boldsymbol{w}$ for $\lambda = 0$. Our main proposition is that the scaling exponents of the Navier-Stokes field $\boldsymbol{u}_+$ and of the passive vector $\boldsymbol{u}_-$ are the same. If true, we can study the anomalous scaling of the Navier-Stokes problem by using the successful tools and the concepts employed to understand the anomalous scaling for passive fields (scalar or vector).

To verify our proposition we present in Fig. 1 the scaling properties of $\boldsymbol{u}_+$ and $\boldsymbol{u}_-$ for $\mu = 1$ obtained by direct numerical simulations of Eqs. (1) and (2). The pseudospectral code used for the simulations has a resolution of 128^3, dealiased with 2/3 rule, with a normal viscosity 0.075, time step 1.6×10^{-3}. The forcing was an Ornstein-Uhlenbeck process with amplitude 0.75 on $|k| = 1$ modes, and with amplitude scaled down dimensionally up to modes with $|k| \leq 2$. In Fig. 1 we show an extended self-similarity plot and a direct plot of the sixth order structure
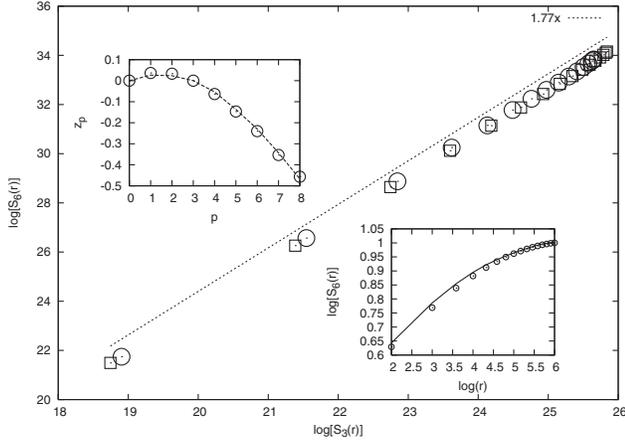
FIG. 1.   Log-log plot of the sixth order structure functions of the fields u_+ and u_- (circles and squares, respectively), for $\mu = 1$, as a function of the third order structure functions. The dashed line corresponds to the best fit in the scaling region with slopes 1.77. Lower inset: the sixth order structure function of the two fields as a function of r. Upper inset: $z_p \equiv \zeta_p/\zeta_3 - p/3$ computed for the structures functions of u_+ (line) and u_- (circles).

functions for both u_- and u_+. One observes convincing scaling behavior with the same exponent for both field. In the upper inset we show the anomalous exponents $z_p \equiv \zeta_p/\zeta_3 - p/3$ for the field u_+ (line) and u_- (circles) computed up to order 8: the agreement is excellent. Thus, the Navier-Stokes field appears to have the same scaling exponents as the passive vector field. For additional strong evidence we turn to shell models [11–13].

To reach a deeper understanding of the relation between the nonlinear and the linear models, and to clearly present the role of the statistically preserved structures, we consider the Sabra shell model which is a truncated description of the Navier-Stokes equations:

$$\left(\frac{d}{dt} + \nu k_n^2\right)u_n = i[k_{n+1}u_{n+1}^*u_{n+2} - \delta k_n u_{n-1}^*u_{n+1}$$
$$+ (1 - \delta)k_{n-1}u_{n-1}u_{n-2}] + f_n. \quad (3)$$

Here u_n are the velocity modes restricted to "wave vectors" $k_n = k_0 2^n$ with k_0 determined by the inverse outer scale of turbulence. The model contains one additional parameter, δ, and it conserves two quadratic invariants (when the force and the dissipation term are absent) for all values of δ. The first is the total energy $\sum_n |u_n|^2$ and the second is $\sum_n (-1)^n k_n^\alpha |u_n|^2$, where $\alpha = \log_2(1 - \delta)$. In this Letter we consider values of the parameters such that $0 < \delta < 1$ where the second invariant contributes only subleading exponents to the structure functions [14,15]. The exponents characterize the structure functions:

$$S_2(k_n) \equiv \langle u_n u_n^* \rangle \sim k_n^{-\zeta_2}, \quad (4)$$

$$S_3(k_n) \equiv \mathrm{Im}\langle u_{n-1}u_n u_{n+1}^*\rangle \sim k_n^{-\zeta_3}, \quad S_p(k_n) \sim k_n^{-\zeta_p}. \quad (5)$$

The values of the scaling exponents were determined accurately by direct numerical simulations. Besides ζ_3 which is exactly unity [16], all the other exponents ζ_p are anoma-

lous, differing from $p/3$. It was established numerically that the scaling exponents are universal, i.e., they are independent of the forcing f_n as long as the latter is restricted to small n [12]. Despite of the much simpler structure of shell models in comparison with Navier-Stokes equations, there are no analytical calculations of scaling exponents. Previous attempts being manly based on stochastic closures [17,18].

