Fully Developed Turbulence and the Multifractal Conjecture

Roberto Benzi · Luca Biferale

Abstract We review the *Parisi-Frisch* (Proc. Int. School of Physics "E. Fermi", pp. 84–87, North-Holland, Amsterdam, 1985) MultiFractal formalism for Navier-Stokes turbulence with particular emphasis on the issue of statistical fluctuations of the dissipative scale. We do it for both Eulerian and Lagrangian Turbulence. We also show new results concerning the application of the formalism to the case of Shell Models for turbulence. The latter case will allow us to discuss the issue of Reynolds number dependence and the role played by vorticity and vortex filaments in real turbulent flows.

Keywords Eulerian and Lagrangian Turbulence · Multifractals

1 Introduction

Turbulent flows are characterized by strong fluctuations of energy transfer from small to large scales [2]. The best known theoretical approach to describe the statistical properties of turbulent flows is due to Kolmogorov in 1941 (K41). The K41 theory assumes that turbulent fluctuations can be considered homogeneous and isotropic at very small scales. Moreover, Kolmogorov assumed that the average (in time) rate of energy dissipation ϵ is Reynolds independent for large enough Reynolds number, $Re \equiv U_0 L_0 / \nu$, where U_0 is some characteristic velocity at large scale L_0 and ν is the kinematic viscosity of the flow. For the sake of simplicity we will set $U_0 = L_0 = 1$ from now on. Thus $Re = \nu^{-1}$. The equations of motions governing hydrodynamical incompressible turbulence are the Navier-Stokes equations, with density set to $\rho = 1$:

$$\partial_t \mathbf{u} + (\mathbf{u}\partial_\mathbf{r})\mathbf{u} = \partial_\mathbf{r} p + \nu \Delta \mathbf{u}, \tag{1}$$

where u_i is a three-dimensional field satisfying $\partial_i u_i = 0$. Note that in terms of u_i the average rate of energy dissipation is given by $\epsilon \equiv \sum_{ij} \nu (\partial_i u_j)^2$. Thus, ϵ becomes independent of Re if at large Reynolds number $\partial_i u_j \sim Re^{1/2}$.

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The statistical properties of the vector field u_i can be studied by using the n-order tensor, given by the correlation functions $\langle u_{i_1}(\mathbf{x}_{a_1}) \ldots u_{i_n}(\mathbf{x}_{a_n}) \rangle$, where the index $i_1, \ldots, i_n$ stands for the spatial directions and $\mathbf{x}_{a_i}$ are n-points in the physical space, and the symbol $\langle \ldots \rangle$ means ensemble average. If we assume, following Kolmogorov, that turbulence becomes isotropic and homogeneous at scales small enough, then we can rewrite the correlation functions in terms of only longitudinal velocity increments, $\delta_r u = \delta \mathbf{u}(\mathbf{r}) \cdot \hat{\mathbf{r}}$ (where $\delta \mathbf{u}(\mathbf{r}) = \mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})$) and transverse increments, $\delta_r w = \delta \mathbf{u}(\mathbf{r}_T)$ (where $\mathbf{r}_T \cdot \mathbf{u} = 0$). For a discussion on the complexities arising when also anisotropic fluctuations are relevant see [3].

One of the main predictions of the Kolmogorov theory is contained in his famous 4/5-equation for homogeneous and isotropic turbulence:

$$\langle (\delta_r u)^3 \rangle = -\frac{4}{5} \epsilon r + 6\nu \frac{d}{dr} \langle (\delta_r u)^2 \rangle. \tag{2}$$

For large Re (i.e. small ν), and at fixed distance r, the 4/5 equation tells us two important information: the statistical properties of $\delta_r u$ have a non zero skewness, and they have a scaling behavior as a functions of r. Note that (2), taken in the limit of vanishing viscosity, is dimensionally consistent with having $\delta_r u \sim r^{1/3}$, which implies for the p-th order longitudinal Structure Functions, $S^{(p)}(r) \equiv \langle (\delta_r u)^p \rangle \sim r^{p/3}$.

It turns out experimentally that the K41 theory is only partially correct, i.e. (2) is very well verified when the reference scale, r, is much smaller than the integral scale, L_0, implying a recovery of homogeneity and isotropy for small enough scales. However, the structure function data are not in agreement with the Kolmogorov theory for all orders p. For instance, K41 theory predicts that the hyper-Flatness, $S^{(p)}(r)/(S^{(2)}(r))^2$, should be constant as a function of r, while it is not. Both experiments and numerics show that it grows when going from large to small scales. This effect is usually referred to as "intermittency" and it is clearly missing in the K41 scenario (for a collection of experimental and numerical data on both Eulerian and Lagrangian data, see [4, 5]).

What is the physical explanation of intermittency? Is intermittency a finite Reynolds effect, i.e. does it disappear asymptotically for large Re? Are scaling properties of the turbulent field spoiled by intermittency? These questions are the fundamental issues discussed by the scientific community in the last 50 years. Unfortunately, there are no exact results about intermittency in Navier-Stokes turbulence. So, we must rely both on experimental and numerical data, or on phenomenological and dynamical modeling of turbulence. In particular, a key problem is to control the effects of viscosity, i.e. how to connect experimental and numerical data—naturally limited in Reynolds numbers—to any theoretical or phenomenological understanding proposed for the asymptotic infinite Reynolds number case. In this paper we will deal with such an issue, focusing on the MultiFractal (MF) phenomenological description of "infinite Reynolds" turbulence and on its extension to the case of finite Reynolds.

