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Mimicking a turbulent signal: Sequential multiaffine processes

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An efficient method for the construction of a multiaffine process, with prescribed scaling exponents, is presented. At variance with the previous proposals, this method is sequential and therefore it is the natural candidate in numerical computations involving synthetic turbulence. The application to the realization of a realistic turbulentlike signal is discussed in detail. The method represents a first step towards the realization of a realistic spatiotemporal turbulent field. [S1063-651X(98)50506-5]

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In recent years the relevance of multifractal measures and multiaffine processes in many fields (mainly fully developed turbulence) has been well understood [1–4]. In different contexts, for instance numerical simulations and comparison of theoretical models with experimental data, a rather natural problem is the construction of artificial signals mimicking real phenomena (e.g., turbulence). In particular, it is important to have efficient numerical techniques for the construction of a multiaffine field $\phi(x)$ whose structure functions scale as

$$\langle |\phi(x+r)-\phi(x)|^q\rangle \sim r^{\zeta_q}, \tag{1}$$

where $\langle\ \rangle$ indicates a spatial (or temporal) average, r varies in an appropriate scaling range, and the exponents ζ_q are given. The most interesting case, and the most physically relevant, is when ζ_q is a nonlinear function of q, that is, a strictly multiaffine field.

Let us first notice that the generation of a multiaffine function is much more difficult than the generation of a multifractal measure, which can be obtained with a simple multiplicative process generalizing the two scales Cantor set.

Up to now, there have existed well established methods for the construction of multiaffine fields [5–8]; see [8] for a short review. All of these methods share the common characteristic of not being sequential: the process is built as a whole in an interval (in space or time) of fixed length. To extend the interval one has to rebuild the process from the beginning. This is an evident limitation if one is interested in constructing a temporal signal mimicking, for example, those obtained by an anemometer measurement. Furthermore, nonsequential algorithms always require a huge amount of stored data.

In this paper we introduce a simple and efficient sequential method for the construction of a multiaffine function of time $u(t)$ with prescribed statistical properties. The guideline of our approach will be the reproduction of a turbulentlike temporal signal. Though the basic idea on the construction of the multiaffine process comes from fully developed turbulence, nevertheless the method is general and can be applied to any signal.

A typical anemometer measurement gives a one-dimensional string of data representing the one-point turbulent velocity $u(t)$ along the direction of the mean flow U. According to the Taylor hypothesis [9], for small turbulence intensities $u \ll U$, the time variations of u can be assumed to be due to the advection (with velocity U) of a frozen turbulent field past the measurement point, so that

$$\begin{aligned}\delta u(\tau) &= u(x,t+\tau)-u(x,t)\\ &= u(x-U\tau,t)-u(x,t)=\delta u(l),\end{aligned} \tag{2}$$

where $l=U\tau$. Therefore, once the spatial scaling (1) is given, we have

$$S_q(\tau)=\langle |u(t+\tau)-u(t)|^q\rangle \sim \tau^{\zeta_q}. \tag{3}$$

The frozen field is the result of the superposition of turbulent patterns (eddies) of many different sizes l, whose contribution to the time variation of the velocity decays with a typical correlation time $\tau_{\text{sweep}} \sim l/U$. For the sake of simplicity, in the following, we shall introduce a set of reference scales $l_n = 2^{-n}$ at which scaling properties will be tested. With this picture in mind, we represent the signal $u(t)$ by a superpo-

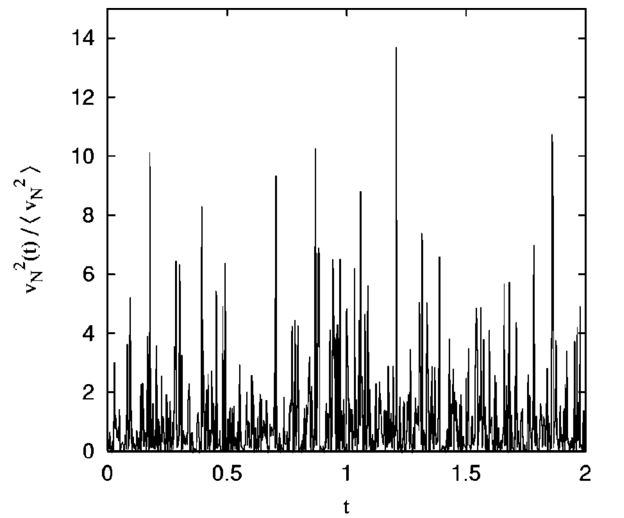

FIG. 1. Time series $v_N^2(t)$ normalized to the average for the model with $N=15$ octaves and $b=0.9$.

