Turbulence on a Fractal Fourier set

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(Dated:)

The dynamical effects of mode reduction in Fourier space for three dimensional turbulent flows is studied. We present fully resolved numerical simulations of the Navier-Stokes equations with Fourier modes constrained to live on a fractal set of dimension D. The robustness of the energy cascade and vortex stretching mechanisms are tested at changing D, from the standard three dimensional case to a strongly decimated case for $D = 2.5$, where only about 3% of the Fourier modes interact. While the direct energy cascade persist, deviations from the Kolmogorov scaling are observed in the kinetic energy spectra. A model in terms of a correction with a linear dependency on the co-dimension of the fractal set, $E(k) \sim k^{-5/3+3-D}$, explains the results. At small scales, the intermittent behaviour due to the vorticity production is strongly modified by the fractal decimation, leading to an almost Gaussian statistics already at $D \sim 2.98$. These effects are connected to a genuine modification in the triad-to-triad nonlinear energy transfer mechanism.

PACS numbers:

Understanding the eddy motions at different scales is key in turbulence. The question is fundamental [1–3] and practical since modeling relies on assumptions invoking scaling invariance and scale-by-scale energy budgets [4, 5]. During the formation of strong turbulent fluctuations, large spatial structures create thin vorticity layers or filaments under both the action of shearing and stretching. Visualizations of the vorticity field, both at moderate and large Reynolds number, show a proliferation of small-scale vortex filaments populating intermittently all regions of the flow (as shown in Fig. 1). A long debate exists whether or not the presence of such geometrical structures can be correlated to the non-Gaussian statistical properties at the dissipative scale [1]. The dichotomy between dynamical and statistical descriptions has been investigated both theoretically and numerically with different strategies. Closures [6, 7] and renormalisation-group approaches [8] are based on a Fourier description of the turbulent motion and focus on the mean spectral properties. The multifractal model has been developed to account for intermittency and anomalous fluctuations using a hierarchy of scale sizes in real space [1]. Besides, many authors have focused on a vortex-by-vortex analysis, looking for the signatures of quasi-singularities or extreme events associated to specific dynamical properties of the Navier-Stokes (NS) equations [9–15].

This paper addresses the problem of the relation between dynamical and statistical properties of small-scale turbulent fluctuations using a novel technique: the NS equations are solved on a pre-selected, multiscale set of Fourier modes and the flow develops fluctuations on a given Fourier skeleton, belonging to a fractal set of dimension $D \leq 3$. For $D = 3$, the standard problem is recovered. As a result of the Fourier decimation, the velocity field is embedded in a three dimensional space, but effectively possesses a number of Fourier modes that grows slower with decreasing D. Degrees of freedom inside a sphere of radius k go as $\#_{dof}(k) \sim k^D$.

This idea has been introduced in [16] to test the hypothesis that two dimensional turbulence in the inverse energy cascade approaches a quasi-equilibrium state, when the

