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Self-scaling properties of velocity circulation in shear flows

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We investigate the scaling properties of the velocity circulation of a turbulent shear flow. We evaluate, using extended self-similarity, the circulation scaling exponents both at maximum and minimum shear regions. We show that the anomalous component of the velocity circulation and the anomalous component of the velocity structure functions are equal. [S1063-651X(97)11202-8]

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Fluctuations of the energy dissipation and intermittency of the velocity-field inertial-range statistics are two of the most important features of fully developed turbulent flows. A quantitative measure of intermittency is usually given by the set of scaling exponents ζ_n of the n-order structure functions, namely,

$$F_n(r) \equiv \langle |\delta_r v|^n \rangle \sim r^{\zeta_n}. \tag{1}$$

According to the original Kolmogorov theory (K41) [1] $\zeta_n = n/3$, while deviations from this law are due to intermittency corrections.

It has been argued that vortex filaments are the basic geometrical objects for describing possible dominant and subdominant contributions to the K41 power laws. In this framework, multifractal deviations to K41 have also been phenomenologically explained in terms of scaling properties of vortex filaments [2].

In this paper we investigate a possible bridge between the velocity-differences intermittency, measured by the scaling exponents of structure functions, and the scaling properties of velocity circulations around a contour C, namely:

$$\Gamma(C) \equiv \oint_C \vec{v} \cdot d\vec{l} = \int_\Sigma \vec{\omega} \cdot d\vec{\sigma}, \tag{2}$$

where $\vec{\omega}$ is the vorticity field and Σ is any surface, lying on the contour C. It has been emphasized that circulation is the ideal observable, able to highlight both velocity and vorticity scaling properties, and eventually linking the two statistics [3]. According to dimensional arguments, the most natural ansatz [4], is that circulation structure functions, $G_n(r)$, scale as

$$G_n(r) = \langle |\Gamma(r)|^n \rangle \sim F_n(r) r^n, \tag{3}$$

where $\Gamma(r)$ means the circulation evaluated around a contour of radius r. We investigate the validity of Eq. (3) which has been recently observed not to be satisfied [3,4].

It is of primary interest to determine whether quantities with the same physical dimension, but different tensorial structure, have the same scaling properties. It is natural to argue, for example, that in cases with strong anisotropic effects, observables with different rotational properties would have different scaling exponents [5].

In this paper we examine circulation scaling properties by using numerical data from a (3D) shear flow simulation [6]. The presence of a shear in the flow allows us to also address questions concerning the not universal character of scaling laws for anisotropic turbulence. We show that velocity difference and circulation structure functions scaling exponents have the same anomalous contribution, even if Eq. (3) is not valid.

First we briefly summarize some details of our simulation, and we present our data analysis. In order to measure the scaling properties of the circulation structure functions we shall use extended self-similarity (ESS) as recently proposed [7–9].

Our data set comes from a simulation [6] of a 3D turbulent shear flow, in a volume of $V = 160^3$ (with our choice of parameters, one lattice spacing is about one Kolmogorov scale η_k and $R_\lambda \sim 40$). The flow is forced such that the unstable static solution of the N-S equations is

$$U_x \sim \sin(k_z z) \quad U_y = 0 \quad U_z = 0, \tag{4}$$

with $k_z = 8\pi/L$ being the wave vector corresponding to the integral scales. In this way the shear has a spatial dependence $S(z) \sim \cos(k_z z)$.

Some analysis of velocity statistics for the same data set have already been published [10–12]. It has been shown that the scaling exponents of the velocity structure function are strongly dependent on the presence of shear. We have evaluated $\Gamma(r)$ according to definition (2) for all squared contours with a fixed area $A = r^2$, with r extending from the dissipative range to the integral scales, at two different z levels corresponding to a maximum and a minimum shear level, respectively. As recently pointed out in [4], we find that the probability distribution function of $\Gamma(r)$ depends only on the area A enclosed by the contour C, independently of the shape of the contour itself.