sition of functions with different characteristic times, representing eddies of various sizes

$$u(t)=\sum_{n=1}^{N} v_n(t). \tag{4}$$

The functions $v_n(t)$ are defined via a multiplicative process

$$v_n(t)=g_n(t)x_1(t)x_2(t)\cdots x_n(t), \tag{5}$$

where the $g_n(t)$ are independent stationary random processes, whose correlation times are the sweeping time scales $\tau_n=l_n/U=2^{-n}$ (assuming $U=1$) and $\langle g_n^2\rangle=l_n^{2h}$, where h is the scaling exponent. For fully developed turbulence $h=1/3$. Scaling will show up for all time delay larger than the UV cutoff τ_N and smaller than the IR cutoff τ_1. The $x_j(t)$ are independent, positive defined, identical distributed random processes whose time correlation decays with characteristic time τ_j. The probability distribution of x_j determines the intermittency of the process.

The origin of Eq. (5) is fairly clear in the context of fully developed turbulence. Indeed according to the refined similarity hypothesis of Kolmogorov [10,11], we can identify v_n with the velocity difference at scale l_n and x_j with $(\varepsilon_j/\varepsilon_{j-1})^{1/3}$, where ε_j is the energy dissipation at scale l_j.

It is easy to show, with a simple argument, that the process constructed according to Eqs. (4) and (5) is multiaffine. Because of the fast decrease of the correlation times $\tau_j=2^{-j}$, the characteristic time of $v_n(t)$ is of the order of the shortest one, i.e., $\tau_n=2^{-n}$. Therefore, the leading contribution to the structure function $S_q(\tau)$ with $\tau\sim\tau_n$ will stem from the nth term in Eq. (4). This can be understood nothing that in the sum $u(t+\tau)-u(t)=\sum_{k=1}^{N}[v_k(t+\tau)-v_k(t)]$ the terms with $k\leqslant n$ are negligible because $v_k(t+\tau)\simeq v_k(t)$ and the terms with $k\geqslant n$ are subleading. Thus one has

$$S_q(\tau_n)\sim\langle|v_n|^q\rangle\sim\langle|g_n|^q\rangle\langle x^q\rangle^n\sim\tau_n^{hq-\log_2\langle x^q\rangle} \tag{6}$$

and therefore for the scaling exponents (3),

$$\zeta_q=hq-\log_2\langle x^q\rangle. \tag{7}$$

The limit of an affine function can be obtained when all the x_j are equal to 1.

The above results can be proved in a rigorous way considering, as a first step, the second order structure function $S_2(\tau)$. Using the definitions (4) and (5) and stochastic independence one obtains

$$S_2(\tau)=2\sum_{n=1}^{N}[\langle v_n(t)^2\rangle-\langle v_n(t)v_n(t+\tau)\rangle]. \tag{8}$$

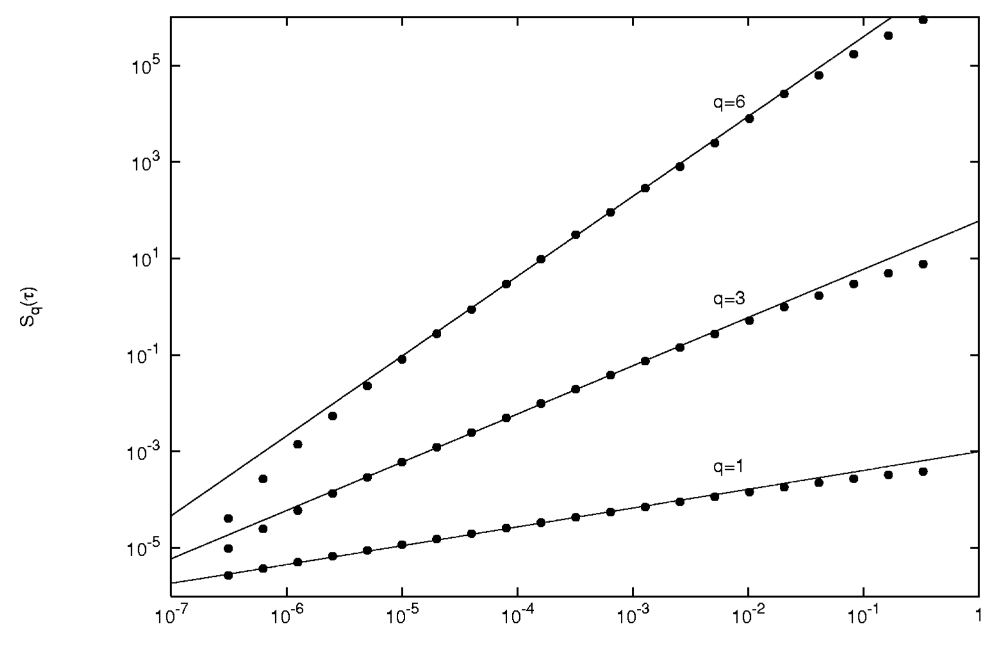

FIG. 2. Numerical (dots) and theoretical (line) structure functions $S_q(\tau)$ for the model with $N=20$ octaves and $b=0.9$. The exponents are $\zeta_1=0.39$, $\zeta_3=1$, $\zeta_6=1.65$. The structure functions are shifted by a multiplicative factor for plotting purposes.