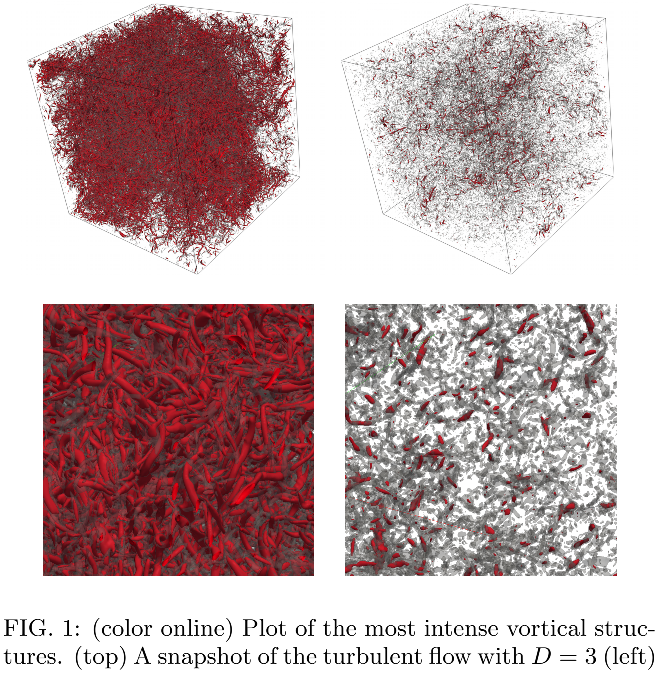

FIG. 1: (color online) Plot of the most intense vortical structures. (top) A snapshot of the turbulent flow with $D = 3$ (left) and a snapshot of the turbulent flow with $D = 2.98$ (right). Isosurfaces of the Q invariant of the velocity gradient tensor are plotted: values $Q/Q_{rms} = 1$ (grey) and $Q/Q_{rms} = 2$ (red). (bottom) A zoom in the top snapshots highlights details of the small scales.

turbulent motion is restricted on a set with $D \sim 4/3$ as suggested in [17]. Fourier decimation methods are not new for hydrodynamics in the direct energy cascade regime: we mention protocols with a specific degree of mode reduction [18–20], and the extreme truncation criterion of shell models for the turbulent energy cascade [21]. Moreover, at small Reynolds numbers, intermittency might strongly depend on the amount of scales resolved in the inertial range, a first evidence of which can be found in Ref. [18].

Fractal mode-reduction is a new route to perform numerical simulations to tackle the problem of intermittency and to develop multi-scale models of turbulence. Being an exquisitely dynamical approach, it is different from *aposteriori* filtering techniques, largely exploited to analyse turbulent data [22]. In the fractally decimated NS equations, a number of dynamical active variables are selected in a self-similar way and they are a function of one tuning parameter only, D. The problem is reformulated on a fractal set, without breaking any symmetry: statistical homogeneity, isotropy and rescaling properties of the inertial terms hold true as in the original NS equations in $D = 3$. Reducing modes in a self-similar way in Fourier space changes the relative weights of local to non-local triadic interactions, by modifying altogether the roles played by the large-scale advection, the non-linear stretching and the turbulent eddy viscosity.

To address these questions, we performed a series of Direct Numerical Simulations (DNS) of fractally decimated NS equations, with 1024^3 or 2048^3 collocations points on a regular cubic grid. The decimation operator $\mathcal{P}^D$ acts in the space of velocity fields as follows [16]. We define $\mathbf{v}(\mathbf{x}, t)$ and $\mathbf{u}(\mathbf{k}, t)$ as the real and Fourier space representation of the velocity field in $D = 3$, respectively. The decimated field, $\mathbf{v}^D(\mathbf{x}, t)$, is obtained as:

$$\mathbf{v}^D(\mathbf{x}, t) = \mathcal{P}^D \mathbf{v}(\mathbf{x}, t) = \sum_{\mathbf{k} \in \mathcal{Z}^3} e^{i\mathbf{k}\cdot\mathbf{x}}\, \gamma_{\mathbf{k}} \mathbf{u}(\mathbf{k}, t)\,. \qquad (1)$$

The random numbers $\gamma_{\mathbf{k}}$ are quenched in time and are:

$$\gamma_{\mathbf{k}} = \begin{cases} 1, & \text{with probability } h_k\,, \\ 0, & \text{with probability } 1 - h_k, k \equiv |\mathbf{k}|\,. \end{cases} \qquad (2)$$

The choice for the probability $h_k \propto (k/k_0)^{D-3}$, with $0 < D \leq 3$ ensures that the dynamics is isotropically decimated to a D-dimensional Fourier space. The factors h_k are chosen independently and preserve Hermitian symmetry $\gamma_k = \gamma_{-k}$ so that $\mathcal{P}^D$ is self-adjoint. The NS equations in the decimated Fourier space are then defined as:

$$\partial_t \mathbf{v}^D = \mathcal{P}^D N(\mathbf{v}^D, \mathbf{v}^D) + \nu\, \nabla^2 \mathbf{v}^D + \mathbf{f}^D\,. \qquad (3)$$

