Analytic Calculation of Anomalous Scaling in Random Shell Models for a Passive Scalar

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(Received 31 January 1997)

An exact nonperturbative calculation of the fourth-order anomalous correction to the scaling behavior of a random shell model for passive scalars is presented. Importance of ultraviolet (UV) and infrared (IR) boundary conditions on the inertial scaling properties are determined. We find that anomalous behavior is given by the null space of the inertial operator and we prove strong UV and IR independence of the anomalous exponent. A limiting case where diffusive behavior can influence inertial properties is also presented. [S0031-9007(97)03479-0]

PACS numbers: 47.27.Gs, 47.27.Jv

Recently, there has been growing evidence, both numerically and experimentally, that fully developed turbulence is characterized by anomalous scaling of the velocity structure functions $F_p \equiv \langle |v(x+r) - v(x)|^p \rangle$. In particular, it has been shown that, within the inertial range (i.e., $\eta \ll r \ll L$, η being the Kolmogorov dissipation scale and L the outer length of external forcing) $F_p(r) \sim r^{z_p}$, where z_p is a nonlinear function of p which do not follow dimensional counting, i.e., $z_p \neq \frac{p}{q} z_q$. One of the most challenging scientific issues is to develop a theory which allows a systematic computation of z_p by using the equation of motions.

Recently [1–5] this issue has been addressed by studying a highly nontrivial "toy model" introduced by Kraichnan, namely, the advection of a passive scalar by a random, Gaussian velocity field, white in time, and whose two-point velocity correlation function is given by $\langle v_i(x,t) \times v_j(x',t') \rangle = \delta(t-t') D_{ij}(|x-x'|)$, with $D_{ij}(x) = D_{ij}(0) - \hat{D}_{ij}(x)$. Here, $\hat{D}$ is the d-dimensional velocity-field structure function: $\hat{D}_{ij}(x) = D_0 |x|^{\xi} [(d-1+\xi)\delta_{ij} - \xi x_i x_j |x|^{-2}]$ where the scaling exponent of the second-order velocity structure function, ξ $(0 \ll \xi < 2)$, is a free parameter. Higher order velocity-field correlation functions are fixed by the Gaussian assumption.

Although such a choice is far from being realistic, many interesting analytical and phenomenological results have been obtained for this toy model. Because of the delta correlation in time, moment equations to all orders are closed. In [1,6], for the first time Kraichnan gave the closed expression for $\tilde{S}_2(r)$, where

$$\tilde{S}_p(r) \equiv \langle |\theta(x) - \theta(x+r)|^p \rangle \sim r^{\zeta_p}, \tag{1}$$

$\theta(x)$ being the scalar field transported by the turbulent velocity field. In the inertial range $\tilde{S}_p(r) \sim r^{\zeta_p}$ and the set of scaling exponents ζ_p fully characterizes intermittency. In [1] a theory for all structure functions is proposed and an explicit formula for ζ_p is derived. The main physical outcome is that all structure functions of order greater than two have intermittent corrections and that intermittency should be connected to some nontrivial matching between advective and diffusive properties of the model.

In [2,3] it has been shown that intermittency of the scalar structure functions is connected to the properties of the null space of the linear operator appearing in the equation of multipoint passive scalar moments. Moreover, in [2,3] a perturbative expression of intermittency correction, as a function of the parameter ξ and of the system dimensionality has been derived. Both in [1] and in [2,3], matching conditions at infrared (IR) and ultraviolet (UV) scales should be taken into account. The same problem for the case of passive vectors was addressed in [7]. Finally, in all cases, universality in the scaling exponents is supposed to be preserved.

In [8] two of us have worked out an even simpler toy model which displays connections to the physics of a passive scalar advected by a random velocity field, being at the same time more tractable both analytically and numerically. In [8] the intermittent properties of a shell model for a passive scalar advected by a delta-correlated random velocity field have been investigated. The main results presented in [8] were that (i) the second-order structure function has no anomalous scaling, i.e., $\zeta_2 = 2 - \xi$; (ii) all structure functions of order larger than two have anomalous corrections; and (iii) anomalous behavior tends to vanish when approaching the laminar regime, $\xi = 2$. For all three points the model agrees with the Kraichnan model.

The importance of UV and IR boundaries for the anomalous scaling was left unanswered in [8]. In particular, numerical simulations were unable to distinguish among contribution coming from the inertial null space and possible singular behavior introduced by the boundary conditions.