The paper is organized as follows. First, we review some of the results so far obtained by employing the MultiFractal theory of turbulence, introduced 25 years ago by Parisi and Frisch [1, 2] (see [6] for a recent review). Then, we discuss subtleties connected on how to introduce in a self-consistent way the effects of viscosity in the MF description, i.e. how to control finite Reynolds effects for both Eulerian and Lagrangian turbulence. Finally, we present new data obtained on a class of dynamical systems for the turbulent energy cascade, the so-called Shell Models [2, 7, 8]. This will allow us to address and test the MF formalism at changing Reynolds numbers. We conclude with some perspectives for further work.

2 Review of the Multifractal Approach of Turbulence

The multifractal approach to turbulence is based on the assumption that the statistical properties of turbulent flows do exhibit scaling properties even if there is intermittency. Let us discuss scaling in a rather more abstract way by using the symmetries underlying Euler and Navier-Stokes equations. The Euler equations have the "standard" symmetries of the Newton laws for inviscid fluid, i.e. isotropy, parity and time reversal. Moreover, the Euler equations display another important set of symmetries of global scale invariance, i.e. invariance under the transformations [9]:

$$r \to \lambda r; \qquad u \to \lambda^h u; \qquad t \to \lambda^{1-h} t. \tag{3}$$

The value of h is arbitrary and it is not fixed by any physical and/or mathematical constraint. The Navier-Stokes equations have no time reversal (due to viscosity) and satisfy a new scale invariance property, namely:

$$r \to \lambda r; \qquad u \to \lambda^h u; \qquad t \to \lambda^{1-h} t; \qquad \nu \to \lambda^{1+h} \nu. \tag{4}$$

Note that the new scaling property implies $\epsilon \to \lambda^{3h-1} \epsilon$. We can now rephrase the K41 theory in the following way: using (2) the energy dissipation rate should be constant and equal to $S^{(3)}(r)/r$ which implies $h = 1/3$. In other words, the scaling transformation $r \to \lambda r$ ($\lambda \ll 1$) is equivalent to a change in the viscous effect and, therefore, to a change of ϵ. If we assume that ϵ is homogeneous in space and time, it follows that h cannot be arbitrary and should be fixed to $h = 1/3$. Let us note that the scale invariance for the Navier-Stokes equations is now restricted to a particular value of h, i.e. there is less "scale" symmetry in the system as naively predicted by the equation of motions. However, one can take a different point of view, namely one can assume that h is still arbitrary, i.e. the scale symmetry holds, although ϵ is constant only on average. The problem is to understand what is meant by "average". This is the conceptual step performed in the Parisi-Frisch paper [1]. The idea is to assume that in the limit of infinite Reynolds numbers, and for any fixed scale, r, the scale invariance holds with some probability $P_r(h)$. It is then assumed that $\delta_r u$ and $P_h(r)$ are scaling function of r, i.e.

$$\delta_r u \sim r^h; \qquad P_h(r) \sim r^{F(h)}. \tag{5}$$

Then, all the correlation functions should be computed by averaging over the probability $P_h(r)$:

$$S^{(p)}(r) \propto \int dh \, r^{F(h)+ph}. \tag{6}$$

The breaking of time reversal still leads to $\epsilon \sim const$ which implies the constrain $S^{(3)}(r)/r \sim const$. Using the definition:

$$\zeta(p) \equiv \min_h [F(h) + ph],$$

we can perform the integral by the saddle point method and obtain $S^{(p)}(r) \sim r^{\zeta(p)}$ with the constraint $\zeta(3) = 1$.

Historically, $F(h)$ has been written as $3 - D(h)$ by assuming that $D(h)$ is the fractal dimension of the set at scale r where $\delta_r u \sim r^h$ and, for this reason, the above picture has been named the "multifractal" approach to turbulence. We note that there is no reason to introduce any fractal dimension, i.e. there is no *a priori* geometrical interpretation in defining $P_h(r)$.

The crucial physical point, which is discussed in this paper, is the role of the viscosity in the multifractal approach. The scaling invariance of the Navier-Stokes equations tells us that by a scale transformation $r \to \lambda r$ nothing changes if the viscosity becomes smaller by a factor λ^{1+h}. In the real world, the viscosity is fixed and therefore it increases as $\lambda^{-(1+h)}$. Eventually, the viscosity becomes so large as to kill any turbulent fluctuations. We can assume that viscosity is relevant if the turbulent characteristic time scale $r/\delta_r u$ is of the same order of the viscous time scale r^2/ν, which defines a fluctuating, i.e. h-dependent, dissipation scale $\eta(h)$:

$$\frac{\eta \delta_\eta u}{\nu} \sim 1 \;\; \to \;\; \eta(h) \sim \nu^{1/(1+h)}. \tag{7}$$

The idea of a fluctuating dissipation scaling has been introduced by Paladin and Vulpiani [10] and it should be considered as a consequence of the scale invariance (4). Let us discuss in more details this point.