The scaling exponents of $G_n(r)$ are defined, in the inertial range, as

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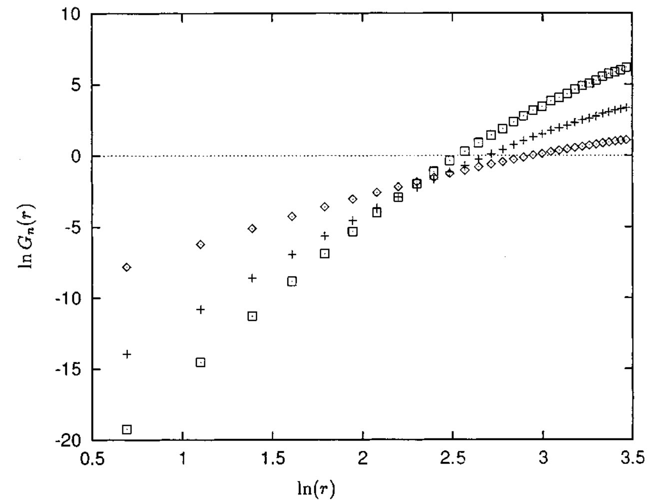

FIG. 1. Log-log plots of $G_n(r)$ vs r, for $n=2$ (diamonds), $n=4$ (crosses), $n=6$ (squares).

$$G_n(r) \sim r^{\chi_n}. \tag{5}$$

Let us first mention that, similarly to what happens for the structure functions, $F_n(r)$, due to the moderate Reynolds number of our simulation we are unable to detect a scaling law of $G_n(r)$ with respect to the scale r [see Fig. 1, where the log-log plots of $G_n(r)$ versus r, for $n=2,4,6$ are shown]. It is therefore useful to use ESS in order to improve the quality and the extension of the scaling regime. In the following we introduce the exponents γ_n defined as

$$G_n(r) \sim [G_3(r)]^{\gamma_n}. \tag{6}$$

In Fig. 2(a) we plot G_5 versus G_3 in the minimum shear zone. As one can see, a good scaling range is detected. The best fit done in the inertial range has a slope $\gamma_5=1.60$, while the best fit done in the dissipative range has the slope $\gamma_5=5/3$, as expected in the laminar zone from standard dimensional analysis.

Similar results have been obtained also for other structure functions. The corresponding scaling exponents are shown in Table I. These values are different from the dimensional prediction $\gamma_n=n/3$, giving the first positive evidence for anomalous scaling of $G_n(r)$. In Fig. 2(b) we plot G_5 versus G_3 at the maximum shear. At variance with the analogous analysis performed on the velocity structure functions (see [6] for a detailed discussion), the circulation exhibits a wide scaling region, allowing us to give an estimate for the scaling exponent. The γ_n for the maximum shear case are also reported in Table I. Comparing Figs. 2(a) and 2(b), we can see that whereas at the minimum shear level the scaling region begins at few Kolmogorov scales, namely, $5\eta_k$, the scaling region at the maximum shear level is smaller, beginning at about $9\eta_k$ (see [6] for a discussion on this point).

