

# Helicity advection in Turbulent Models

L. Biferale<sup>1,2</sup>, D. Pierotti<sup>3</sup> and F. Toschi<sup>2,4</sup>

<sup>1</sup> Dip. di Fisica, Università di Tor Vergata, Via della Ricerca Scientifica 1,  
I-00133, Roma, Italy.

<sup>2</sup> Istituto Nazionale di Fisica della Materia, unità di Tor Vergata.

<sup>3</sup> Dept. of Chemical Physics, Weizmann Institute of Science,  
76100, Rehovot, Israel.

<sup>4</sup> Dip. di Fisica, Università di Pisa, Piazza Torricelli 2,  
I-56126, Pisa, Italy.

## Abstract

Helicity transfer in a shell model of turbulence is investigated. In particular, we study the scaling behavior of helicity transfer in a dynamical model of turbulence lacking inversion symmetry. We present some phenomenological and numerical support to the idea that Helicity becomes -at scale small enough- a passively-advected quantity.

*“ουτοι συνεχθειν, αλλα συμφιλειν εφυν”*

ANTIGONH

*“non sono nata per condividere l’odio, ma l’amore”*

## 1 Introduction

One of the most intriguing problems in three dimensional fully developed turbulence (FDT) is related to the appearance of anomalous scaling laws at high Reynolds numbers, i.e. in the limit when Navier-Stokes dynamics is dominated by the non-linear interactions.

The celebrated 1941 Kolmogorov theory (K41) was able to capture the main phenomenological ideas by performing dimensional analysis based on the energy transfer mechanism. Kolmogorov postulated that the energy cascade should follow a self-similar and homogeneous process entirely dependent on the energy transfer rate,  $\epsilon$ . This idea, plus the assumption of local isotropy and universality at small scales, led to a precise prediction on the statistical properties of the increments of turbulent velocity fields:  $\delta v(r) \sim |v(x+r) - v(x)| \sim (r \cdot \epsilon(r))^{1/3}$ . From this the scaling of moments of  $\delta v(r)$ , the structure functions, can be determined in terms of the statistics of  $\epsilon(r)$ , i.e.

$$S_p(r) \equiv \langle (\delta v(r))^p \rangle = C_p \langle (\epsilon(r))^{p/3} \rangle r^{p/3}, \quad (1)$$

where  $C_p$  are constants and the scale  $r$  is supposed to be in the inertial range, i.e. much smaller than the integral scale and much larger than the viscous dissipation cutoff. If  $S_p(r) \sim r^{\zeta(p)}$  and  $\langle \epsilon^p(r) \rangle \sim r^{\tau(p)}$  then

$$\zeta(p) = p/3 + \tau(p/3). \quad (2)$$

In the K41 the  $\epsilon(r)$  statistic is assumed to be  $r$ -independent, that is  $\tau(p) = 0$ , implying  $\zeta(p) = \frac{p}{3}, \forall p$ . On the other hand, there are many experimental and numerical [1, 2] results telling us that K41 scenario for homogeneous and isotropic turbulence is quantitatively wrong. Strong intermittent bursts in the energy transfer have been observed and non trivial  $\tau(p)$  set of exponents measured.

Many different authors have focused their attention on the possible rôle played by helicity, the second global invariant of 3D Navier-Stokes eqs. [4, 5, 8, 9], for determining leading or sub-leading scaling properties of correlation functions in the inertial range.

Recently [3, 10], an exact scaling equation for the third order velocity correlations entering in the helicity flux definition has been derived under two hypothesis: (i) there exists a non-vanishing helicity flux, (ii) the flux becomes Reynolds independent in the limit of FDT. This relation predicts a  $r^2$ -scaling for the particular third order velocity correlation entering in the definition of helicity flux, at difference from the celebrated linear behavior in  $r$  showed by the third order velocity correlations entering in the definition of energy flux.

This simple fact tells us that different velocity correlation with the same physical dimension but with different tensorial structure may show different *leading* scaling properties.

Moreover, even if overwhelming evidences indicate that the main physics is driven by the energy transfer, there can be some sub-leading new intermittent statistics hidden in the helicity flux properties.