Consider then a passive advected field which in the discrete shell space has the complex amplitudes w_n. The dynamical equations for this field are linear and constructed under the following requirements: (i) the structure of the equations is obtained by linearizing the nonlinear problem and retaining only such terms that conserve the energy; (ii) the resulting equation is identical with the sabra model when $w_n = u_n$; (iii) the energy is the only quadratic invariant for the passive field in the absence of forcing and dissipation. These requirements lead to the following linear model:

$$\frac{dw_n}{dt} = \frac{i}{3}\Phi_n(u, w) - \nu k_n^2 w_n + f_n, \quad (6)$$

where the advection term is defined as

$$\Phi_n(u, w) = k_{n+1}[(1 + \delta)u_{n+2}w_{n+1}^* + (2 - \delta)u_{n+1}^*w_{n+2}]$$
$$+ k_n[(1 - 2\delta)u_{n-1}^*w_{n+1} - (1 + \delta)u_{n+1}w_{n-1}^*]$$
$$+ k_{n-1}[(2 - \delta)u_{n-1}w_{n-2}$$
$$+ (1 - 2\delta)u_{n-2}w_{n-1}]. \quad (7)$$

Observe that when $w_n = u_n$ this model reproduces the Sabra model, and also that the total energy is conserved because $\sum_n \mathrm{Im}[\Phi_n(u, w)w_n^*] = 0$. The second quadratic invariant is not conserved by the linear model. Finally, both models have the same "phase symmetry" in the sense that the phase transformations $u_n \to u_n \exp(i\phi_n)$ and $w_n \to w_n \exp(i\theta_n)$ leave the equations invariant iff $\phi_{n-1} + \phi_n = \phi_{n+1}$, $\theta_{n-1} + \theta_n = \theta_{n+1}$. This identical phase relationship guarantees that the nonvanishing correlation functions of both models have precisely the same forms. Thus, for example, the only second and third correlation functions in both models are those written explicitly in Eqs. (4) and (5).

As already remarked, the anomalous scaling of w_n can be investigated in terms of the SPS [7,9,10]. For example, for the second order correlation function denote the propagator $P_{n,n'}^{(2)}(t|t_0)$; this operator propagates any initial condition $\langle w_n w_{n'}^* \rangle(t_0)$ (with average over initial conditions, independent of the realizations of the advecting field u_n) to the decaying correlation function (with average over realizations of the advecting field u_n)

$$\langle w_n w_n^* \rangle(t) = P_{n,n'}^{(2)}(t|t_0)\langle w_{n'} w_{n'}^* \rangle(t_0). \quad (8)$$

The second order SPS, $Z_n^{(2)}$, is the left eigenfunction with eigenvalue 1,

$$Z_{n'}^{(2)} = Z_n^{(2)} P_{n,n'}^{(2)}(t|t_0). \quad (9)$$

Note that $Z_n^{(2)}$ is time independent even though the operator

$P_{n,n'}^{(2)}(t|t_0)$ is time dependent. Each order correlation function is associated with another propagator $\mathbf{P}^{(p)}(t|t_0)$ and each of those has an SPS, i.e., a *left* eigenfunction $\mathbf{Z}^{(p)}$ of eigenvalue 1. These nondecaying eigenfunctions scale with k_n, $\mathbf{Z}^{(p)} \sim k_n^{-\xi_p}$, and the values of the exponents ξ_p are anomalous. Finally, one can show that these SPS are also the leading scaling contributions to the structure functions of the *forced* problem (6) [7,9]. Thus the scaling exponents of the linear problem are independent of the forcing f_n, since they are determined by the SPS of the decaying problem.

Using Eqs. (6), we can now write the (Sabra) shell model version of Eqs. (1) and (2):

$$\frac{du_n}{dt} = \frac{i}{3}\Phi_n(u,u) + \frac{i\lambda}{3}\Phi_n(w,u) - \nu k_n^2 u_n + f_n, \quad (10)$$

$$\frac{dw_n}{dt} = \frac{i}{3}\Phi_n(u,w) + \frac{i\lambda}{3}\Phi_n(w,w) - \nu k_n^2 w_n + \tilde{f}_n. \quad (11)$$

For $\lambda = 0$ we recover the equations for the nonlinear and a linear models, Eqs. (3) and (6). At this point we present strong evidence that the scaling exponents of either field exhibit no jump in the limit $\lambda \to 0$. Accordingly, the scaling exponents of either field can be obtained from the SPS of the linear problem.