Using Eq. (3), in the inertial range we obtain

$$\chi_n=\zeta_n+n. \tag{7}$$

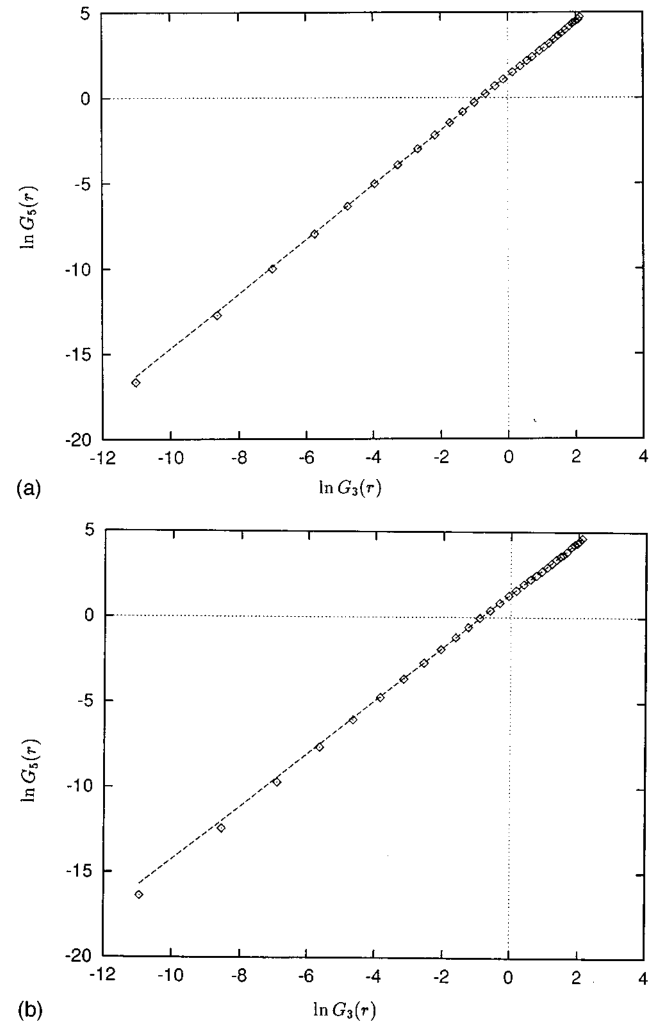

FIG. 2. (a) G_5 vs G_3, in the minimum shear zone. (b) G_5 vs G_3, in the maximum shear zone. The straight lines correspond to the best fit done in the inertial range, the slopes are $\gamma_5=1.60$ in (a) and $\gamma_5=1.56$ in (b).

In Ref. [4] there has been a first attempt to understand whether Eq. (7) is verified or not. Here we show that there is a relationship between ζ_n and χ_n, although this relation is not given by Eq. (7).

We cannot directly check Eq. (7) because neither $G_n(r)$ nor $F_n(r)$ show a clear scaling range with respect to r. However, according to Eq. (3), the quantity $H_n \equiv G_n/F_n$ must show a dimensional scaling, namely,

$$H_n(r) \sim H_m(r)^{n/m} \tag{8}$$

Let us remark that Eq. (8) is a condition weaker than Eq. (3) or Eq. (7), namely, Eq. (8) is satisfied for any functional relation of the form

TABLE I. Scaling exponents γ_n: at the minimum shear (first line), and at the maximum shear (second line). The error in the estimates of the exponents is 2%.

	γ_1	γ_2	γ_4	γ_5	γ_6
min sh	0.35	0.68	1.30	1.60	1.89
max sh	0.36	0.69	1.29	1.56	1.81

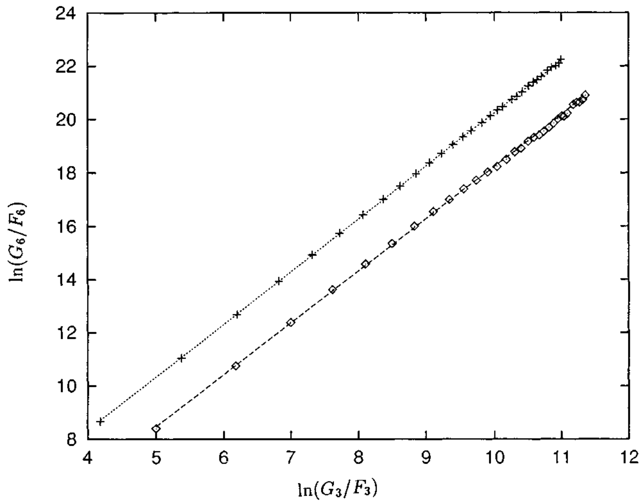

FIG. 3. Log-log plot of relation (8) for $n=6$, $m=3$ at the maximum (crosses) and minimum shear (diamonds). Data at the maximum shear have been shifted along the y axis of one unity. The straight lines correspond to the best fit over all scales, the slopes are $d(6,3)=1.97$ for the maximum shear and $d(6,3)=1.98$ for the minimum shear.

$$H_n(r)=\mathcal{F}\left(\frac{r}{\eta_k}\right)^n. \tag{9}$$

In Fig. 3, we plot H_6 versus H_3 for the minimum and maximum shear. As one can see, there is a wide scaling region extending from the smallest to the integral scale of motion. The corresponding scaling exponents have been found to satisfy the simple dimensional scaling (8) [$d(6,3)=1.98$ for the minimum shear and $d(6,3)=1.97$ for the maximum shear]. We have found that Eq. (8) is satisfied (within 2%) for all $n,m\in[1,6]$. This is our main result.

According to Eq. (9) and to the results so far obtained, one may argue that in the inertial range the function $\mathcal{F}(v/\eta_{\parallel})$ behaves as r^{α_s}. It follows that in the inertial range, Eq. (9) becomes

$$H_n(r)\sim r^{\alpha(n)}, \tag{10}$$

with $\alpha(n)\equiv\alpha_s n$. Surprisingly enough, scaling (10) holds rather clearly in our simulation.