A possible interpretation of the scale invariance would suggest that the velocity difference $\delta_r u$ depends on h in the following manner:

$$\delta_r u \sim g(r/\eta_{k41}) f(r/\eta_{k41}))^h, \tag{8}$$

where η_{k41} is the Kolmogorov—not fluctuating—dissipative scale estimated as $\eta_{k41} \equiv (\nu^3/\epsilon)^{1/4}$. The appropriate asymptotic behaviors for the functions $g(x)$ and $f(x)$ must be: $g(x) \sim const$ and $f(x) \sim x$ for $x \gg 1$, while $g(x) \sim x$ and $f(x) \sim const$ for $x \ll 1$. Thus $f(x)$ and $g(x)$ can be interpreted as cutoff functions on the inertial-range scaling behavior introduced by the dissipation. Let us remark that we can also expect a cutoff function entering into the definition of the probability distribution $P_h(r)$, i.e. we should expect that

$$P_h(r) \sim [f(r/\eta_{k41})]^{F(h)}. \tag{9}$$

It is interesting to remark that in the range of scales where

$$g(x) \sim const, \tag{10}$$

we have, using (9) that $S^{(p)}(r) \sim [f(r/\eta)]^{\zeta(p)}$. Recalling that $\zeta(3) = 1$, the previous expression can be rewritten as $S^{(p)}(r) \sim (S^{(3)}(r))^{\zeta(p)}$. It turn out that the range of scales where the equality (10) holds is much more extended then the usual inertial range, i.e. from experimental and numerical data one can safely deduce that the relation (10) is valid down to scales $x \simeq 10$; i.e. where viscosity is already important. As a result, both function $f(x)$ and $g(x)$ are not constant anymore at those scales, but the viscous corrections on both functions are almost the same. This empirical fact is known as ESS (Extended Self Similarity), it has been extensively used in literature [11, 12] to estimate the scaling exponents $\zeta(p)$ with good accuracy. Equation (10) cannot hold also for very small scales, $x \ll 1$, where viscosity starts to have a dominant role. In order to take into account also of these effects, we need to consider relation (7) and its consequences. The generalization can be done following [13]

$$\delta_r u \sim g(r/\eta(h))(f_h(r/\eta(h))^h; \qquad P_h(r) \sim [f(r/\eta(h))]^{F(h)}. \tag{11}$$

In Sect. 3.2, we will discuss some explicit form of the function $f(x)$ and $g(x)$. Equations (11) tells us how to introduce in a self-consistent way the effect of finite viscosity, i.e. breaking of scale invariance within the scale invariant MF approach. We will refer to the whole picture as the *multifractal conjecture*. More precisely, we must expect that for all quantities that are invariant with respect to the Navier-Stokes symmetries, the scaling properties can be

obtained by using the same universal function $F(h)$, including the effect of finite viscosity. Thus, all the Reynolds number effects on the statistical properties of turbulence should be predicted using the function $F(h)$, the scaling invariance *and* (7). From this point of view, the multifractal conjecture should be able to predict how different quantities (for instance moments of the velocity gradients) depend on Re. The explicit functional form for the interpolating functions $f(x)$ and $g(x)$ may be safely considered not particularly relevant, as soon as the asymptotic scaling limits are preserved (see also [14, 15] for some preliminary attempts to determine them using constraints from the Navier-Stokes equations).

Let us make some remarks on the above picture, mostly in a historical perspective. From the theoretical point of view, the major challenge is to compute the function $F(h)$. In some particular, linear cases, this computation can be done [16], although no one knows how to perform such calculations for the Navier-Stokes equations. There has been a lot of effort to clarify, using both experiments and numerical simulations, whether the statistical properties of turbulence exhibit scaling behavior. So far, there is a wide consensus that this is the case. Knowing the exponents $\zeta(p)$ one can easily compute the function $F(h)$ by some reasonable "fitting" procedure. Thus, we can verify whether the multifractal conjecture is satisfied or not, i.e. whether there exists or not some quantity (invariant with respect the group of Navier-Stokes symmetry) which cannot be predicted knowing the function $F(h)$. For non-invariant quantities, the multifractal conjecture tells us nothing. Also, there have been some approaches in the past claiming that for large enough Re, the statistical properties of turbulence are asymptotic to the K41 theory. Experiments and numerical simulations do not support such claims [4, 5, 17].

3 Predictions Concerning the Fluctuating Dissipative Scale

3.1 Eulerian Framework

The multifractal conjecture allows us to develop a number of different predictions. Let us assume that we know the function $F(h)$ or, equivalently, the scaling exponents $\zeta(p)$. It is then possible to predict the scaling behavior of the following non trivial quantities. Some of these predictions are based on the basic idea that the dissipation scale of turbulence is a fluctuating quantity satisfying (7). It is then possible to compute the probability distribution of the velocity gradients by using the estimate [10, 18, 19]:

$$\partial_r u \sim \frac{\delta_\eta u}{\eta}. \tag{12}$$

Since $\delta_{\eta(h)}u \sim (\eta(h))^h$ and $\eta(h) \sim \nu^{1/(1+h)}$, we have:

$$\langle(\partial_r u)^q\rangle \sim \int dh \nu^{(q(h-1)+F(h)))/(1+h)} \sim Re^{\chi(q)}, \tag{13}$$

which predicts the scaling properties of the gradient as function of $Re \sim 1/\nu$, and where the last expression is obtained by the saddle point method when the Reynolds number $Re \sim \nu^{-1}$ tends to infinity. Here

$$\chi(q) \equiv \min_h[(2q(h-1)+F(h))/(1+h)].$$

Note that the prediction is highly non trivial and it is connects $\chi(p)$ with the exponents $\zeta(p)$ via a non-linear relation valid for any function $F(h)$ (see [2]):

$$\chi(q) = \frac{p - \zeta(p)}{2}; \quad \text{with } q = \frac{\zeta(p) + p}{2}. \tag{14}$$