At each iteration of the numerical integration, the non linear term, $N(\mathbf{v}, \mathbf{v}) = -\mathbf{v} \cdot \nabla \mathbf{v} + \nabla p$, is projected on the quenched fractal set, to constrain the dynamical evolution to evolve on the same Fourier skeleton at all times. In the (L^2) norm, $\|\mathbf{v}\| \propto \int |\mathbf{v}(\mathbf{x})|^2 d^3x$, the self-adjoint operator $\mathcal{P}^D$ commutes with the gradient and viscous operator. Since $\mathcal{P}^D \mathbf{v}^D = \mathbf{v}^D$, it then follows that both energy and helicity are conserved in the inviscid and unforced limit, exactly as in the original problem.

A pseudo-spectral spatial method is adopted to solve eqs. (3), fully dealiased with the two-thirds rule; time stepping is implemented with a second-order Adams-Bashforth scheme. The flow is stationary, statistically isotropic and homogeneous. A large-scale forcing [23] keeps the total kinetic energy constant in a range of shells, $0.7 \leq |\mathbf{k}| < 1.7$. We performed several runs at changing the fractal dimension $2.5 \leq D \leq 3$, the spatial resolution and the realization of the fractal mask. Table I summarises the relevant parameters.

A visualisation of the most intense vortical structures reveals the effect of decimation on turbulent flows. In Figure 1, we plot isosurfaces of the Q invariant of the velocity gradient tensor [24]: the $D = 3$ case shows a large number of structures of both large and small-scale vortex filaments. The $D = 2.98$ clearly differs because structures are smaller and more spherical-like, also they are much less abundant, indicating a less intermittent spatial distribution of structures.

To disentangle the relation between large and small scales, the starting point is the shell-to-shell energy transfer in the Fourier space. Following the notation adopted in Ref. [2], we write the energy spectrum for a generic flow in dimension D as:

$$E^D(k) = \int_{|\mathbf{k}_1|=k} d^3k_1 \gamma_{\mathbf{k}_1} \int d^3k_2 \gamma_{\mathbf{k}_2} \langle \mathbf{u}(\mathbf{k}_1)\, \mathbf{u}(\mathbf{k}_2) \rangle\,, \quad (4)$$

where the decimation factor $\gamma_{\mathbf{k}}$ takes into account that the Fourier mode $\mathbf{k}$ is active with probability $h_{\mathbf{k}}$. Similarly, we can write for the energy flux across a Fourier mode k, $\Pi^D(k) = \int_{|\mathbf{k}_1|<k} d^3k_1 \partial_t E(k_1)$:

$$\Pi^D(k) = \int_{|\mathbf{k}_1|<k} d^3k_1 \gamma_{\mathbf{k}_1} \int d^3k_2 d^3k_3 \gamma_{\mathbf{k}_2} \gamma_{\mathbf{k}_3} S(\mathbf{k}_1|\mathbf{k}_2, \mathbf{k}_3), \qquad (5)$$

where the explicit form of the symmetric triadic correlation function is [25]: $S(\mathbf{k}_1|\mathbf{k}_2, \mathbf{k}_3) = -Im[\langle(\mathbf{k}_1 \cdot \mathbf{u}(\mathbf{k}_3))(\mathbf{u}(\mathbf{k}_1) \cdot \mathbf{u}(\mathbf{k}_2))\rangle + \langle(\mathbf{k}_1 \cdot \mathbf{u}(\mathbf{k}_2))(\mathbf{u}(\mathbf{k}_1) \cdot \mathbf{u}(\mathbf{k}_3))\rangle]$. Supposing a self-similar behaviour of the velocity fluctuations $u(k) \sim k^{-a}$, we can estimate the scaling behaviour of the energy flux as $\Pi^D(\lambda k) \sim \lambda^{3D+1-3a} \Pi^D(k)$. In this expression, the rescaling factor λ^{3D} is due to the integral over the variables $(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$, while λ^{1-3a} comes from the triadic non-linear term.