Homogeneous and isotropic turbulence has, by definition, always a vanishing mean helical flux. Nevertheless, both fluctuations about the zero-mean, in isotropic cases, and/or net non-zero fluxes, in cases where inversion symmetry is explicitly broken, can be of some interest for the understanding of fully developed turbulence.

In this letter, we analyze the helical transfer mechanism in dynamical models of turbulence [11, 9, 12], built such as to explicitly consider helicity conservation in the inviscid limit.

In a previous publication [6] we found the first strong numerical evidence that a Reynolds-independent helicity flux is present in cases where the forcing mechanism explicitly breaks inversion symmetry. In this paper we present a phenomenological argument which supports the idea that helicity behaves as a quantity passively transferred toward small scales by the mean energy flux, in agreement with both numerical findings in the true Navier-Stokes eqs. [7] and with our previous numerical evidences about the strong intermittent properties of the helicity flux [6].

In the following, we briefly summarize the main motivation behind the introduction of Shell Models for turbulence. We summarize the main results obtained in [6] and we present the new argument about the passive character of helicity fluctuations.

## 2 Shell Models

Shell models have demonstrated to be very useful for the understanding of many properties connected to turbulent flows [13]-[19]. The most popular shell model, the Gledzer-Ohkitani-Yamada (GOY) model ([13]-[19]), has been shown to predict scaling properties for  $\zeta(p)$  (for a suitable choice of the free parameters) similar to what is found experimentally.

The GOY model can be seen as a severe truncation of the Navier-Stokes equations: it retains only one complex mode  $u_n$  as a representative of all Fourier modes in the shell of wave numbers  $k$  between  $k_n = k_0 2^n$  and  $k_{n+1}$ .

It has been pointed out that GOY model conserves in the inviscid, unforced limit two quadratic quantities. The first quantity is the *energy*, while the second is the equivalent of *helicity* in 3D turbulence [18]. In two recent works [9, 12] the GOY model has been generalized in terms of shell variable,  $u_n^+, u_n^-$ , transporting positive and negative helicity, respectively. It is easy to realize that only 4 independent classes of models can be derived such as to preserve the same helical structure of Navier-Stokes equations [8]. All these models have at least one inviscid invariants non-positive defined which is similar to the 3D Navier-Stokes helicity. In the following, we will focus on the intermittent properties of one of them which has already been extensively investigated (see [12, 20] for more details). The time evolution for positive-helicity shells reads [12]:

$$\dot{u}_n^+ = ik_n (A_n [u, u])^* - \nu k_n^2 u_n^+ + \delta_{n,n_0} f^+, \quad (3)$$

with the equivalent eqs. for  $u^-$ . In (3)  $\nu$  is the molecular viscosity,  $f^+$ ,  $f^-$  are the large scale forcing and  $A[u, u]$  refers to the non-linear terms of the model. Namely:

$$A_n[u, u] \equiv u_{n+2}^- u_{n+1}^+ + b_3 u_{n+1}^- u_{n-1}^+ + c_3 u_{n-1}^- u_{n-2}^-. \quad (4)$$

It is easy to verify that for the choice  $b_3 = -5/12$ ,  $c_3 = -1/24$  there exists two global inviscid invariants [12]: the energy,

$$E = \sum_{i=1}^N (|u_i^+|^2 + |u_i^-|^2)$$

and helicity,

$$H = \sum_{i=1}^N k_i (|u_i^+|^2 - |u_i^-|^2).$$

The equations for the fluxes throughout shell number  $n$  are:

$$\frac{d}{dt} \sum_{i=1,n} E_i = k_n \langle (uuu)_n^E \rangle - \nu k_n^2 \sum_{i=1,n} E_i + E_{in}, \quad (5)$$

$$\frac{d}{dt} \sum_{i=1,n} H_i = k_n^2 \langle (uuu)_n^H \rangle - \nu k_n^2 \sum_{i=1,n} H_i + H_{in}, \quad (6)$$

where  $E_i$  and  $H_i$  are the energy and helicity of the  $i$ -th shell, respectively:  $E_i = \langle |u_i^+|^2 + |u_i^-|^2 \rangle$ ,  $H_i = k_i \langle |u_i^+|^2 - |u_i^-|^2 \rangle$ .  $E_{in}$  and  $H_{in}$  are the input of energy and helicity due to forcing effects,  $E_{in} = \Re(\langle f^+(u_1^+)^* + f^-(u_1^-)^* \rangle)$ ,  $H_{in} = \Re(k_1 \langle f^+(u_1^+)^* - f^-(u_1^-)^* \rangle)$ .