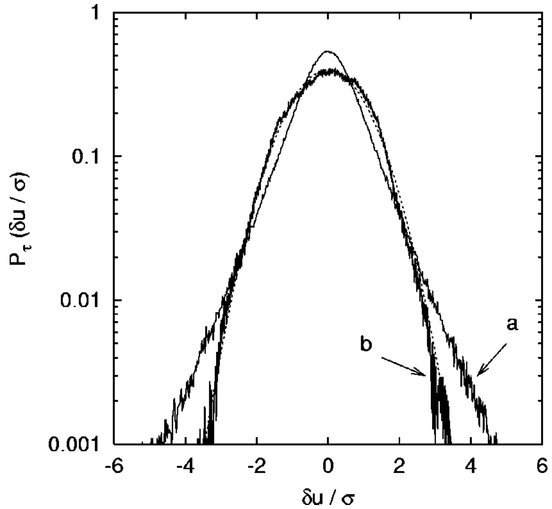

FIG. 3. Probability density functions for the normalized velocity differences $\delta u(\tau)/\sigma$, where $\sigma=\langle \delta u^2\rangle^{1/2}$, for different τ. For large $\tau=10$ (b) the PDF is nearly Gaussian (dashed curve). For very small $\tau=0.001$ (a) large tails are evident. The parameters are $N=15$ octaves and $b=0.9$.

Let us now introduce the normalized correlation functions for $g_n(t)$ and $x_j(t)$,

$$C\left(\frac{s}{\tau_n}\right)=\frac{\langle g_n(t')g_n(t)\rangle}{\langle g_n^2\rangle}; \quad F\left(\frac{s}{\tau_j}\right)=\frac{\langle x_j(t')x_j(t)\rangle}{\langle x_j^2\rangle}, \tag{9}$$

where we have set $t'=t+s$. Plugging into Eq. (8) the definition (5), one obtains

$$S_2(\tau)=2\sum_{n=1}^{N}\langle g_n^2\rangle\langle x^2\rangle^n\left[1-C\left(\frac{\tau}{\tau_n}\right)F\left(\frac{\tau}{\tau_1}\right)\cdots F\left(\frac{\tau}{\tau_n}\right)\right]. \tag{10}$$

For $\tau_N\ll\tau\ll\tau_1$ one can neglect the UV and the IR cutoffs, so we have

$$S_2(\tau)=2\sum_{n=2}^{\infty}\langle g_n^2\rangle\langle x^2\rangle^n\left[1-C\left(\frac{\tau}{\tau_n}\right)\prod_{j=2}^{n}F\left(\frac{\tau}{\tau_j}\right)\right]+O\left(\frac{\tau}{\tau_1}\right). \tag{11}$$

where we have used the expansion $C(\tau/\tau_1)\sim F(\tau/\tau_1)=1-O(\tau/\tau_1)$. Writing now Eq. (10) for $\tau\rightarrow 2\tau$ and by shifting the summation index, $n\rightarrow n-1$, one obtains for $\tau\ll\tau_1$

$$S_2(2\tau)=2^{2h}\langle x^2\rangle^{-1}S_2(\tau)+O(\tau/\tau_1) \tag{12}$$

which leads, as long as $\zeta_2<1$, to the scaling behavior:

$$S_2(\tau)\sim\tau^{\zeta_2} \quad \text{with} \quad \zeta_2=2h-\log_2\langle x^2\rangle. \tag{13}$$

The key point in the above arguments is that the dominant contribution to the structure function $S_q(\tau)$ comes from octaves n such that $\tau_n\sim\tau$, that is, locality.

The constraints for locality can be captured with a simple argument [12]. At a generic τ, the UV convergence requires that for $\tau_n\ll\tau$ the quantities $\langle|v_n(t+\tau)-v_n(t)|^q\rangle\sim\langle|v_n|^q\rangle\sim 2^{-n\zeta_q}$ have to be bounded for $n\rightarrow\infty$ and therefore $\zeta_q>0$. Similarly, when $\tau_n\gg\tau$ we have that $\langle|v_n(t+\tau)-v_n(t)|^q\rangle\sim(\tau/\tau_n)^{q/2}\langle|v_n|^q\rangle\sim 2^{-n(\zeta_q-q/2)}$, for stochastic processes such that $C(x)=1-O(x)$ and $F(x)=1-O(x)$. Therefore, the IR convergence in the latter case requires $\zeta_q<q/2$. We observe that the last condition is different from the usual locality condition $\zeta_q<q$ [12], which holds for differentiable processes where $C(x)=1-O(x^2)$ and $F(x)=1-O(x^2)$.

