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Some Recent Results on a Shell Model for 3D Turbulence

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Abstract. In this note we report a series of recently published results (by R. Benzi, M.H. Jensen, G. Paladin, G. Parisi, A. Vulpiani and myself) , on a Shell Model for the three dimensional cascade of energy in Fully Developed Turbulence. We present numerical and analitycal computations of the intermittent corrections to the K41 scaling of the velocity increments in the inertial range.

Key words: fully developed turbulence – dynamical systems – multifractals

1. Introduction

According to Kolmogorov theory [1] in turbulent flows there is a cascade of energy toward dissipative scales.

The presence of a range of length scales where inertial forces are dominant and where viscous effects as well as the external forcing can be neglected suggests the existence of (universal) scaling laws.

Assuming a constant rate of non-linear transfer of energy one obtains the classical Kolmogorov results for moments of the velocity difference in the inertial range: $< \delta u(\ell)^p > \propto \ell^{\zeta_p}$ with, $\zeta_p = p/3$ Nevertheless there are many experimental and numerical evidences [2, 3] that strong fluctuations of the energy transfer and dissipation are present, leading to the existence of a whole spectrum of possible singularities and to a non-linear ζ_p.

Only phenomenological arguments have been proposed in order to explain intermittency in the high Reynolds number limit of Navier-Stokes quations, while a direct derivation from their dynamic structure is still far from being achieved.

For this reason, it is useful to analyze particular models of the energy cascade process, instead of the complete Navier-Stokes equations, using an approach to the intermittency problem firstly proposed in [4] and developed by Grappin et al. [5]. We thus hope to reproduce the main characteristics of the small scale statistics of turbulence by a chaotic dynamical systems with a limited number of degrees of freedom.

In this model [6, 7] the Fourier space is divided in N shells. Each shell k_n ($n = 1, 2, ...N$) consists of the wavenumbers k such that $K_0 2^n < k \leq K_0 2^{n+1}$. The velocity difference over a length scale $\approx k_n^{-1}$ is given by u_n. The energy is $E = \sum |u_n|^2/2$ and its power spectrum is $E(k_n) = |u_n|^2/(2k_n)$. The Navier

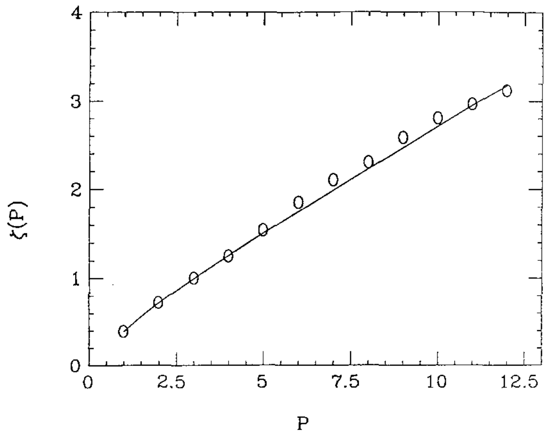

Fig. 1. Circles are the values of $\zeta(p)$ obtained from a numerical integration of equation (1) [7]. The solid line is computed from our closure ansatz by inserting in equation (6) the values of Π_1, Π_2 coincident with the correspondent numerical results [7].

Stokes equations are thus approximated by a dynamical system with $2N$ differential equations:

$$(\frac{d}{dt}+\nu k_n^2)\, u_n = i(a_n u_{n+1}^* u_{n+2}^* + b_n u_{n-1}^* u_{n+1}^* + c_n u_{n-1}^* u_{n-2}^*) + f\delta_{n,4} \quad (1)$$

where ν is the viscosity, f is a forcing (here on the fourth mode). The coefficients a_n, b_n and c_n are fixed by demanding energy and phase space conservation [7]. The unstable fixed point of eqs. (1) when $\nu = f = 0$ is given by the Kolmogorov scaling $u_n \propto k_n^{-1/3}$. The approximation of considering only first and second nearest neighbours for the interactions between shells follows from the locality of the cascade in the $k-$space.

>From a direct numerical integration of (1) [7] the energy spectrum is observed to scale as $k^{-\alpha}$, in the inertial range, with an exponent $\alpha = 1 + \zeta_2$ not exactly equal to the value $5/3$ expected by applying dimensional arguments. In fig. 1 one sees that the exponents ζ_p are not linear in p. This data show intermittent corrections to the K41 theory in quantitative and qualitative agreement with exponents extracted from real turbulent flows [3] The intermittency of the energy dissipation exhibited by the model is consistent with the multifractal approach [8, 9]. Our hope now consist in finding out some simple ansatz on the probability distribution function for the shell variables in the inertial range consistent with the closure equation of motion and able to reproduce the numerical results for the scaling laws [10].

2. The Closure Ansatz

In order to understand which is the real nature of intermittency in this model we will concentrate on the stationary probability $P[u]$, where we denote by $[u]$ the set of all u's. The knowledge of $P[u]$, in the region of large j and in the zero viscosity limit (i.e. fully developed turbulence), is enough to obtain all the relevant information.

For the $P[u]$ one can make, in the fully turbulent regime, the following Ansatz: $P[u] \propto exp(-\sum_j H_j)$, where H_j is given by $H(u_j, u_{j-1}, u_{j+1}, u_{j-2}, u_{j+2}.....)$. It is easy to verify that the above expression is compatible with the closure equations. In other words we assume that the Hamiltonian, $H = \sum_j H_j$, corresponding to the stationary distribution is invariant under scale transformations (i.e translations with respect to j).