If a constant energy flux develops in the inertial range of scales, the following dimensional relation holds:

$$a = D + 1/3 \rightarrow E^D(k) \sim k^{3-D}\, E^{K41}(k) \qquad (6)$$

where $E^{K41}(k) \sim k^{-5/3}$ is the Kolmogorov spectrum expected for the standard case in $D = 3$, possibly corrected because of intermittency [26]. Dimensional prefactors have been omitted for simplicity. The relation (6)

D	3	3	2.999	2.99	2.99	2.98	2.98	2.8	2.8	2.5
N	1024	1024	1024	1024	2048	1024	2048	1024	1024	1024
M_r	100%	100%	99%	93%	92%	87%	85%	25%	25%	3%
η	0.75	0.95	0.75	0.95	0.70	0.75	0.70	0.90	0.40	0.65
$\mathcal{N}_T$	10	6	15	15	7	15	8	10	15	8

TABLE I: DNS parameters. The fractal dimension D; the grid resolution per spatial direction N; the percentage of surviving Fourier modes M_r; Kolmogorov length scale η in grid spacing units, where the grid spacing is $\Delta x = 2\pi/N$; the number of large-scale eddy-turnover-times, $\mathcal{N}_T$.

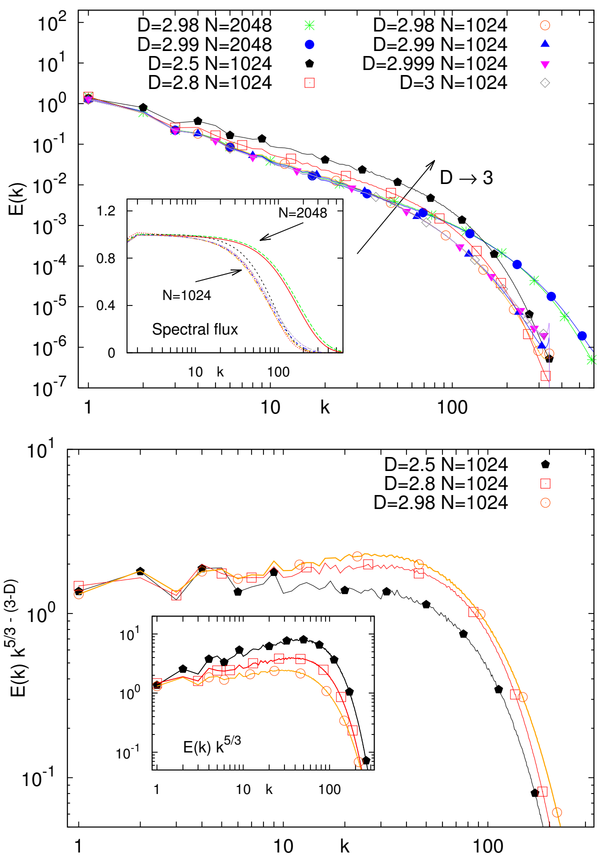

FIG. 2: (Upper panel): log-log plot of the mean kinetic energy spectra at changing D; in the inset, the mean kinetic energy fluxes. (Lower panel): Compensated energy spectra $E^D(k)\,k^{5/3-3+D}$ vs the wavenumber k; in the inset, the compensation is done with the Kolmogorov K41 prediction, $E^D(k)\,k^{5/3}$.

is obtained by noticing that because of homogeneity, we have that $\langle\, \mathbf{u}(\mathbf{k}_1)\cdot\mathbf{u}(\mathbf{k}_2)\,\rangle \propto F(\mathbf{k}_1)\delta^3(\mathbf{k}_1+\mathbf{k}_2)$, and by also noticing that the decimation projector verifies the identity $(\gamma_{\mathbf{k}})^2 = \gamma_{\mathbf{k}}$. As a result, the dynamical effect of Fourier fractal decimation is to make the energy spectrum shallower than the Kolmogorov prediction for three-dimensional turbulence, predicting the existence of a critical dimension $D = 7/3$, when the spectrum becomes ultraviolet divergent. By decreasing D in the presence of a forward energy cascade, the system has fewer modes available to transfer the same amount of energy (see Table I), and the velocity field becomes increasingly rougher.

