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Scaling property of turbulent flows

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We discuss a possible theoretical interpretation of the self-scaling property of turbulent flows [extended self similarity (ESS)]. Our interpretation predicts that, even in cases when ESS is not observed, a generalized self-scaling must be observed. This prediction is checked on a number of laboratory experiments and direct numerical simulations.

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The small scale statistical properties of turbulent flows are usually described in terms of the probability distribution of the velocity increments $\delta v(r) \equiv v(x+r)-v(x)$, where $v(x+r)$ and $v(x)$ are velocities along the x axis at two points separated by a distance r. By assuming locally statistical homogeneity and isotropy and a constant rate ϵ of energy transfer from large to small scales, Kolmogorov (1941) (K41) [1] predicted the existence of an inertial range, i.e., $\eta \ll r \ll L$, L being the integral scale of turbulence and $\eta \equiv (\nu^3/\epsilon)^{1/4}$ the inner or Kolmogorov scale, where the probability distribution function (PDF) of $\delta v(r)$ depends only on r and ϵ. It then follows that

$$\langle \delta v(r)^p \rangle \sim \epsilon^{p/3} r^{p/3}. \tag{1}$$

Although the basic assumptions behind K41 are usually considered to be correct, there is rather large evidence that K41 prediction (1) is violated in fully developed turbulence, namely, one finds in the inertial range

$$\langle \delta v(r)^p \rangle \sim r^{\zeta(p)} \tag{2}$$

where $\zeta(p\}$ is a nonlinear convex function of p and $\zeta(3)=1$ [2]. Scaling (2) is referred to as anomalous scaling because it cannot be deduced by simple dimensional considerations.

Recently it has been pointed out that scaling (2) can be generalized in the following way:

$$\langle \delta v(r)^p \rangle \sim \langle \delta v(r)^3 \rangle^{\zeta^*(p)} \tag{3}$$

where $\zeta^*(p) \approx \zeta(p)$ [3]. Scaling (3) has been observed both at low and moderate Reynolds number and for a wider range of scales r with respect to scaling (2) [4]. Because of these properties, the self-scaling property (3) of the velocity field has been named extended self similarity (ESS). The aim of this Rapid Communication is to propose an interpretation of ESS. Moreover, our interpretation predicts a generalized form of ESS which should hold also for nonisotropic and nonhomogeneous turbulence and for any scale r. These predictions are supported by experimental and numerical results.

Our starting point is the multifractal interpretation of anomalous scaling (2), namely,

$$\langle \delta v(r)^p \rangle \sim \int d\mu(h) r^{hp} r^{3-D(h)}, \tag{4}$$

$$\zeta(p) = \inf_h [hp+3-D(h)], \tag{5}$$

where $D(h)$ is assumed to be the fractal dimension of the set of points where $\delta v(r) \sim r^h$.

Many phenomenological multifractals models for $D(h)$ have been proposed. Among them, we shall consider those models which are consistent with an infinitely divisible distribution of random multipliers [5,6]. For all these models $D(h)$ can be written as

$$D(h) = 3 - d_0 f\left(\frac{h-h_0}{d_0}\right) . \tag{6}$$

Different models give different shapes of the function $f(x)$ and suggest different physical interpretations of h_0 and d_0.

By using (6) into (5) we obtain

$$\zeta(p) = h_0 p + d_0 H(p) \tag{7}$$

where

$$H(p) = \inf_x [px + f(x)]. \tag{8}$$

In order to clarify the following discussion, let us consider the She-Leveque model which is in remarkably good agreement with existing experimental and numerical data [7]. In this model h_0 characterizes the most singular behavior of the velocity field and $D_0 \equiv 3-d_0$ the corresponding fractal dimension.