We consider a slight different form of the model equations (1). Let us take the variables u_j in the polar representation $u_j = k_j^{-1/3} \rho_j exp(-\theta_j)$. The equations of the moduli ρ_j become:

$$\begin{aligned}(\frac{d}{dt} + \nu k_j^2)\rho_j &= k_j^{2/3}(\rho_{j+1}\rho_{j+2} sin(\theta_j + \theta_{j+1} + \theta_{j+2}) \\ &\quad -\frac{1}{2}\rho_{j-1}\rho_{j+1} sin(\theta_j + \theta_{j+1} + \theta_{j-1}) \\ &\quad -\frac{1}{2}\rho_{j-2}\rho_{j-1} sin(\theta_j + \theta_{j-1} + \theta_{j-2})) \end{aligned} \tag{2}$$

In the following we will use the variables $\Delta_j = \theta_{j-2} + \theta_{j-1} + \theta_j$ in order to simplify the notation. In the inertial range we will set $\nu = 0$. In this case the Kolmogorov solution correspond to $\rho_j = const.$

It is easy to realize that the trasfer of energy is mainly gouverned by the sign of $sin(\Delta_{j+1})$, we argue that in order to have a cascade of energy from large scales to small scales it should be negative, at least in the average.

It is possible to prove that the phases θ_j must have a flat distribution between $[0, 2\pi]$ while the same is not necessarly true for the sum of three of them [10]. Thus even in the Kolmogorov picture we must introduce some phase coerency between different scales in order to satisfy the requirement of a forward cascade of energy. Next we shall consider the time average $< ... >$ for the moment of order p of ρ_j. In the inertial range ($\nu = 0$), we obtain:

$$\begin{aligned} 0 &= < \rho_j^p d/dt \rho_j > = < \rho_j^p \rho_{j+1}\rho_{j+2} S_{j+2} > \\ &\quad -\frac{1}{2} < \rho_j^p \rho_{j+1}\rho_{j-1} S_{j+1} > -\frac{1}{2} < \rho_j^p \rho_{j-1}\rho_{j-2} S_j > \end{aligned} \tag{3}$$

where we have introduced the variables $S_j = sin(\Delta_j)$. Our aim is to solve equations (3) for all p by using a multiplicative process.

Our starting point is the hypothesis that $\rho_{j+1} = a_{j+1}\ \rho_j$ where a_j is a random variable to be specified. By substituting the above equation into (3) we obtain:

$$\begin{aligned} < a_{j+2} a_{j+1}^2 a_j^{p+2} a_{j-1}^{p+2} S_{j+2} > &- \frac{1}{2} < a_{j+1} a_j^{p+1} a_{j-1}^{p+2} S_{j+1} > \\ &- \frac{1}{2} < a_j^p a_{j-1}^{p+1} S_j > = 0. \end{aligned} \tag{4}$$

In order to solve these equations we have to specify the correlation among the a_j and the S_j. We first assume that a_j are uncorrelated variables (among themselves). This is a quite strong assumption which should be considered to be a first order approximation to the real solution. Next we assume that

$$a_j = C \ (1 \ - \ \beta \ S_j). \tag{5}$$

As a consequence of these two assumptions the S_j are uncorrelated variables. The assumption (5) gives the Kolmogorov scaling law for $C = 1$ and $\beta = 0$. Introducing the moments:$\Pi_p =< (1 \ - \ \beta \ S_j)^p >$, where $< ... >$ should be considered the average on the stochastic process $\beta \ S_j$ and by using this definition in (4), we obtain:

$$2C^6\Pi_{p+2}^2\Pi_2 < (1-\beta S)S > -C^3\Pi_{p+1}\Pi_{p+2} < (1-\beta S)S > \\ -\Pi_{p+1} < (1-\beta S)^p S >= 0. \tag{6}$$

Given the probability distribution of S we can consider (6) as a set of equations $F_p(\beta) = 0$. It turns out that beside the exact results $\zeta(3) = 1$, it is possible to close all the equations starting from the first two moments Π_1 and Π_2. In fig. (1) we have superimposed the $\zeta(p)$ curve obtained from eq. (6) via the relation $\zeta(p) = p/3 - log_2(C^p\Pi_p)$. and using as external input the values of Π_1 and Π_2 obtained numerically [7]. As we can see the numerical agreement is quite good. This tells us that our assumption on the multiplicative process could be considered to be a good first approximation.

We next go back to equation (6) and try to specify more about the probability distribution of S. Our starting point is to assume Δ_j to be uniformly distributed in the interval $[-\pi, 0]$, in this way we are describing a *forward* cascade of energy. Under this hypothesis we can compute the functions $F_p(\beta)$. It turns out that the values of β at which the equations $F_p(\beta)$ are solved fall in to a narrow interval for any p. This suggest us the possibility to improve the result by allowing β to fluctuate. We still consider Δ_j to be uniformly distributed. On the other hand we consider β not a constant rather a stochastic variable independent of S with probability distribution:

$$P(\beta) = q\delta(\beta - \beta_0) + (1-q)\delta(\beta). \tag{7}$$

Using eq. (7) into (6), we obtain a set of equations $F_p(\beta_0, q) = 0$. We have found that for $q = .95$ these equations (for $p = 0, ..., 4$) are nearly simultaneously satisfied. In conclusion we can say that for the closure equation obtained from $\frac{d}{dt} < \rho_n^p >= 0$, the assumptions made about the multiplicative process seem to satisfy all the functional constraints imposed by the equation of motion.

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