In the upper panel of Figure 2, we plot the kinetic energy spectra and the associated energy fluxes, at changing the fractal dimension. It shows that at increasing the grid resolution for fixed D, from $N = 1024$ to $N = 2048$, no appreciable differences are observed, indicating that the presence of a forward energy cascade appears robust and Reynolds independent. In the lower panel we also show that the spectra compensate very well with the prediction (6), while they fail to satisfactorily compensate with the classical K41 prediction when $D < 3$. Figure 2 (upper inset) shows that at decreasing the fractal decimation, the mean energy transfer towards small scales is almost unchanged, i.e. the hypothesis leading to the prediction (6) is well verified. On the other hand, temporal fluctuations of the kinetic energy flux decrease with the fractal dimension D (not shown).

It might be argued that the effect of fractal Fourier decimation is purely geometrical and that the main dynamical processes are unchanged. To show that this is not the case, it is useful to analyse the effect of a *static* Fourier decimation. This can be done by considering snapshots of standard $D = 3$ turbulence, and applying the fractal decimation as an *aposteriori* filter. It is immediate to realise that the effect of the static decimation on the spectrum is $E^D_{st}(k) \sim k^{D-3}\,E^{K41}(k)$, implying that the geometrical action of the decimation goes in the opposite direction of the dynamical one.

We now consider the dynamical effect of the fractal Fourier decimation on the small-scale structures, by focusing on the statistics of the vorticity field in the real space. In Fig. 3 we plot the probability density function (PDF) of the vorticity field, normalised with its standard deviation. It is striking to note that already at $D = 2.99$, vorticity fluctuations have changed their intensity of one order of magnitude, despite the fact that the mean enstrophy is practically unchanged. Even more strikingly, intermittent fluctuations disappear already at $D = 2.8$, where a quasi-Gaussian vorticity PDF is measured. The transition towards a Gaussian behaviour is better quantified considering the vorticity kurtosis. In Figure 4, we compare results of the fractally decimated NS equations, with those obtained from the application of the *aposteriori* static mask on three-dimensional turbulence, as previously done for the kinetic energy spectra. The dynamical fractal decimation makes a very fast transition towards a Gaussian behaviour, such that at $D = 2.98$ the kurtosis has decreased by 30%, to already approach the Gaussian value at $D = 2.8$. In the case of the *aposteriori* static decimation, vorticity kurtosis assumes the Gaussian value only at $D = 2.5$, while staying almost unchanged in the range $D \geq 2.98$. Such a strong difference clearly indicates that constraining the dynamics to a

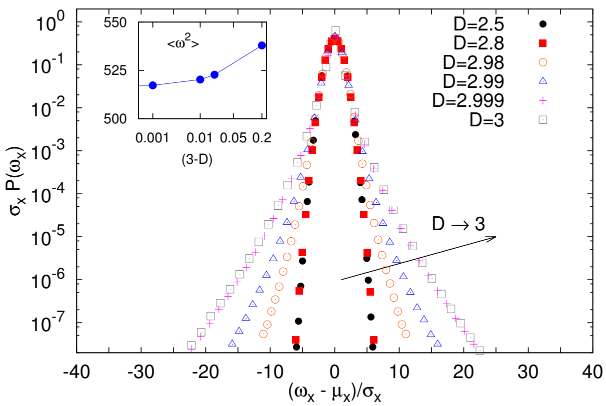

FIG. 3: Probability density function of the vorticity component ω_x, normalised to its standard deviation. Data refer to simulations at resolution $N = 1024$. In the inset, mean square vorticity $\langle w^2 \rangle$ versus the fractal dimension deficit, $3 - D$.