A similar computation can be performed for the higher order structure functions. The generic $S_q(\tau)$ can be expressed as a linear combination of terms scaling as $\tau^{\zeta_{m_1}}\cdots\tau^{\zeta_{m_k}}$ with $m_1+\cdots+m_k=q$. From the convexity of ζ_q [13] it follows that the leading contribution to $S_q(\tau)$ for small τ is given by $S_q(\tau)\sim\tau^{\zeta_q}$, with the exponents ζ_q as defined in Eq. (7).

Regular behavior for very short time delays $\delta u(\tau)\sim\tau$, physically related to the presence of dissipation, can be simply achieved in our model by smoothing $g_n(t)$ and $x_n(t)$ over a time interval smaller then the UV cutoff τ_N.

The numerical implementation of the method proposed above is very simple. The stochastic process $x_j(t)$ can be easily generated via the nonlinear Langevin differential equations:

$$dx_j=-\frac{1}{\tau_j}\frac{dV}{dx_j}dt+\sqrt{\frac{2}{\tau_j}}dW_j, \tag{14}$$

where $V(x)=\infty$ for $x<a$ (a positive constant) and $V(x)\rightarrow\infty$ for $x\rightarrow\infty$. It is clear that the x_j so obtained have the same probability density function independent of τ_j.

Similarly for the g_n one can use the evolution law

$$dg_n=-\frac{1}{\tau_n}\frac{dY}{dg_n}dt+\sigma_n\sqrt{\frac{2}{\tau_n}}dW_n, \tag{15}$$

where $Y(g)\rightarrow\infty$ as $|g|\rightarrow\infty$ and $\sigma_n=l_n^h$.

Numerical tests have been performed adopting for the stochastic differential equations (14) and (15) the following potentials:

$$V(x)=-2\ln x \quad \text{for} \quad (1-b)^{1/3}<x<(1+b)^{1/3} \tag{16}$$

and $V(x)=\infty$ otherwise, where $0<b<1$. The choice (16) corresponds to having a uniform distribution for x^3 between $1-b$ and $1+b$. In this way, the moments $\langle x^q\rangle$ can be easily computed. In our numerical tests the g_n processes have been chosen to be defined by the following simple potential:

$$Y(g)=\frac{1}{2}g^2. \tag{17}$$

For $h=1/3$, these choices insure that $\zeta_3=1$ according to the scaling prescribed by Kolmogorov's law. The parameter b tunes the intermittency of the signal: when $b\downarrow 0$ we recover an affine process. The choice (17) gives a nonskewed signal and a Gaussian velocity field in the limit $b\downarrow 0$. In Fig. 1 we show the quantity $v_N^2(t)$, which can be considered as the energy density dissipation of the turbulent signal. As one can see, high intermittency is detected.

The theoretical and numerical scaling laws are compared in Fig. 2. The computed scaling exponents are in perfect agreement with those given by Eq. (7). Figure 3 shows the probability density function (PDF) of the velocity differences $\delta u(\tau)=u(t+\tau)-u(t)$ for different τ. At large $\tau\sim 1$ the

PDF is nearly Gaussian, whereas at small delays the PDF is increasingly peaked around zero with high tails corresponding to large fluctuations with respect to their rms value.

For a specific problem with a nonzero skewness, as in turbulence, $Y(g)$ must be chosen as an asymmetric function; see [8] for a suitable choice according to experimental data.

In this paper we have introduced an efficient sequential algorithm for the generation of multiaffine processes. This method, at variance with previous proposals, is not based on hierarchical construction, and can be applied to any multiaffine signals with specified scaling laws. Furthermore, no huge amount of memory is required for the numerical implementation.

A possible, relevant, application of such a signal would be to use it for describing the temporal part of a synthetic turbulent velocity field. The spatial part can be implemented by using any hierarchical constructions previously proposed, [5–8] Nevertheless, this way to glue together spatial and temporal multiaffine fluctuations would not be realistic, due to the absence of a real sweeping of small scales by large scales. This is connected to the fact that in our temporal signal, the Taylor hypothesis is introduced by hands without any real direct dynamical (stochastic) coupling between large and small scales.

These difficulties in reproducing an Eulerian spatio-temporal field are absent if one considers the velocity statistics in quasi-Lagrangian coordinates [14]. In this framework, a pure temporal signal would correspond to the velocity field felt in the moving reference frame attached to a fluid particle. The sweeping effect is thus removed and the characteristic time scales are the dynamical eddy turnover times. Work in this direction is in progress.

Another possible interesting investigation would be to check whether our signal defines a Markov process for the energy cascade as it seems to be the case for experimental turbulent signals [15].

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