In (5) and (6) we have introduced the triple correlation:

$$\langle (uuu)_n^E \rangle = (\Delta_{n+1}^+ + \Delta_{n+1}^-) + (b_3 + 1/2)(\Delta_n^+ + \Delta_n^-) \quad (7)$$

$$\langle (uuu)_n^H \rangle = (\Delta_{n+1}^+ - \Delta_{n+1}^-) + (b_3 + 1/4)(\Delta_n^+ - \Delta_n^-) \quad (8)$$

and

$$\Delta_n^+ = \langle \Im(u_{n+1}^- u_n^+ u_{n-1}^+) \rangle. \quad (9)$$

Assuming that there exists a stationary state we have  $\frac{d}{dt} \Pi_n^E = \frac{d}{dt} \Pi_n^H = 0$ , where  $\Pi_n^E = k_n \langle (uuu)_n^E \rangle$  and  $\Pi_n^H = k_n^2 \langle (uuu)_n^H \rangle$ . Moreover, in the inertial range we can neglect the viscous contribution in (5) and (6), obtaining:

$$\langle (uuu)_n^E \rangle = k_n^{-1} E_{in}, \quad \langle (uuu)_n^H \rangle = k_n^{-2} H_{in}. \quad (10)$$

In [6] we have shown that in the case of a large scale forcing breaking inversion symmetry (i.e.  $f^+ \neq f^-$ ) energy and helicity fluxes coexist in the systems, both being Reynolds-independent. In the inertial range we may therefore neglect the viscous contribution and we obtain for the energy-triple-correlations and for the helicity-triple-correlation:

$$\langle (uuu)_n^E \rangle \sim k_n^{-1}, \quad \langle (uuu)_n^H \rangle \sim k_n^{-2}. \quad (11)$$

Relation (10) is the equivalent of what found for helical Navier-Stokes turbulence in [3, 10].

Let us remark that the coexistence of both energy and helicity fluxes is only possible due to the non-positiveness of helicity; in 2D turbulence, for example, a similar result, concerning enstrophy and energy cascades, is clearly a priori forbidden.

In [6] we have measured the statistics of energy and helicity transfers, by defining the two sets of scaling exponents:

$$\Sigma_E^{(p)} \equiv \left\langle \left( (uuu)_n^E \right)^{(p/3)} \right\rangle \sim k_n^{-\zeta(p)} \quad (12)$$

$$\Sigma_H^{(p)} \equiv \left\langle \left( (uuu)_n^H \right)^{(p/3)} \right\rangle \sim k_n^{-\psi(p)}. \quad (13)$$

