Multiscale Velocity Correlations in Turbulence

R. Benzi,[1] L. Biferale,[2,4] and F. Toschi[3,4]
[1]*AIPA, Via Po 14, 00100 Roma, Italy*
[2]*Dipartimento di Fisica, Università di Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Roma, Italy*
[3]*Dipartimento di Fisica, Università di Pisa, Piazza Torricelli 2, I-56126 Pisa, Italy*
[4]*INFM-Unitá di Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Roma, Italy*
(Received 3 December 1997)

Multiscale correlation functions in high Reynolds number experimental turbulence and synthetic signals are investigated. Fusion rule predictions as they arise from multiplicative, almost uncorrelated, random processes for the energy cascade are tested. Leading and subleading contributions, in both the inertial and viscous ranges, are well captured by assuming a simple multiplicative random process for the energy transfer mechanisms. [S0031-9007(98)05773-1]

PACS numbers: 47.27.Eq

In stationary turbulent flows, a net flux of energy establishes in the inertial range, i.e., from forced scales, L, down to the dissipative scale r_d. Energy is transferred through a statistically scaling-invariant process, which is characterized by a strongly non-Gaussian (intermittent) activity. Understanding the statistical properties of intermittency is one of the most challenging open problems in three-dimensional fully developed turbulence.

Intermittency in the inertial range is usually analyzed by means of the statistical properties of velocity differences, $\delta_r u(x) = u(x) - u(x + r)$. In particular, in the last twenty years [1], overwhelming experimental and theoretical works focused on structure functions: $S_p(r) = \langle[\delta_r u(x)]^p\rangle$. A wide agreement exists on the fact that structure functions show a scaling behavior in the limit of very high Reynolds numbers, i.e., in the presence of a large separation between integral and dissipative scales, $L/r_d \to \infty$:

$$S_p(r) \sim \left(\frac{r}{L}\right)^{\zeta(p)}. \tag{1}$$

The velocity fluctuations are anomalous in the sense that $\zeta(p)$ exponents do not follow the celebrated dimensional prediction made by Kolmogorov, $\zeta(p) = p/3$. In fact, $\zeta(p)$ are observed to be a nonlinear function of p, which is the most important signature of the intermittent transfer of fluctuations from large to small scales.

In order to better characterize the transfer mechanism, it is natural to look also at correlations among velocity fluctuations at different scales. Multiscale correlation functions should play in turbulence the same role played by correlation functions in critical statistical phenomena.

Recently, some theoretical work [2,3] and an exploratory experimental investigation [4] have been devoted to the behavior of multiscale velocity correlations:

$$\begin{aligned} F_{p,q}(r,R) &\equiv \langle[u(x + r) - u(x)]^p[u(x + R) - u(x)]^q\rangle \\ &\equiv \langle[\delta_r u(x)]^p[\delta_R u(x)]^q\rangle \end{aligned} \tag{2}$$

with $r_d < r < R < L$. When the smallest among the two scales r goes beyond the dissipative scales, r_d, new properties of the correlation functions (2) may arise due to the nontrivial physics of the dissipative cutoff. From now on, we will mostly concentrate on correlation functions with both r and R in the inertial range. Moreover, in order to simplify our discussion, we will confine our analysis for the case of longitudinal velocity differences.

Stochastic cascade processes are simple and well known useful tools to describe the leading phenomenology of the intermittent energy transfer in the inertial range. Both anomalous scaling exponents and viscous effects [1,5] can be reproduced by choosing a suitable random process for the multiplier, $W(r,R)$, which connects velocity fluctuations at two different scales, $R > r$.

The main finding of this Letter is that experimental multiscale correlations (2) are in *quantitative* agreement, for any separation of scale r/R, with the prediction one obtains by using a pure uncorrelated multiplicative process for the energy cascade.