Equations (10) and (11) were solved numerically for 25 shells with $k_1 = 2$, $\nu = 10^{-7}$, $\delta = 0.5$ and $\lambda = 10^{-1}$, 10^{-3}, 10^{-4}, 10^{-5}, 0. In Fig. 2 we show, for example, results for the sixth order objects $\langle|u_{n-1}u_n u_{n+1}^*|^2\rangle$ and $\langle|w_{n-1}w_n w_{n+1}^*|^2\rangle$. Plotted are double-logarithmic plots of these object as a function of k_n. For the shown values of λ the scaling exponent ζ_6 was measured using the range $n = 4$–12, with the results 1.71 ± 0.05, 1.71 ± 0.03, 1.71 ± 0.02, 1.73 ± 0.02, and 1.73 ± 0.02, respectively. We see that the exponents of the linear and nonlinear model at $\lambda = 0$ are the same and they coincide with the exponents of the two coupled models (10) and (11) for $\lambda > 0$. The same results were obtained for all the exponents up to order 10.

We stress at this point that the two problems do not share *exactly* the same statistics; the linear problem, being symmetric in $w_n \to -w_n$ has an even probability distribution function (PDF) and thus zero prefactors for all the odd structure functions. The statement is only about the identity of the scaling exponents, neither the trajectory in phase space nor the PDF. In the inset of Fig. 2 we also demonstrate that the linear and the nonlinear problems share the same scaling properties for correlations that depend on more than one shell. The data pertain to $F_{p,q}(k_n, k_m) \equiv \langle|u_n|^p|u_m|^q\rangle$, with $p = 2$, $q = 2$ for both models. Finally, we comment that the limit $\lambda \to 0$ can be considered mathematically for the shell model equations (10) and (11), to prove that it is not singular [19].
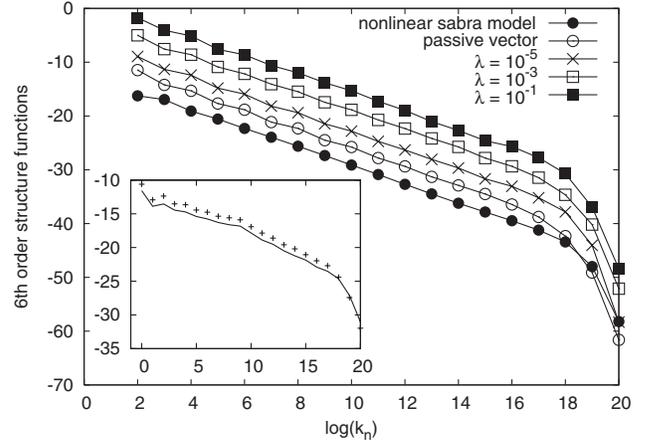
FIG. 2.   The sixth order structure function of the field w_n in Eqs. (11) for $\lambda = 10^{-1}$, 10^{-3}, and 10^{-5}, together with the sixth order structure function for the Sabra model (3) and for the linear model (6), respectively. The structure functions of the field u_n for $\lambda > 0$ are not shown since they are indistinguishable from those of the w_n. Inset: log-log plot of the fourth-order correlation function $F_{2,2}(k_n, k_7)$ vs k_n calculated for the linear field $(+)$ and for the nonlinear field (solid line) at $\lambda = 0$.

The greatest asset of the present approach is that we can now forge a connection between the SPS of the linear model and the *forced* correlation function of the nonlinear problem. This underlines the anomaly of the scaling properties of the latter model, and allows us to determine ζ_2. We start with the second order quantities. We can project a generic second order decaying correlation function of the linear model onto the second order SPS, thus creating a statistically conserved quantity [9]:

$$I^{(2)} \equiv \sum_n Z_n^{(2)}\langle w_n w_n^*\rangle(t) = \sum_{n,n'} Z_n^{(2)} P_{n,n'}^{(2)}(t|t_0)\langle w_{n'} w_{n'}^*\rangle(t_0),$$