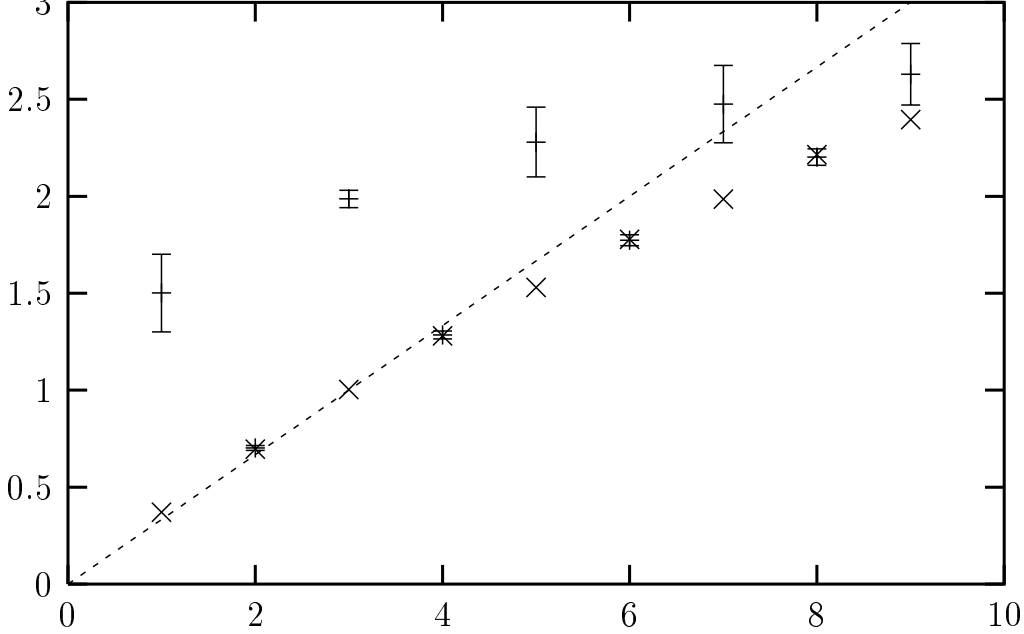


Figure 1: Anomalous exponents for the helicity flux,  $\psi_p$  (+), and for the energy flux,  $\zeta_p$  ( $\times$ ), for  $p = 1, \dots, 9$  obtained with  $N = 26$  and  $\nu = 2 \cdot 10^{-9}$ .

As one can see in Figure 1 we found that the even part of the two statistics coincides, i.e.  $\zeta(2p) = \psi(2p)$ . On the other hand, the scaling exponents of odd moments are different. In particular the helicity exponents show a strong intermittent shape reminding to what one finds, for example, for the statistics of a passive scalar advected by a turbulent flow. Phenomenological and numerical evidences supporting the idea that helicity may be safely be considered as a passive advected quantity, at small enough scales, have already been presented in [7].

### 3 Helicity advection

Let us push further this idea by analyzing in more details the equation of motion describing the evolution of shell variables. In order to highlight the energy and helicity dynamics it is better to project the equation of motion into two new variables feeling the even and the odd part of the statistics respectively:

$$w_n = \frac{u_n^+ + u_n^-}{2} \quad \lambda_n = \frac{u_n^+ - u_n^-}{2}. \quad (14)$$

Let us notice that due to the transformation properties with respect to the symmetry  $u_n^+ \leftrightarrow u_n^-$  it is obvious that  $w_n$  will mainly feel the energy statistics while  $\lambda_n$  will be highly sensible to the helical properties of the flow. The total energy and helicity in the shell  $n$  will be  $E_n \propto |w_n|^2 + |\lambda_n|^2$  and  $H_n \propto \Re(w_n \lambda_n^*)$ .

Writing now the inertial-time evolution for these two variables we obtain:

$$\dot{w}_n = (w_{n+2}w_{n+1} + b_3w_{n+1}w_{n-1} + c_3w_{n-1}w_{n-2}) + (-\lambda_{n+2}\lambda_{n+1} - b_3\lambda_{n+1}\lambda_{n-1} + c_3\lambda_{n-1}\lambda_{n-2}) \quad (15)$$

$$\dot{\lambda}_n = (w_{n+2}\lambda_{n+1} + b_3w_{n+1}\lambda_{n-1} - c_3w_{n-1}\lambda_{n-2}) + (-\lambda_{n+2}w_{n+1} - b_3\lambda_{n+1}w_{n-1} - c_3\lambda_{n-1}w_{n-2}). \quad (16)$$

Equations (15) and (16) are identical to the set of coupled equations describing - in the shell world - an active scalar  $\lambda_n$  advected by a turbulent velocity field  $w_n$ . The active character of  $\lambda_n$  is clearly

dictated by the quadratic terms in the RHS of (15).

We now want to argue that at small enough scale the  $O(\lambda^2)$  part in the RHS of (15) becomes negligible and therefore we end up with a set of eqs describing the passive advection of the field  $\lambda_n$  by the velocity field  $w_n$ .