The main idea turns around the hypothesis that small scale statistics is fully determined by a cascade process conditioned to some large scale configuration:

$$\delta_r u(x) = W(r,R)\delta_R u(x), \tag{3}$$

where, requiring homogeneity along the cascade process, the random function W should depend only on the ratio r/R. Structure functions are then described in terms of the W process: $S_p(r) = C_p\langle[W(r/L)]^p\rangle$, where $C_p = \langle[\delta_L u(x)]^p\rangle$ if the stochastic multiplier may be considered almost uncorrelated with the large-scale velocity field. Pure power laws arise in the high Reynolds regime: in this limit we must have $\langle[W(\frac{r}{R})]^p\rangle \sim (\frac{r}{R})^{\zeta(p)}$. In the same framework, it is straightforward to give the leading prediction for the multiscale correlation functions (2):

$$F_{p,q}(r,R) \sim \left\langle\left[W\left(\frac{r}{R}\right)\right]^p\left[W\left(\frac{R}{L}\right)\right]^{p+q}\right\rangle, \tag{4}$$

 0031-9007/98/80(15)/3244(4)$15.00

which becomes in the hypothesis of negligible correlations amongmultipliers:

$$F_{p,q}(r,R) = C_{p,q}\left\langle\left[W\left(\frac{r}{R}\right)\right]^p\right\rangle\left\langle\left[W\left(\frac{R}{L}\right)\right]^{p+q}\right\rangle \sim \frac{S_p(r)}{S_p(R)}\,S_{p+q}(R)\,. \qquad (5)$$

This expression was for the first time proposed in [6,7] and later extensively studied in [2]. In the latter paper the expression (5) was considered to rigorously express the leading behavior of (2) when $r/R \to 0$ as long as some weak hypothesis of scaling invariance and of universality of scaling exponents in Navier-Stokes equations hold. Let us notice that, besides any rigorous claim, expression (5) is also the zeroth order prediction starting from any multiplicative uncorrelated random cascade satisfying $\langle[W(\frac{r}{R})]^p\rangle \equiv S_p(r)/S_p(R)$.

In this Letter we address three main issues: (i) whether the prediction (5) gives the correct leading behavior in the limit of large separation of scales $r/R \sim 0$, (ii) if this is the case, what one can say about subleading behavior for separation $r/R \sim O(1)$, and (iii) what happens to those observable for which the "multiplicative prediction" (5) is incorrect because of symmetry reasons. Indeed, let us notice that for a correlation like

$$F_{1,q}(r,R) = \langle(\delta_r u)\,(\delta_R u)^q\rangle\,, \qquad (6)$$

the multiplicative prediction gives

$$F_{1,q}(r,R) = \frac{S_1(r)}{S_1(R)}\,S_{1+q}(R)\,.$$

Such a prediction is wrong because, if homogeneity can be assumed, $S_1(r) = 0$ for all scales r. In this case prediction (5) does not represent the leading contribution.

In this Letter we propose a systematic investigation of (2) in high Reynolds number experiments and synthetic signals. The main purpose consists in probing whether multiscale correlation functions may show new dynamical properties (if any) which are not taken into account by the standard simple multiplicative models for the energy transfer.

Experimental data have been obtained in a wind tunnel (Modane) with $\mathrm{Re}_\lambda = 2000$. The integral scale is $L \sim 20$ m and the dissipative scale is $r_d = 0.31$ mm. Synthetic signals are built in terms of wavelet decomposition with coefficients defined by a pure uncorrelated random multiplicative process [8]. Such a signal should therefore show the strong fusion rules prediction (5) and it will turn out to be a useful tool for testing how many deviations from (5), observed in experiments or numerical simulations, are due to important dynamical effects or only to unavoidable geometrical corrections.

First of all, let us notice that for any one-dimensional string of numbers (such as the typical outcome of laboratory experiments in turbulence) the multiscale correlations (2) feel strong geometrical constraints. In particular we may always write down "Ward identities" (WI):

$$S_p(R-r) \equiv \langle\{[u(x+R) - u(x)] - [u(x+r) - u(x)]\}^p\rangle \qquad (7)$$

$$= \sum_{k=0,p} b(k,p)\,(-)^k F_{k,p-k}(r,R)\,, \qquad (8)$$

where $b(k,p) = p!/[k!(p-k)!]$.

For example, for $p = 2$ we have

$$2F_{1,1}(r,R) \equiv S_2(r) + S_2(R) - S_2(R-r) \sim \left[\left(\frac{r}{R}\right)^{\zeta(2)} + O\left(\frac{r}{R}\right)\right]S_2(R)\,, \qquad (9)$$

where the latter expression has been obtained by expanding $S_2(R-r)$ in the limit $r/R \to 0$. For $p = 3$ we have

$$S_3(R-r) = S_3(R) - S_3(r) + 3F_{2,1}(r,R) - 3F_{1,2}(r,R)\,.$$

The Ward identities will turn out to be useful for understanding subleading predictions to the multiplicative cascade process. One may argue that in a geometrical setup different from the one specified in (2) the same kind of constraint will appear with eventually different weights among different terms.