The above relation implies the existence of dissipative anomaly, i.e. $\chi(2) = 1$, for any spectrum such that $\zeta(3) = 1$:

$$\lim_{Re \to \infty} \epsilon \sim Re^{-1} \langle (\partial_r u)^2 \rangle \to const. \tag{15}$$

Such non-trivial multifractal spectrum for velocity gradients may be used to define a multifractal spectrum of dissipative scales, each-one defining the different cross-over from the inertial range to the dissipative range for different moments. To implement this, we may follow two slightly different methods. First, let us define the n-th order dissipative scale as the intersection between the extrapolation for large scales of the differential smooth behavior of $S^{(n)}(r)$ with the extrapolation for small scales of the inertial behavior:

$$S^{(n)}(r) \sim r^n \langle (\partial_r u)^n \rangle \quad \text{if } r \ll \eta_{k41}, \tag{16}$$

$$S^{(n)}(r) \sim r^{\zeta(n)} \quad \text{if } r \gg \eta_{k41}. \tag{17}$$

The intersection between the two power-law behavior can be considered a good estimate for the typical dissipative scale, $\eta_{(n)}$, of the n-th order structure functions:

$$(\eta_{(n)})^n \langle (\partial_r u)^n \rangle = (\eta_{(n)})^{\zeta(n)}. \tag{18}$$

Using expression (14) for the scaling of gradients, we get:

$$\eta_{(n)} \sim Re^{-\chi(n)/(n - \zeta(n))}. \tag{19}$$

Notice that according to this definition we have for the dissipative cut-off of the second order structure function:

$$\eta_{(2)} \sim Re^{-\chi(2)/(2 - \zeta(2))}. \tag{20}$$

Another, slightly different, way to define the moment-dependent viscous cut-off is the one followed in [20, 21], where a balancing inside the equations of motion between inertial and dissipative terms is used to obtain $\eta_{(n)}$. Following [20, 21], it is easy to derive the dimensional balancing between the inertial terms and the dissipative contribution in the evolution of the n-th order generic structure functions, from (1):
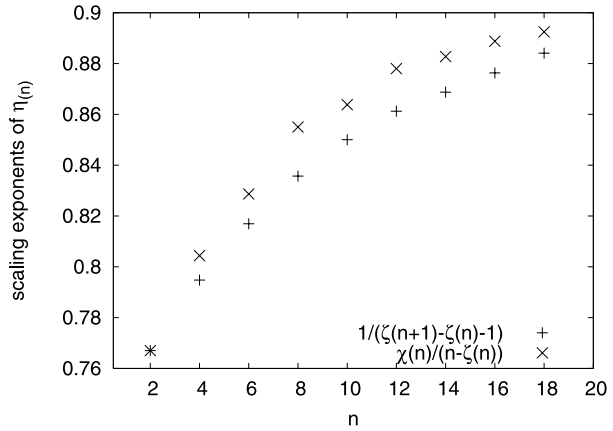
$$S^{(n+1)}(r)/r \sim \nu S^{(n)}(r)/r^2. \tag{21}$$

If the n-th order dissipative scale is now obtained by asking it to be the scale where the previous matching holds, we easily obtain, another expression for the cut-offs [20]:

$$\eta'_{(n)} \sim Re^{-1/(\zeta(n+1) - \zeta(n) + 1)}. \tag{22}$$

Let us notice that the two expressions (19) and (22) are exactly equivalent for the cut-off entering in the dissipative anomaly $\eta_{(2)} \equiv \eta'_{(2)} = Re^{-1/(2 - \zeta(2))}$, for any choice of $F(h)$, while they are slightly different for high-order moments. The discrepancies are however very

small, as can be directly checked by plugging some possible shape of the $\zeta(p)$ exponents that fit the experimental data (for example the log-Poisson expression derived in [22], as shown in Fig. 1). Recently, ad-hoc advanced computations have been performed by changing the numerical resolution, in order to test the existence of such non-trivial fluctuations at dissipative scales [23]. The numerical data have also been successfully compared with the MF prediction (7) in [24]. Note that it is quite difficult to get good laboratory data for velocity fluctuations at scales much smaller than η_{k41} due to the intrusive nature of the experimental probes. Nevertheless, in recent years very interesting and promising experimental techniques have been developed to track particles in turbulent flows, accessing temporal fluctuations instead of spatial fluctuations [25–30]. The quality of predictions based on the multifractal conjecture is even more striking by performing the statistical analysis of the velocity field in the Lagrangian framework.