In this Letter we show how to compute exactly and nonperturbatively the inertial scaling behavior of the fourth-order structure functions. The main result is that the scaling properties are completely dominated by the null space of the inertial linear operator and strongly universal. The signature of UV and IR cutoffs is due to the presence of subdominant terms which weakly perturb the pure-scaling behavior of the inertial operator. Anomalous scaling is calculated for any $\xi > 0$. We also present some results which support the strong singular nature of the limit $\xi \to 0$, In this limit, due to the nonlocal nature of interactions, it is not possible to neglect UV effects if the diffusive scale, k_d, is taken fixed.

We first introduce a simplified version of the shell model discussed in Ref. [8]. The model is defined in terms of a shell discretization of the Fourier space in a set of wave numbers defined on a geometric progression $k_n = \lambda^n$, with $\lambda > 1$. Passive increments at scale $r_n = k_n^{-1}$ are described by a complex variable $\theta_n(t)$. The time evolution is obtained according to the following criteria: (i) The linear term is a purely diffusive term given by $-\kappa k_n^2 \theta_n$, (ii) the advection term is a combination of the form $k_n \theta_{n'} u_{n''}$, (iii) interacting shells are restricted to nearest neighbors of n, and (iv) in the absence of forcing and damping the model preserves the volume in the phase space and the passive energy $E = \sum_n |\theta_n|^2$. Properties (i), (ii), and (iv) are valid also for the original equation of a passive scalar advected by a Navier-Stokes velocity field in the Fourier space, while property (iii) is an assumption of locality of interactions among modes. This assumption is rather well founded as long as $0 \ll \xi \ll 2$. The model is defined by the following equations ($m = 1, 2, \ldots$)

$$\left[\frac{d}{dt} + \kappa k_m^2\right]\theta_m(t) = i[c_m \theta_{m+1}^*(t) u_m^*(t) + b_m \theta_{m-1}^*(t) u_{m-1}^*(t)] + \delta_{1m} f(t), \quad (2)$$

where the star denotes complex conjugation and $b_m = -k_m$, $c_m = k_{m+1}$ for imposing energy conservation in the zero diffusivity limit. Boundary conditions are defined as $u_0 = \theta_0 = 0$. The forcing term is Gaussian and delta correlated: $\langle f(t) f(t') \rangle = F_1 \delta(t - t')$ acts only on the first shell. In numerical implementations, the model is truncated to a finite number of shells N with the additional boundary conditions $\theta_{N+1} = 0$.

Following Kraichnan [6] we assumed that the velocity variables $u_m(t)$ and the forcing term $f(t)$ are independent complex Gaussian and white in time, with scaling law: $\langle u_m(t) u_n^*(t') \rangle = \delta(t - t') \delta_{nm} d_m$, $d_m = k_m^{-\xi}$. Because of the delta correlation in time, we can close the equations of motion for all structure functions. Numerical simulations show that the model has the same qualitative intermittency of the model studied in [8].

In this Letter we concentrate on the nonperturbative analytic calculation of the fourth-order structure function $P_{mm} = \langle (\theta_m \theta_m^*)^2 \rangle \propto k_m^{-\zeta_4}$ (the lowest order with nontrivial anomalous scaling). The closed equation satisfied by $P_{mq} = \langle (\theta_m \theta_m^*) (\theta_q \theta_q^*) \rangle$ is

$$\begin{aligned}\dot{P}_{mq} = {} & (\delta_{1,m} E_m + \delta_{1,q} E_q) F_1 - \kappa (k_m^2 + k_q^2) P_{mq} \\ & + [-P_{mq} c_m^2 d_m ((1 + \delta_{q,m+1}) + \lambda^{\xi-2}(1 + \delta_{q,m-1})) + P_{m+1,q} c_m^2 d_m (1 + \delta_{q,m}) \\ & + P_{m-1,q} b_m^2 d_{m-1} (1 + \delta_{q,m}) + (q \leftrightarrow m)], \end{aligned} \quad (3)$$

where $E_n = \langle \theta_n \theta_n^* \rangle$. We can symbolically represent Eq. (3) as

$$\dot{P}_{mq} = \mathcal{I}_{mq,lp} P_{lp} + \kappa \mathcal{D}_{mq,lp} P_{lp} + \mathcal{F}_{mq}, \quad (4)$$

where $\mathcal{I}$ and $\mathcal{D}$ are the inertial and the diffusive fourth-order tensor and $\mathcal{F}$ is the forcing term.