The main result presented in this Letter is that all multiscale correlation functions are well reproduced in their leading term, $\frac{r}{R} \to 0$, by a simple uncorrelated random cascade (5) and that their subleading contribution, $\frac{r}{R} \sim O(1)$, is fully captured by the geometrical constraint previously discussed, namely, the Ward identities.

The recipe for calculating multiscale correlations will be the following: first, apply the multiplicative guess for the leading contribution and look for geometrical constraints in order to find out subleading terms. Second, in all cases where the leading multiplicative contribution vanishes because of underlying symmetries, look directly for the geometrical constraints and find out what is the leading contribution applying the multiplicative random approximation to all, nonvanishing, terms in the WI.

Let us check the strong fusion rules prediction (5) for moments with $p > 1, q > 1$. In Fig. 1 we have checked the large scale dependency by plotting $F_{p,q}(r,R)/S_{p+q}(R)S_p(R)$ as a function of R at fixed r, for different values of p, q.

The expression (5) predicts the existence of a plateau (independent of R) at all scales R where the leading multiplicative description is correct.

From Fig. 2 one can see that, in the limit of large separation $R \to L$ at fixed r, $F_{p,q}(r,R)/S_{p+q}(R)S_p(R)$ shows a tendency toward a plateau. On the other hand, there are clear deviations for $r/R \sim O(1)$. Such deviations show a very slow decay as a function of the scale separation.

In order to understand the physical meaning of the observed deviations to the fusion rules (5), we compare, in Fig. 1, the experimental data against the equivalent

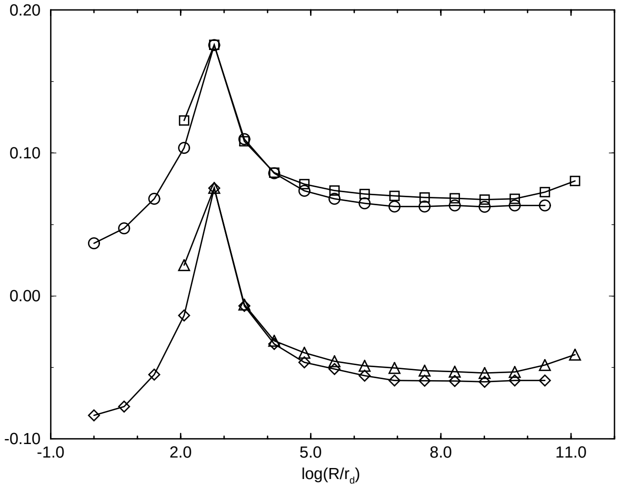

FIG. 1. Experimental and numerical $F_{p,q}(r,R)/S_{p+q}(R)S_p(r)$ at fixed r and changing the large scale R. Circles correspond to $p = 2, q = 2$ and diamonds to $p = 4, q = 2$ for the experimental data. Squares correspond to $p = 2, q = 2$ and triangles to $p = 4, q = 2$ for the synthetic signal. Small scale r is fixed to $r = 16$ in units of the Kolmogorov scale. The data for $p = 4, q = 2$ have been shifted along the vertical axis for the sake of presentation.

quantities measured by using the synthetic signal. We notice an almost perfect superposition of the two data sets, indicating that the deviations observed in real data can hardly be considered a "dynamical effect."

Using the WI plus our multiplicative recipe for $p = 4$ we quickly read that the leading contribution to $F_{2,2}$ is $O(r^{\zeta(2)})O(R^{\zeta(4)-\zeta(2)})$, while subleading terms scale as $O(r^{\zeta(4)})$, and as $O(r^{\zeta(3)})O(R^{\zeta(4)-\zeta(3)})$.