3.2 Lagrangian Framework

Given the importance of particle dynamics in turbulent flows, numerous numerical and experimental studies have flourished in the last few years [26, 31–36] (see also [30] for a recent review). Neutrally buoyant particles, advected by a turbulent velocity field $\mathbf{u}(\mathbf{x}, t)$, follow the same path as fluid molecules and evolve according to the dynamics $\dot{\mathbf{X}}(t) = \mathbf{v}(t) \equiv \mathbf{u}(\mathbf{X}(t), t)$, where the Lagrangian velocity $\mathbf{v}$ equals the Eulerian one $\mathbf{u}$ computed at the particle position $\mathbf{X}$. Such particles constitute a clear-cut indicator of the underlying turbulent fluctuations. Recently, it has been shown, by comparing the different numerical studies and different experimental results [5, 37], that Lagrangian turbulence is universal, intermittent, and well described by a suitable generalization of the Eulerian Multifractal formalism to the Lagrangian domain [38–40].

Lagrangian Structure Functions (LSF) are defined as:

$$\mathcal{S}_{\mathrm{L},i}^{(p)}(\tau) \equiv \langle [v_i(t + \tau) - v_i(t)]^p \rangle = \langle (\delta_\tau v_i)^p \rangle, \tag{23}$$

where $i = x, y, z$ runs over the three velocity components, and the average is defined over the ensemble of particle trajectories evolving in the flow. From now on, we will assume isotropy and therefore drop the dependency from the spatial direction i. The presence of long spatial and temporal correlations suggests, in analogy with critical phenomena, the

existence of scaling laws for time scales larger than the dissipative Kolmogorov time and smaller than the typical large-scale time, $\tau_\eta \ll \tau \ll T_L$:

$$\mathcal{S}_L^{(p)}(\tau) \sim \tau^{z(p)}. \tag{24}$$

Straightforward dimensional arguments *à la* Kolmogorov predict $z(p) = p/2$, independently of the flow properties. However, it is known that LSF experience show strong variations when changing the time lags τ, as highlighted by the increasingly non-Gaussian tails characterizing the probability density functions of $\delta_\tau v$ for smaller and smaller τ's [26]. This leads to a breakdown of the dimensional argument; correspondingly, there is a growth of the Lagrangian flatness when going to smaller and smaller time lags. In [5, 37] it has been shown that the scaling exponent $\kappa(4) = z(4)/z(2)$, entering in the evolution of the fourth-order flatness:

$$\frac{\mathcal{S}_L^{(4)}(\tau)}{(\mathcal{S}_L^{(2)}(\tau))^2} \sim (\mathcal{S}_L^{(2)}(\tau))^{\kappa(4)-2} \tag{25}$$

does not depend on the experimental or numerical large-scale set-up. In other words, the high frequency fluctuations are universal.

It is possible to obtain a link between Eulerian and Lagrangian MF formalism via the dimensional relation [38, 39]:

$$\tau \sim r/\delta_r u. \tag{26}$$

Indeed, if we substitute (26) into (5) and into (6) we obtain a Lagrangian prediction for LSF once the Eulerian one is known via its $F(h)$ spectrum [39], namely

$$\mathcal{S}_L^{(p)}(\tau) \sim \int dh \tau^{(ph+F(h))/(1+h)} \sim \tau^{z(p)}, \tag{27}$$

where

$$z(p) = \min_h [(ph + F(h))/(1+h)].$$

This result is obtained by the saddle point method for inertial-range time intervals: $\tau_\eta \ll \tau \ll T_L$. Let us notice that the above relation (27) can be read as a prediction for the Lagrangian domain, once the Eulerian statistics is known. This is because we are using the same MF functions $F(h)$ for both domains. The suggested road map should be the following: (i) first measure the Eulerian scaling exponents $\zeta(p)$; (ii) then via an inverse Legendre transform extract the $F(h)$-spectrum; (iii) finally, apply the relation (27) and calculate the Lagrangian scaling. Such procedure is working well, at least within the statistical limitation and the Reynolds number limitations allowed by numerical and experimental state-of-the-art techniques [5] (see also [41] for a recent theoretical attempt).

Even more interesting, in [5], the same argument, leading to the spatial dissipative fluctuating scale (7), has been extended into the Lagrangian domain to obtain an expression for the fluctuating dissipative time scale [42]:

$$\tau_\eta(h) \sim \nu^{(1-h)/(1+h)}. \tag{28}$$

Following the same ideas discussed in Sect. 2 about the viscous modification to the MF formalism, we may now introduce two functions, $\tilde{f}(\frac{\tau}{\tau(h)})$ and $\tilde{g}(\frac{\tau}{\tau(h)})$, which take into account the viscous corrections to the Lagrangian inertial scale (27). It is customary to use for

the two cross-over functions a Batchelor-Meneveau functional form [5, 13, 40]. The global description for velocity increments becomes then:

$$\delta_\tau v(h, \tau_\eta(h)) \sim \frac{\tau/T_L}{[(\frac{\tau}{T_L})^\beta + (\frac{\tau_\eta(h)}{T_L})^\beta]^{\frac{1-2h}{\beta(1-h)}}}, \tag{29}$$

in which β is a free parameter which controls the crossover from dissipative to inertial time lags. It is easy to realize that the above expression reproduces the asymptotic regimes, $\delta_\tau v \sim \tau$ for $\tau \ll \tau_\eta$ and $\delta_\tau v \sim \tau^{h/(1-h)}$ in the inertial range. As in the Eulerian case, each exponent h must be weighted with its probability, which is also modeled to include dissipative-range physics, mimicking what has already been discussed in the introduction for the Eulerian case:

$$P_h(\tau, \tau_\eta(h)) = \mathcal{Z}^{-1}(\tau) \left[\left(\frac{\tau}{T_L} \right)^\beta + \left(\frac{\tau_\eta(h)}{T_L} \right)^\beta \right]^{\frac{F(h)}{\beta(1-h)}}. \tag{30}$$