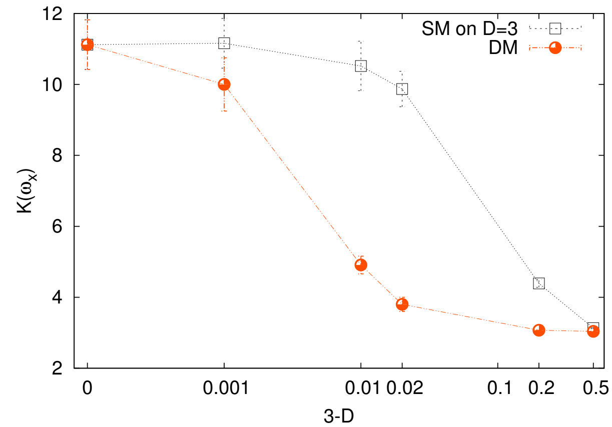

FIG. 4: Lin-log plot of vorticity kurtosis vs the dimension deficit $3 - D$. The upper curve (SM) is obtained from the application of the *aposteriori* static mask on $D = 3$ velocity field snapshots; the lower curve (DM) is obtained from the Fourier decimated DNS.

sub-set of modes is critical for the complete development of intermittency in real space. The presence or absence of some of the Fourier modes strongly modify the fluctuations of all the others, suggesting the possibility that intermittency is the result of percolating dynamical properties across the whole Fourier lattice [27].

Conclusions. We have numerically studied turbulent flows resulting from the evolution of the NS equations solved on a fractal skeleton, characterized by a unique control parameter, the dimension D. Fractal Fourier decimation modifies the relative weight between local and non-local Fourier triads [28], as well as the phase correlation between Fourier modes [29, 30]. The first result is that the decimation does not alter the energy flux, i.e. an inertial range of scales with a constant-flux solution is observed at changing D, at least in the the parameter range investigated here. This is in agreement with the observation that Galerkin truncations do not alter the inviscid conservation of quadratic quantities, preserving the existence of exact scaling solution for suitable third-order correlation functions (see appendix of Ref. [31]). Second and most striking, the mode reduction has two important effects. The Fourier spectrum of the surviving modes gets a power law correction, and small-scale intermittency is quickly reduced for $D < 3$ and it is observed to almost vanish already at $D = 2.98$.

Because of the spectrum modification, the scaling exponent of the second order longitudinal structure function becomes $\zeta_2 + (D - 3)$, where ζ_2 is the scaling exponent measured in the standard $D = 3$ case. This observation would suggest that, for the dimension deficit $3 - D < 1$, one may obtain corrections to all anomalous exponents proportional to $3-D$, and the anomalous exponents might be computed perturbatively in the dimension deficit. If this is the case, the critical dimension D_c is estimated as the value of the fractal dimension D where the Kolmogorov 1941 scaling is recovered, namely $\zeta_2 + D_c - 3 = 2/3$. This gives $D_c \sim 2.96$ not far from the value of D at which intermittency is observed to vanish in our simulations. However, there is no reason to assume that anomalous exponents can be computed perturbatively in $3 - D$. In fact, intermittency might also be understood as the result of a global, multiple-scale interaction in Fourier space, needing all degrees of freedom to develop: any tiny decimation would kill these singular solutions of the NS dynamics. In such case, anomalous exponents can not be obtained perturbatively, and phenomenological cascade models [1] would be unable to explain the results. Finally, we want to mention that the effect of Fourier decimation on the dynamics could also be interpreted as a modification of the non-linear term of the NS equations, exactly removing the statistical contribution of each decimated Fourier mode. As such, fractal decimation might introduce at all scales self-similar fluctuations that dominate the scaling properties, similarly to what happens for NS equations stirred by a random, power-law forcing [32–36]. As reported in Ref.[35], when the external energy injection directly affects the cascade and becomes the dominant statistical contribution in the inertial range, a transition to a Gaussian statistics for velocity increments in the inertial range is observed. All these possibilities are open, and might be key to explain the strong departure from the non-Gaussian statistics of standard $D = 3$ turbulence. Given the state-of-the-art of numerical simulations, it is hardly possible to discriminate between these different effects.

We acknowledge useful discussions with S. Musacchio and P. Perlekar, who collaborated with us in the first part of the work. DNS were done at CINECA(Italy), within the EU-PRACE Project Pra04, N.806. This work is part of the activity of the ERC AdG NewTURB, N.339032. We thank F. Bonaccorso and G. Amati for technical support. We thank the COST-Action MP1305 for support.

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