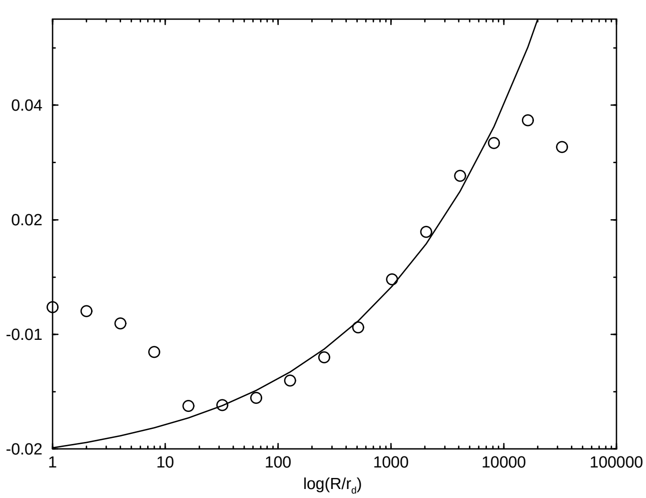

FIG. 2. Experimental $F_{1,2}(r,R)$ at fixed $r = 16r_d$ and at varying R. The integral scale $L \sim 1 \times 10^4 r_d$. Let us remark that the observed change of sign in the correlation implies the presence of at least two power laws. The continuous line is the fit in the region $r < R < L$ obtained by using only the first two terms in (10).

This superposition of power laws is responsible for the slowly decaying correlations in Fig. 1. The result so far obtained, i.e., that both the experimental data and the synthetic signal show the same quantitative behavior, is a strong indication that multiscale correlation functions, at least for $p > 1, q > 1$, are in good agreement with the random multiplicative model for the energy transfer.

For multiscale correlations where the direct application of the random-cascade prediction is useless, like $F_{1,q}(r,R)$, we use the WI plus the multiplicative prediction applied to all terms, except the $F_{1,q}$. One obtains the expansion

$$F_{1,q}(r,R) \sim \left[O\left(\frac{r}{R}\right)^{\zeta(2)} + O\left(\frac{r}{R}\right)^{\zeta(3)} + O\left(\frac{r}{R}\right)^{\zeta(4)} + \cdots + O\left(\frac{r}{R}\right)^{\zeta(q+1)}\right] S_{q+1}(R)\,, \qquad (10)$$

which coincides when $q = 1$ with the exact result (9) using $\zeta(3) = 1$.

In Fig. 2 we show the experimentally measured $F_{1,2}$ and the fit that we obtain by keeping only the first two terms of the expansion in (10). The fit has been performed by imposing the value for the scaling exponents $\zeta(2), \zeta(3)$ measured on the structure functions, i.e., only the coefficients in front of the power laws have been fitted. As one can notice, the fit works perfectly in the inertial range. Let us remark that the correlation changes sign in the middle of the inertial range, which is a clear indication that a single power-law fit (neglecting subleading terms) would completely miss the correct behavior.

Next we consider the WI for $p = 3$. Because of the fact that $S_3(r) \sim r$ in the inertial range, one can easily show that the WI forces $F_{12} \sim F_{21}$. Therefore we can safely state that also correlation functions of the form $F_{p,1}$ feel nontrivial dependency from the large scale R, at variance with the prediction given in [3] using isotropic arguments.

Let us summarize what the framework is that we have found up to now. Whenever the simple scaling ansatz based on the uncorrelated multiplicative process is not prevented by symmetry arguments, the multiscale correlations are in good asymptotic agreement with the fusion rules prediction even if strong corrections due to subleading terms are seen for small-scale separation $r/R \sim O(1)$. Subleading terms are strongly connected to the WI previously discussed, i.e., to geometrical constraints. In the other cases [i.e., $F_{1,q}(r,R)$] the geometry fully determines both leading and subleading scaling.

All these findings led us to the conclusion that multiscale correlations functions measured in turbulence are fully consistent with a multiplicative, almost uncorrelated, process.

Also the analysis of the energy dissipation statistics may show important correlations due to unavoidable geometrical overlaps between observable at different scales [9]. In [5] it has been discussed in detail whether the refined Kolmogorov hypothesis (RKH) for the energy dissipation is consistent with a random multiplicative process for the velocity increments. It has been shown that, at least on the synthetic signal, RKH is satisfied.