Here $\mathcal{Z}$ is a normalizing factor. We notice that again everything is expressed in terms of the same MF function $F(h)$. Only a new free parameter, β, enters to describe the whole Lagrangian behavior, at all time scales, once the Eulerian MF spectrum, $F(h)$, is known. Putting everything together, we obtain for the LSF the following MF prediction:

$$\langle (\delta_\tau v)^p \rangle \sim \int dh \, P_h(\tau, \tau_\eta)[\delta_\tau v(h)]^p. \tag{31}$$

The previous formula for the Lagrangian domain predicts a highly non trivial shape for the local scaling exponents entering in the Flatness behavior:

$$\kappa(4, \tau) \equiv \frac{d \log(\mathcal{S}_L^{(4)}(\tau))}{d \log(\mathcal{S}_L^{(2)}(\tau))}. \tag{32}$$

As shown in Fig. 2, where we plot it for a series of different Reynolds numbers and for a given choice of the multifractal spectrum $F(h)$ (see caption for details). The functional dependence shown in Fig. 2 was tested in [5] only for $\kappa(4)$ and for a limited set of Reynolds numbers, due to the unavoidable limitations in experiments and numerical simulations. We want now to test it further, by boosting the Reynolds number by orders of magnitudes. To achieve this, we switch to a dynamical surrogate of the Navier-Stokes equations.

3.3 Shell Models

Let us now discuss in more details the behavior of the local scaling exponents for the Lagrangian structure functions (32). A decrease of the local scaling exponent simply means that intermittency is growing in the dissipation range. A detailed analysis of the flow configurations, which can be performed in high-resolution numerical simulations, show that the tip in the scaling exponents is strongly related to the presence of coherent three-dimensional vortices [34, 43]. This is an important observation which needs to be investigated carefully. The physical question we are discussing, concerns the possibility that coherent structures, i.e. vortices, are responsible for intermittency in three-dimensional turbulence, not just in the dissipative region but for the whole inertial range. After all, if the largest intermittent fluctuations of velocity gradients are due to vortices, it is somehow reasonable to think of
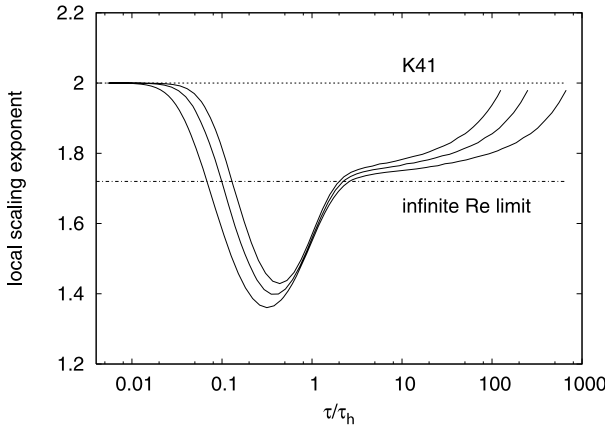
Fig. 2 Fourth-order local scaling exponents, $\kappa(4, \tau) \equiv \dfrac{d \log(\mathcal{S}_L^{(4)}(\tau))}{d \log(\mathcal{S}_L^{(2)}(\tau))}$, from the MF formalism (27) for three different Reynolds numbers, $Re = 2 \times 10^4, 9 \times 10^4, 7 \times 10^5$ (from *top* to *bottom*) The *two horizontal lines* correspond respectively to the non intermittent, K41 case, $\kappa(4) = 2$, and to the infinite Reynolds number limits, $\kappa(4) = 1.72$. The $F(h)$ spectrum used to derive this values has a log-Poisson shape as proposed in [22]. The value of the free parameter is $\beta = 4$. Notice that the inertial range extension, identified as the region where the local exponent is constant, becomes larger and larger when increasing the Reynolds number. Notice also the strong increase in the intermittency, measured by the deviation from the K41 value, $\kappa = 2$, for time scales across the viscous domain

the effect of coherent structure as a key feature for explaining intermittency. If this is true, the multifractal conjecture may be misleading, i.e. scale invariance is not a fundamental properties of three-dimensional turbulence. Nevertheless, in [5] it has been shown that the multifractal conjecture does predict the tip in the local scaling exponents for the structure functions.