$$(12)$$

where the average is over different initial conditions for the linear fields and different realization of the advecting velocity field. To show that the forced second order correlation of the nonlinear field is dominated by $Z^{(2)}$, we use this forced correlation function *instead of* $Z^{(2)}$ in Eq. (12). The test is whether $I^{(2)}$ remains constant on a time window which increases with Reynolds number. This is shown in Fig. 3. The success of this test demonstrates that (i) there exists a SPS for the linear problem; (ii) the SPS is well represented by the *forced nonlinear* second order correlation functions. This is a direct demonstration that the correlation function of the nonlinear model scales with the same anomalous exponent as $Z^{(2)}$. An even more stringent test can be made using SPS of orders large than 2, where also correlations between different shells are relevant for the decaying properties [9,10]. For example, $I^{(4)}$ is given by the weighted sum of three contributions:

$$I^{(4)} = \sum_{n,m} Z_{n,m}^{(a,4)}\langle|w_n|^2|w_m|^2\rangle(t) + \sum_n [Z_n^{(b,4)}\langle w_n w_{n+1}^2 w_{n+3}^*\rangle(t) + \text{c.c.}] + \sum_n [Z_n^{(c,4)}\langle w_n w_{n+1} w_{n+3} w_{n+4}^*\rangle(t) + \text{c.c.}], \quad (13)$$
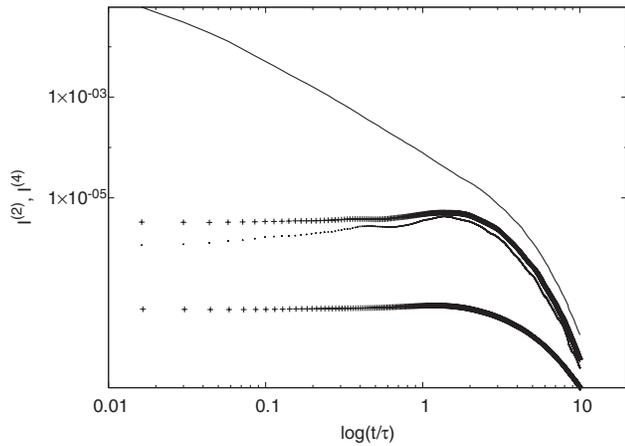
FIG. 3.   With the symbols (+) the constants $I^{(2)}$ (bottom) and $I^{(4)}$ (top) constructed by projecting the decaying structure function of the linear model on the *forced* structure function of the nonlinear model. To emphasize the importance of using the correct SPS, we also show the result for $I^{(4)}$ using the dimensional Kolmogorov prediction for Z^4 (small dots) and $Z^4 = 1$ (solid line).

where all the terms allowed by the phase symmetry were employed. In Fig. 3 we show results for $I^{(4)}$ where again we swapped the SPS of the linear problem for the measured *forced* correlations of the nonlinear problem: $Z_{n,m}^{(a,4)} \rightarrow \langle |u_n|^2 |u_m|^2 \rangle$ and the corresponding expressions for $Z_n^{(b,4)}$ and $Z_n^{(c,4)}$. We thus conclude that the scaling exponents of a given nonlinear shell model can be understood from the SPS of an appropriately constructed linear problem. To make this point crystal clear, we have used in fact the forced structure functions of the nonlinear model as approximants for $Z^{(2)}$, $Z^{(4)}$ in the calculation of $I^{(2)}$ and $I^{(4)}$ shown in Fig. 3. The constancy of both demonstrates that the forced correlation function of the nonlinear model are very well approximated by the SPS of the linear model. This demonstration can be repeated with higher order correlation functions with the same (or better) degree of success. Finally, the existence of a conserved quantity $I^{(2)}$ can be used to *calculate* $\xi_2 = \zeta_2$. Starting from a given arbitrary initial condition (say a δ function on one shell) and computing Eq. (12) with many realizations of the advecting velocity field, one finds that there exists a *sharply defined* ξ_2, $Z_n^{(2)} \sim k_n^{-\xi_2}$, for which $I^{(2)}$ is indeed constant. The same approach can be used to determine ζ_3 but we know that $\zeta_3 = 1$. Unfortunately, this simple approach cannot be used for higher order exponents, because the corresponding SPS depend on more than one k_n, and cannot be represented as a simple power law.

In conclusion, the anomalous scaling of nonlinear models of turbulence, either the Navier-Stokes equations or shell models, are determined by the eigenfunctions of the inertial operator, which are precisely the SPS of the linear problem. Thus, although the concept of eigenfunctions cannot be applied directly in nonlinear problems, we are able to argue that the mechanism leading to anomalous scaling in Navier-Stokes equations and other nonlinear models is identical to the one recently discovered for passively advected fields. This conclusion may open the way to a deeper understand of intermittency in turbulent flows and to a direct computation of the anomalous exponents.

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