At low Reynolds number or, equivalently, at small scales r, the effect of viscosity ν may become relevant. In simple phenomenological models, the effect of viscosity is usually represented as a cutoff in the energy transfer. Here we consider an alternative point of view: the energy transfer, as well as its fluctuations responsible for intermittency effect, continues to hold and, because of viscosity, the probability distribution of the velocity increments acquires a dependence on the ratio (r/η). If this is the case, both h_0 and d_0 may acquire a (smooth) dependence on r. Indeed, we expect that the role of the viscosity should increase the value of h_0 (i.e., reduce the strength of maximum singularity) and reduce the number of structures where $\delta v(r) \sim r^{h_0(r)}$. Thus, the probability P to observe a local scaling $\delta v(r) \sim r^{h_0(r)}$ should decrease. Because $P \sim r^{3-D_0} \sim r^{d_0}$, we deduce that d_0 should be an increasing function of r.

If our picture is qualitatively correct, ESS simply states that $h_0(r)/d_0(r)=$const, i.e., the dependence on r of h_0 and d_0 is the same. Indeed, one has

$$\frac{\zeta(p)}{\zeta(q)} = \frac{\frac{h_0}{d_0} p + H(p)}{\frac{h_0}{d_0} q + H(q)}$$

which does not depend on r. Let us remark that a smooth dependence on r of h_0 and d_0 does not spoil the saddle point integration (4) on $d\mu(h)$. Also, let us note that this interpretation of ESS allows us to generate a synthetic turbulence signal, by a random multiplicative process, which shows ESS. Eventually at very small scales the effect of viscosity is strong enough to destroy ESS. In homogeneous and isotropic turbulence, ESS is indeed broken at small scales of order of few (5–6) Kolmogorov lengths [4].

Our interpretation of ESS is based upon the assumptions that the statistical properties of turbulence at low Re or at small scales are controlled by (4) with h_0 and d_0 smooth functions of r and h_0/d_0=const. This implies a (delicate) balance between the scaling of the most singular structures in a turbulent flow and the number of these structures. This balance can be broken in different ways. For instance, near boundary layers or in strong shear flow conditions, energy production and momentum transfer can significantly change the slope and the number of the most singular structures. In these cases ESS should not be observed [8,9]. However, even in cases where ESS is not observed, our theoretical interpretation could still be valid and we think it is very important to check any possible prediction.

To this aim, let us consider the following dimensionless quantity:

$$G_p(r) = \frac{\langle \delta v(r)^p \rangle}{\langle \delta v(r)^3 \rangle^{p/3}} . \tag{9}$$

Let us notice that in the above expression any other structure function in place of the third order structure function in the denominator would work as well if properly normalized. According to (7) and (8) we obtain

$$G_p(r) = r^{d_0[H(p)-(p/3)H(3)]}. \tag{10}$$

Our theoretical interpretation of ESS suggests that $G_p(r)$ should always satisfy the self-scaling properties:

$$G_p(r) \sim G_q(r)^{\rho(p,q)} \tag{11}$$

regardless of any boundary layer, shear flow, or viscosity which can spoil ESS. In (11) $\rho(p,q)=[H(p)-(p/3)H(3)]/[H(q)-(q/3)H(3)]$ and does not depend on d_0.

We have checked (11) in a variety of turbulent flows. We have found that (11) is always satisfied within the accuracy of statistical errors. In Fig. 1 we plot $G_6(r)$ against $G_5(r)$, in a log-log scale, for a few cases three of which do not show ESS. Details of the experimental and numerical setup can be found in [8,13,14]; typical length separations are $\eta < r < 50\eta$ where with η we mean the Kolmogorov scale. In Fig. 2 we plot $G_8(r)$ versus $G_5(r)$ in a log-log scale for the shear flow numerical simulation; also for this higher couple of moments (11) is satisfied.

In all cases, we have found that (11) is satisfied down to the smallest scale available in our laboratory experiments or numerical simulations. Also, we have found that (11) holds also in the limit where $\delta v(r) \sim r$. This means that, in terms of the self-scaling properties of $G_p(r)$, no evidence of a viscous cutoff has been observed.