So far, we have discussed the statistical properties of homogeneous and isotropic turbulence by checking the multifractal predictions against numerical and experimental data. In the last twenty years, it has been shown that there exists a class of simplified models, named shell model, which shows multifractal intermittency (anomalous scaling) similar qualitatively and quantitatively to what it is observed in Navier-Stokes turbulence. Among many different shell models, we shall consider the shell model proposed in [44] (see also [8] for a review). Shell models of turbulence, can be seen as a truncated description of the Navier-Stokes dynamics, preserving some of the structure and conservation laws of the original equations but *destroying* all spatial structures. They are described by the following set of ODEs:

$$\left(\frac{d}{dt} + \nu k_n^2\right)u_n = i(k_{n+1}u_{n+1}^* u_{n+2} - \delta k_n u_{n-1}^* u_{n+1}$$
$$+ (1 - \delta)k_{n-1}u_{n-1}u_{n-2}) + f_n. \tag{33}$$

Here the u_ns are the velocity modes restricted to 'wavevectors' $k_n = k_0 2^n$ with k_0 determined by the inverse outer scale of turbulence. The model contains one free parameter, δ, and it conserves two quadratic invariants (when the force and the dissipation term are absent) for all values of δ. The first is the total energy $\sum_n |u_n|^2$ and the second is a sort of generalized *helicity* $\sum_n (-1)^n k_n^\alpha |u_n|^2$, where $\alpha = \log_2(1 - \delta)$ [8]. Here we consider values

of the parameters such that $0 < \delta < 1$. The scaling exponents characterize the shell model structure functions, defined as

$$S^{(2)}(k_n) \equiv \langle u_n u_n^* \rangle \sim k_n^{-\zeta(2)}, \tag{34}$$

$$S^{(3)}(k_n) \equiv \mathrm{Im}\langle u_{n-1} u_n u_{n+1}^* \rangle \sim k_n^{-\zeta(3)}, \tag{35}$$

$$S_p(k_n) \sim k_n^{-\zeta(p)}.$$

The values of the scaling exponents have been determined accurately by direct numerical simulations. Besides $\zeta(3)$ which is exactly unity, because a relation similar to (2) holds, all the other exponents $\zeta(p)$ are anomalous, differing from $p/3$. For $\delta = 0.4$, the value of the $\zeta(p)$ are close to the scaling exponents of the Navier-Stokes equations. In applying the multifractal conjecture to the shell model, we shall assume $u_n \sim k_n^{-h}$ with probability $k_n^{-F(h)}$, while the fluctuating dissipative scale $k_d(h)$ is defined by the relation $u_d/(k_d\nu) \sim 1$, where $u_d = k_d^{-h}$. In the dissipation range, the behavior of the shell model is roughly consistent with $u_n \sim k_n \exp(-k_n/k_d)$.[1] By matching between the inertial range (i.e. $u_n \sim k_n^{-h}$) and the dissipation range, we then obtain:

$$u_n \sim k_n^{-h}(1 + A(k_n/k_d(h))^{1+h}) \exp(-k_n/k_d(h)) \tag{36}$$

where A is a Reynolds independent quantity. Consequently, the probability distribution $P_h(k_n)$ should be modified in order to take into account the dissipation effects, namely:

$$P_h(k_n) \sim [k_n(1 + A(k_n/k_d(h)))^{-1}]^{-F(h)}. \tag{37}$$

Knowing $F(h)$, which as usual can be estimated from the knowledge of $\zeta(p)$, we can compute the k_n dependency of $S^{(p)}(k_n)$ both in the inertial and in the dissipative range and compare our findings with numerical simulations of (33).

In Fig. 3 we plot the local scaling exponents

$$\kappa(4, k_n) \equiv \frac{d \log(S_4(k_n))}{d \log(S_2(k_n))},$$

computed from the numerical simulations of the shell model (upper left panel) and predicted by the multifractal conjecture (upper right panel). Different symbols refer to different Reynolds numbers. The first striking result is that the numerical simulations of the model clearly show a well defined tip in the dissipative region (i.e. large value of n in the figure), similar to what it is observed in experiments and in the numerical simulations. The tip is increasing towards small scales as the Reynolds number increases and it deepens. The straight line in the figure is a qualitative fit on the behavior of the tip as a function of Re. Clearly, the increase of intermittency in the dissipative region is scaling as $\log(Re)$. Notice that this scaling behaviour was not visible in the experimental and numerical data shown in [5, 37] because of the limited range of Reynolds spanned in those cases. In the right panel, we show $\kappa(4, k_n)$ as predicted by using the multifractal conjecture with $A = 0.8$. As one can observe, the qualitative and quantitative predictions are in remarkable agreement with the numerical simulations.