The strong and slowly decaying subleading corrections to the naive multiplicative fusion rule predictions are particularly annoying for any attempts to attack analytically the equation of motion for structure functions; in that case, multiscale correlations at almost coinciding scales are certainly the dominant contributions in the nonlinear part of the equations [3]. Indeed, as shown in an analytical calculation for a dynamical toy model of random passive-scalar advection [10], fusion rules are violated at small scale separation and the violations are relevant for correctly evaluating the exact behavior of structure functions at all scales.

When the smallest distance r is inside the viscous length, one can use the approach of multiplicative processes with multiscaling viscous cutoff [11]. Namely, for the correlation $D_{1,q}(R) = \langle (\partial_x u)^2 (\delta_R u)^q \rangle$ one obtains

$$D_{1,q}(R) \sim \left\langle (\delta_R u)^q \left(\frac{\delta u(r_d)}{r_d} \right)^2 \right\rangle, \qquad (11)$$

where r_d is the dissipative scale. In the multifractal interpretation we say $\delta_{r_d} u = (r_d/R)^h \delta_R u$ with probability $P_h(r_d, R) = (r_d/R)^{3-D(h)}$. Following [11] we have

$$\delta u(r_d) r_d \sim \left(\frac{r_d}{R} \right)^h \delta_R u r_d \sim \nu \,. \qquad (12)$$

Inserting the last expression in the definition of $D_{1,q}(R)$, we finally have

$$D_{1,q}(R) \sim \int d\mu(h)\, (\delta_R u)^{q+2} R^{-2} \times \left(\frac{\nu}{R\,\delta_R u} \right)^{\frac{2(h-1)+3-D(h)}{1+h}} \sim \frac{S_{q+3}(R)}{\nu R}\,, \qquad (13)$$

where we have used the fact that the multifractal process is such that $\nu \langle (\partial_x u)^2 \rangle \to O(1)$ in the limit $\nu \to 0$. Expression (13) coincides with the prediction given in [3]. The above computations are easily generalized for any $\langle (\partial_x u)^p (\delta_R u)^q \rangle$.

Finally, we remark that the standard multiplicative process may not be the end of the story, i.e., the dynamics may be more complex than summarized here. For example, one cannot exclude that also subleading (with respect to the multiplicative ansatz) dynamical processes are acting in the energy transfer from large to small scales. This dynamical correction must be either negligible with respect to the geometrical constraints or, at the best, of the same order.

A possible further investigation of such an issue would be to perform a wavelet analysis of experimental turbulent data. From this analysis one may hope to minimize geometrical constraints focusing only on the dynamical transfer properties.

Other possible candidates to investigate the above problem are shell models for turbulence, where geometrical constraints do not affect the energy cascade mechanism. Work in both directions is in progress.

We acknowledge useful discussions with A. L. Fairhall, V. L'vov, and I. Procaccia. M. Pasqui is kindly acknowledged for his help in the analysis of the synthetic signal. We are indebted to S. Ciliberto, R. Chavarria, and Y. Gagne for having allowed us access to the experimental data. L. B. and F. T. have been supported by INFM (PRA TURBO).

[1] U. Frisch, *Turbulence. The Legacy of A. N. Kolmogorov* (Cambridge University Press, Cambridge, England, 1995).

[2] V. S. L'vov and I. Procaccia, Phys. Rev. Lett. **76**, 2898 (1996).

[3] V. S. L'vov and I. Procaccia, Phys. Rev. E **54**, 6268 (1996).

[4] A. L. Fairhall, B. Druva, V. S. L'vov, I. Procaccia, and K. S. Sreenivasan (to be published).

[5] R. Benzi, L. Biferale, S. Ciliberto, M. V. Struglia, and R. Tripiccione, Physica (Amsterdam) **96D**, 162 (1996).

[6] G. Eyink, Phys. Lett. A **172**, 355 (1993).

[7] G. Eyink, Phys. Rev. E **48**, 1823 (1993).

[8] R. Benzi, L. Biferale, A. Crisanti, G. Paladin, M. Vergassola, and A. Vulpiani, Physica (Amsterdam) **65D**, 352 (1993).

[9] M. Nelkin and G. Stolovitsky, Phys. Rev. E **54**, 5100 (1996).

[10] R. Benzi, L. Biferale, and A. Wirth, Phys. Rev. Lett. **78**, 4926 (1997).

[11] U. Frisch and M. Vergassola, Europhys. Lett. **14**, 439 (1991).