In Fig. 4(a) we show, in a log-log plot, the ratio $H_n(r)$ versus r for $n=5$. We can easily recognize two scaling regions: the first one is in the dissipative region, where the best fit has a slope close to 5, the second one is in the inertial range, with a slope 3.20. In Fig. 4(b) we plot the exponents $\alpha(n)$ vs n, for the minimum and maximum shear. In both cases $\alpha(n)$ falls on a straight line $\alpha(n)=\alpha_s n$, with α_s depending on the shear. The best fits in the figure correspond to $\alpha_s=0.68$ for the minimum shear and $\alpha_s=0.54$ for the maximum shear. We have no clear explanation for the dependency of α_s on the shear strength.

In this paper we have mainly investigated the self-scaling properties of the velocity circulation. Indeed at the maximum shear, the velocity-field structure functions has a poor scaling behavior [6], whereas the scaling of $G_n(r)$ vs G_3 is much clearer. Within numerical error we do not see any strong differences between scaling properties of circulation structure functions at minimum and maximum shear.

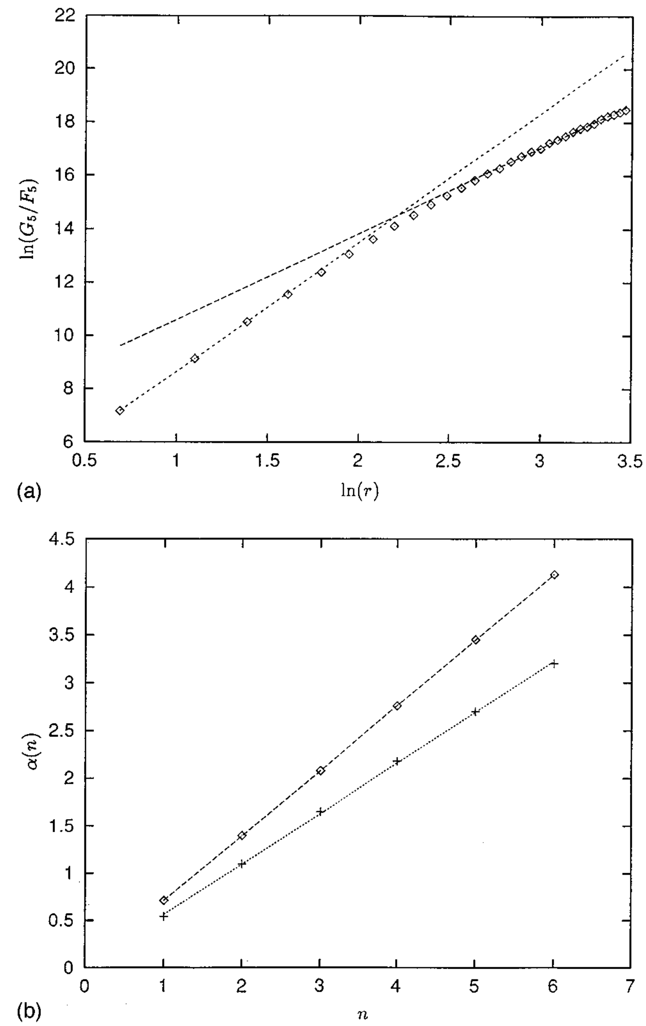

FIG. 4. (a) Plot of $\ln[G_5(r)/F_5(r)]$ vs $\ln r$. The two straight lines correspond to the best fit in the dissipative range and to the best fit in the inertial range. (b) Plot of $\alpha(p)$ vs p for the maximum (squares) and minimum (circles) shear. The best fits (dashed lines) have the following slopes: $\alpha_s=0.68$ for the minimum shear and $\alpha_s=0.54$ for the maximum shear.

From Fig. 4(b) we conclude that the anomalous scaling of velocity circulation is equal to the anomalous scaling of velocity structure functions, in the sense that

$$\chi_n-\zeta_n=\alpha_s n. \tag{11}$$

Figure 4(b) shows that α_s is a nonuniversal quantity, its value may depend on geometrical constraints. One may argue, that such dependency is due to the stretching and folding of vorticity structures induced by the shear. Nevertheless, because the nonlinear dependency of χ_n from n is always the same, one may argue that the analysis of intermittency, in terms of the scaling exponents of the velocity circulation, does not need different physical interpretations with respect to those already proposed for the velocity structure function.

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