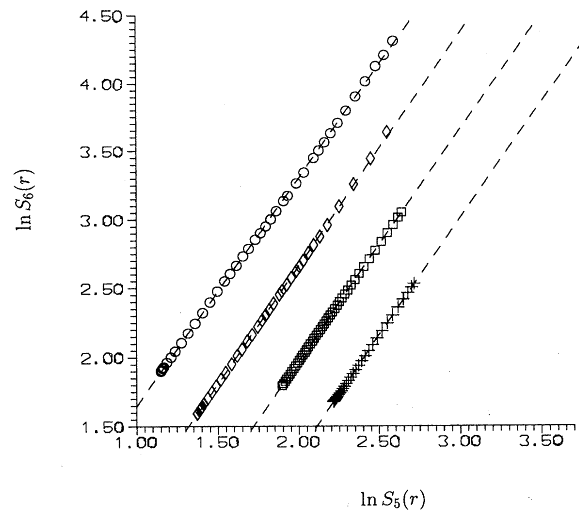

FIG. 1. The figure shows log $G_6(r)$ plotted against log $G_5(r)$ for four different experimental and numerical cases. (0) refers to the wake of a cylinder of 10 cm diameter taken at 60 cm downstream. In this case no ESS is observed (see [8] for further details). Diamonds refers to the hot wire measurement taken at $z=7$ mm of a boundary layer (courtesy of G.R. Chavarria). Also in this case ESS is not observed. Squares refers to a direct numerical simulation of turbulent convection at Ra$\approx 10^7$ [14]. Crosses refers to direct numerical simulation of a Kolmogorov flow [13] at $\mathrm{Re}_\lambda \approx 40$.

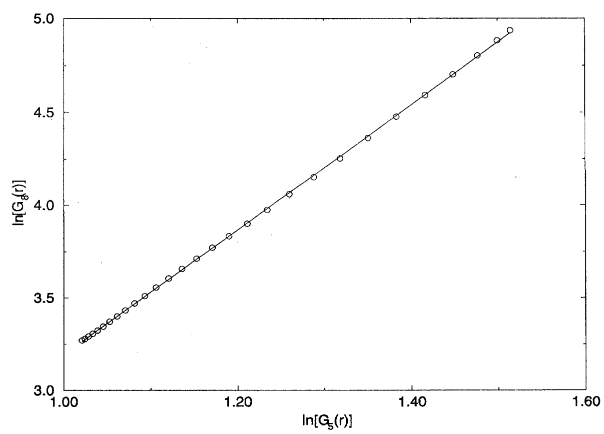

FIG. 2. Ln-Ln plot of $G_8(r)$ versus $G_5(r)$ for the shear flow simulation [13]. The continuous line is the best linear fit giving $\rho(8,5)=3.35$.

The validity of (11) (which we refer to as generalized ESS) may have important theoretical consequences. Indeed, it has been observed in [4,10,12] that the following ESS form of the Kolmogorov refined similarity hypothesis is always satisfied in turbulent flows:

$$\langle \delta v(r)^p \rangle \sim \langle \epsilon(r)^{p/3} \rangle \langle \delta v(r)^3 \rangle^{p/3} \tag{12}$$

where $\epsilon(r)$ is defined as the local energy dissipation averaged on a box of side r. Because of (12), Eq. (11) tells us that $\epsilon(r)$ displays ESS on all scales regardless of the effect of boundary layers and shear flows. This gives strong constraints on how a turbulent flow can dissipate energy on small scales. In particular, viscous effects do not change the anomalous scaling in $\epsilon(r)$ in any appreciable way, at variance with existing theoretical and phenomenological models of turbulence [2,11].

A more systematic presentation of our results, including a simple model to generate a statistical signal in agreement with Eqs. (11) and (12), is in preparation.

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