Our main result is derived by using the following ansatz: the *symmetric* matrix P_{mq}, which fully determines the scaling properties for any fourth-order quantity in the model, can be described as

$$P_{n,n+l} = C_l P_{n,n} \ (l \geq 0), \qquad P_{n,n-l} = D_l P_{n,n} \ (l \geq 0). \quad (5)$$

The independency of C_l and D_l from n is equivalent to demand absence of strong boundary effects, i.e., the matrix is formally infinite dimensional. Clearly this must be verified *a posteriori* showing that the solution we are going to present is UV and IR stable.

Using (5) we obtain

$$\frac{C_{l+1}}{C_l C_1} = \frac{D_{l+1}}{D_l D_1},$$

which is equivalent to write $P_{n+l,n+l} = k_l^{-\zeta_4} P_{n,n}$, where $C_l / D_l = k_l^{-\zeta_4}$ and $\zeta_4 = 2(2 - \xi) - \rho_4$. As usual we indicate by ρ_4 the anomalous correction to the scaling exponent.

Let us notice that (5) does not force the solution to have global scaling invariance: only the diagonal part is requested to have pure scaling.

Let us proceed by analyzing (3) restricted to the inertial operator and for the diagonal ($m = q$) and subdiagonal terms ($q = m - 1$):

$$\dot{P}_{m,m} = 2 P_{m,m} c_m^2 d_m [-1 - x + 2(C_1 + D_1 x)], \quad (6)$$

$$\dot{P}_{m,m-1} = 2 P_{m,m-1} c_m^2 d_m \times \left(-1 - 4x - x^2 + \frac{x}{D_1} + \frac{x + C_2 + \frac{x^2 C_2}{R}}{C_1}\right), \quad (7)$$

where we have posed $x = \lambda^{\xi-2}$ and $R = C_1 / D_1$. By plugging the scaling (5) in (7) one obtains two equations in three unknowns which can be taken to be C_1 and the ratios C_2 / C_1 and R. Numerical investigation suggests the following "scaling ansatz":

$$P_{n,n+l} = C_l P_{n,n}, \quad \text{with } C_l = C_1 k_{l-1}^{\xi-2} \tag{8}$$

and

$$P_{n,n-l} = D_l P_{n,n}, \quad \text{with } D_l = D_1 k_{l-1}^{-(\xi-2)-\rho_4}, \tag{9}$$

where anomalous correction is felt only in the IR part of the matrix. By plugging this scaling in (7) we end up with two equations in two unknowns and we can calculate ρ_4. Let us anticipate that this (wrong) assumption gives results in very good agreement with the numerical simulations, indicating that the true solution is not very far from having pure scaling behavior. In order to solve the full problem, without imposing any "pure scaling" behavior, we analyze the other entry of the matrix $P_{n,q}$ with $q \neq n$ and $q \neq n - 1$. Let us put $\gamma_l = D_{l+1}/D_l$, $\delta_l = C_{l+1}/C_l$.

It is then possible to show that for $l > 1$ by plugging the scaling (5) in the inertial part of (3) and studying the equation for $\dot{P}_{n,n\pm l}$ we obtain two recursion equations:

$$-(1 + x)(1 + x^l) + \gamma_l(R + x^{l+1}) + \frac{1}{\gamma_{l-1}}\left(x^l + \frac{x}{R}\right) = 0, \tag{10}$$

$$(1 + x)(1 + x^l) - \lambda^{\zeta_4}\delta_l(R + x^{l+1}) - \frac{\lambda^{-\zeta_4}}{\delta_{l-1}}\left(x^l + \frac{x}{R}\right) = 0. \tag{11}$$

These two relations can indeed be seen as two maps connecting successive values of γ_l and δ_l, respectively. By iterating forward (backward) the map (10) we move from the diagonal (IR boundary) to the IR boundary (diagonal) along a row of the matrix $P_{l,n}$. By iterating forward (backward) the map (11) we move from the diagonal (UV boundary) to the UV boundary (diagonal) along a row of the matrix $P_{l,n}$.

Let us first note that the two maps are not independent; i.e., they satisfy our scaling ansatz $\delta_l = R\gamma_l$ and therefore we are going to consider only one of the two in what follows. In order to test stability under weak perturbation of boundary conditions in the map (10) we are interested in the behavior by backward iterations, i.e., iterating from $l = \infty$ to $l = 0$. In the limit $l \to \infty$ and $\xi \neq 2$ the map (10) has only two fixed points corresponding to $\gamma_1^* = x/R$ and $\gamma_2^* = 1/R$. It turns out that γ_1^* is stable for back iterations, i.e., iterating from the IR boundary ($l \gg 1$) to the diagonal ($l = 0$). The global solution can now be obtained by a self-consistent method. First, let us take as initial value for R the value that one would have guessed from imposing pure scaling as discussed previously, then we can iterate (10) from the boundary toward the diagonal and find the value for C_2/C_1. This value can be used to close (7) exactly. Next, with the improved value for R, one can restart the full procedure getting a new improved value of R and so on up to the moment when the new value of R reaches its fixed point.