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[1]Balancing of the nonlinear and viscous terms in the far dissipation range actually gives to leading order $u_n \sim k_n \exp(-k_n^\alpha)$ with $\alpha \simeq 0.69$ solution of $2^{-\alpha} + 4^{-\alpha} = 1$.
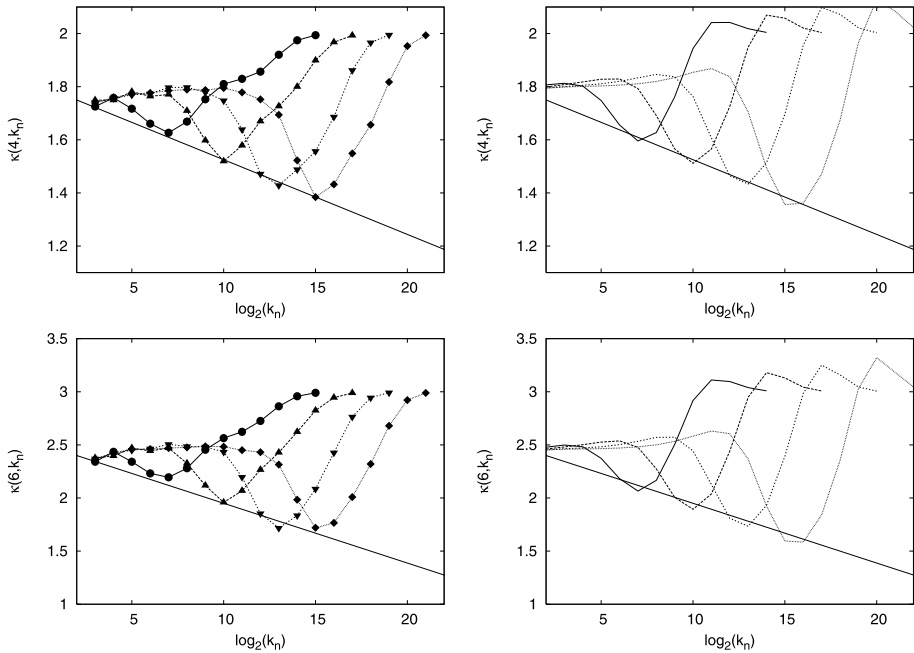
Fig. 3 *Top*: local scaling exponents for fourth order structure functions, $\kappa(4, k_n)$, in the Shell Model, versus $\log_2(k_n)$. *Left* numerical results at four Reynolds numbers in the range $Re \in [10^2 : 10^4]$. *Right*, the MF prediction using relations (36–37); the *straight line* is a fit for the behavior of the bottleneck (maximum of intermittency), the slope of the line is -0.028. *Bottom*: the same for the sixth order scaling exponents $\kappa(6, k_n)$. The *straight line* has the slope -0.056

The quality of the result does not change by increasing the order of the structure functions. In Fig. 3 we also show behavior of $\kappa(6, k_n)$ (i.e. the local scaling exponents of the sixth order structure functions) and compare the numerical simulations with the multifractal prediction. Again, the results confirms what we found for the lower order. Notice again, the widening of the inertial range when increasing the Reynolds number and the power-law behavior of the wavenumber where we observe the highest intermittency, i.e. the minimum in the local scaling exponents.

From the above analysis we can draw some interesting conclusions. First of all, the increase of intermittency in the dissipative range is not due to coherent vortices (there are no vortices in the shell model). Moreover, the increase of intermittency is predicted by the multifractal conjecture because of the fluctuations of the dissipative scale. Translating back these results to the real Navier-Stokes equation, we are tempted to conclude that coherent structures exist but their dynamics is nor relevant to explain intermittency in turbulent flows. They are the tail more than the dog (see Chap. 8.9 of [2]). Actually, one can take the opposite point of view: the matching between inertial range (scale invariance) intermittency and the dissipation range produces an increase in fluctuations and, consequently, an increase of vorticity. This effect is dominant at low Reynolds number, as it is clearly observed in the shell model: i.e. at low Reynolds numbers no scaling behavior is observed and intermittency is strongly dominated by the fluctuations in the dissipation range. Let us also point out that our observations is in qualitative agreement to the known phenomenology of boundary layer turbulence. Near the wall, the local Reynolds number is relatively low and strong intermittency

is observed together with a rich dynamic behavior of coherent structures (hairpin vortices). The full dynamics is dominated by strong intermittent fluctuations and it is tempting to relate our previous discussion to this specific well known turbulent flows.

What is remarkable in the above discussion, is that the change in the intermittency level, is explained by the same MF spectrum $F(h)$, at all scales and at all Reynolds numbers. In other words, the study of low Reynolds turbulence, and of the transition from viscous to inertial range [45, 46], may teach us a lot with respect to the high Reynolds limits [47].

4 Conclusions and Perspectives

Let us summarize our main points. First, we have reviewed the ideas about how to introduce, in a self-consistent way, dissipative effects in the Multifractal description of turbulence, both Eulerian and Lagrangian. The MF formalism, predicts an enhancement of intermittency in the so-called intermediate viscous range [48], as measured by the local scaling exponents (see Fig. 2). We have commented on the fact that such a trend is in very good agreement with real Lagrangian Turbulence data [5], at least concerning low order moments and moderate Reynolds numbers, i.e. up to the numerical and experimental state-of-the-art. Second, in order to test the formalism also for high Reynolds numbers, we switched to a class of Shell Models of turbulence. Here, thanks to the much simpler structures of the model, the Reynolds dependence of the MF prediction can be also tested. We have specialized our discussion on the enhancement of intermittency measured around dissipative scales. The existence of the very same phenomenon observed in real Navier-Stokes equations, lead us to conclude that the viscous intermittency is not due to vortex filaments. Many problems remain opened. For example, the Batchelor-Meneveau structure presented in (29) is not compatible with the requirement that velocity fluctuations follow a simple multiplicative local—in scales—process from large to small scales. This is due to the fact that the fluctuating temporal scale $\tau_\eta(h)$ appears in the definition of the velocity increments and it depends on them (in (28)–(29) the local scaling exponent h is the same). In other words, the functional relation (29) introduces non-local correlation between inertial and dissipative scales. A local-in-scale multiplicative process which takes into account the fluctuating cutoff can be introduced somewhat empirically by building a multiplicative cascade and stopping it according to the criteria (7). It would be interesting to see if such a procedure reproduces the Navier-Stokes data and the Shell Model data as nicely as (29).

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