In Fig. 1 we show the computation of ρ_4 obtained by numerical integration of Eq. (2) as a function of ξ. In the same figure we plot the ρ_4 as a function of ξ obtained by the analytical solution previously discussed. As one can see, the agreement is perfect. Let us notice that it is impossible to go by numerical simulations to values of ξ very near zero because of strong diffusive effects which completely destroy scaling behavior.

Let us notice that it is the strong stability under UV and IR perturbations that allows us to iterate consistently the procedure. We have therefore proved that anomalous scaling comes only from the inertial operator and that it shows a very strong degree of universality as a function of the forcing and dissipative mechanisms, at least as far as the situation with $\xi = \text{const} > 0$ and molecular diffusivity $\kappa \to 0$ is considered.

Some new and interesting phenomena happen when we are in the other possible asymptotic limit ($\xi \to 0$) and fixed molecular diffusivity. A simple dimensional argument tells us that the following relation holds:

$$\kappa = \text{const} \times k_d^{-\xi}, \tag{12}$$

where with k_d we mean a preassigned diffusive scale such that all the inertial dynamics is at $k \ll k_d$. One can show that in such a situation the diffusive operator $\mathcal{D}$ in (4) gives a contribution of the form

$$\dot{P}_{n,n} = -\text{const} \times (k_n/k_d)^{\xi} k_n^{2-\xi} P_{n,n}. \tag{13}$$

Therefore, the diffusive perturbation is absolutely negligible in the case when $\xi \neq 0$ is fixed and $k_d \to \infty$ while it becomes a singular perturbation when k_d is kept fixed and $\xi \to 0$. A detailed analysis of relations (7) shows that in the singular limit $\xi \to 0$ there appear an infinitesimal interval of values of $\xi \sim 0$, where $\rho_4 \to 0$; i.e., the anomalous correction tends to vanish with a particular shape which depends on the constants appearing in relation (12).

As for the limit $\xi \to 2$, where nonlocal effects should also be expected, we interpret the realistic behavior of

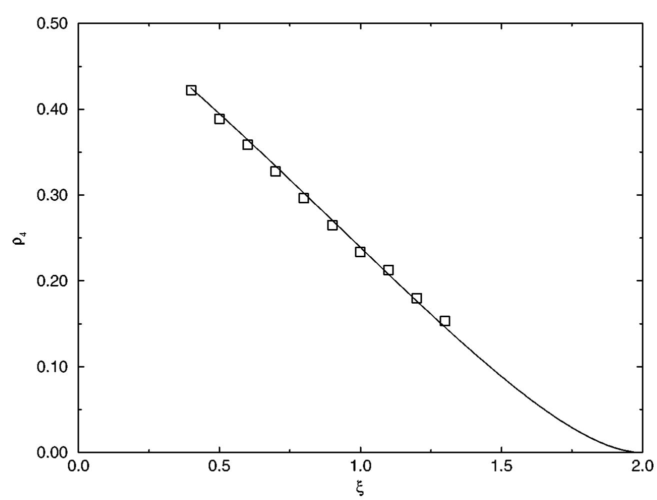

FIG. 1. Analytical ansatz (continuous line) and numerical results (squares) for ρ_4 are plotted for various values of ξ.

our model in terms of the strong constraints imposed by conservation of energy. As in the Kraichnan model, the energy contained in each shell shows a regular scaling: $E_n \sim k_n^{\xi-2}$. Therefore for $\xi \to 2$ the inertial scales are forced to be at equipartition, i.e., the most singular status allowed by the statistics. Hence, equipartition for all other structure functions follows.

Let us remark that the perfect agreement of our inertial null-space solution with the numerical simulation performed with finite diffusivity and in the presence of forcing is the clear demonstration that the scaling behavior is completely dominated by the inertial operator. For any finite system, IR and UV effects weakly perturb the pure scaling solution. Our result shows that the inertial operator is perfectly suitable for picking all anomalous aspects but in the case where strong nonlocal interactions (dynamically produced) completely destroy inertial properties introducing diffusive effects at all scales ($\xi \to 0$ and k_d fixed).

Discussions with B. Dubrulle, G. Falkovich, U. Frisch, I. Procaccia, and M. Vergassola are kindly acknowledged.

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