Let us start by noticing that for the variables  $\Delta^\pm$  which enter in the definition of the fluxes we have:

$$\Delta^+ + \Delta^- \propto w_{n+1} [w_n w_{n-1} - \lambda_n \lambda_{n-1}] + \lambda_{n+1} [w_n \lambda_{n-1} - \lambda_n w_{n-1}] \quad (17)$$

and

$$\Delta^+ - \Delta^- \propto w_{n+1} [w_n \lambda_{n-1} - \lambda_n w_{n-1}] + \lambda_{n+1} [w_n w_{n-1} - \lambda_n \lambda_{n-1}]. \quad (18)$$

Furthermore, by imposing now two general scaling laws for the field  $w_n \sim k_n^{-\alpha}$  and  $\lambda_n \sim k_n^{-\beta}$  we obtain:

$$\langle (uuu)_n^E \rangle \sim \Delta^+ + \Delta^- \sim w_n^3 + w_n \lambda_n^2 + w_n^2 \lambda_n \sim k_n^{-3\alpha} + k_n^{-\alpha-2\beta} + k_n^{-2\alpha-\beta} \quad (19)$$

and

$$\langle (uuu)_n^H \rangle \sim \Delta^+ - \Delta^- \sim w_n^2 \lambda_n + w_n \lambda_n^2 \sim k_n^{-2\alpha-\beta} + k_n^{-\alpha-2\beta}. \quad (20)$$

By requiring now the asymptotic matching of (19) and of (20) with the known behavior for the energy and helicity fluxes we have:

$$\langle (uuu)_n^E \rangle \sim k_n^{-3\alpha} + k_n^{-\alpha-2\beta} + k_n^{-2\alpha-\beta} \sim k_n^{-1} \quad (21)$$

$$\langle (uuu)_n^H \rangle \sim k_n^{-2\alpha-\beta} + k_n^{-\alpha-2\beta} \sim k_n^{-2}. \quad (22)$$

In order to satisfy both matchings we must to require that

$$w_n \sim k_n^{-1/3} \quad \lambda_n \sim k_n^{-4/3}. \quad (23)$$

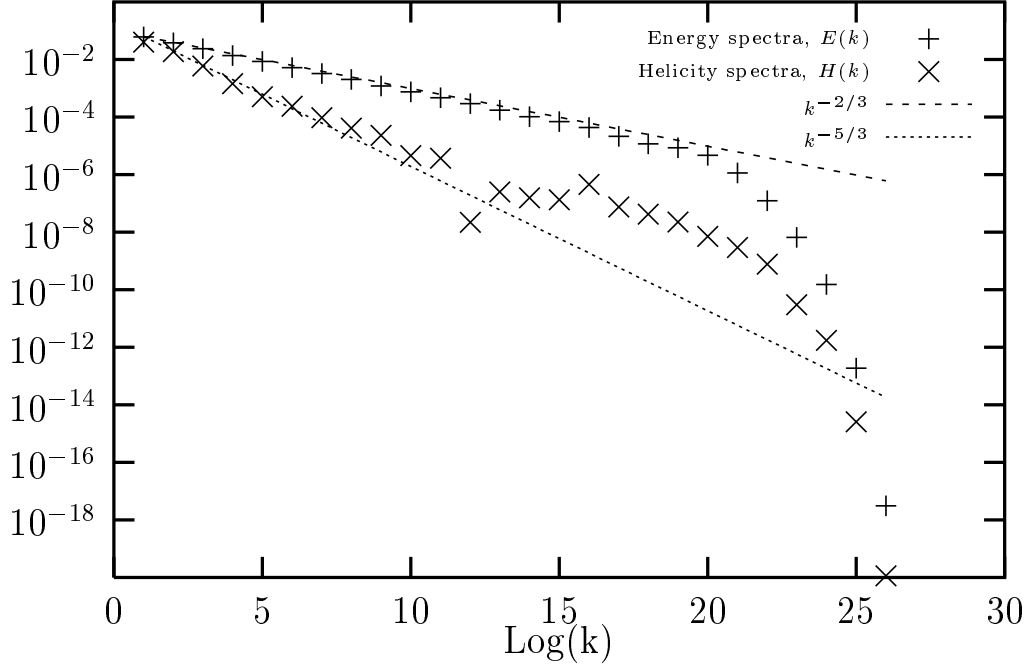


Figure 2: Log-log plot of Energy,  $E(k)$ , and helicity,  $H(k)$ , spectra compared with theoretical expectations.

By plugging the above scalings in the equation of motion we discover that the time evolution for the field  $w_n$  is mainly governed by the field itself, i.e. the coupling with the field  $\lambda_n$  is sub-dominant

at small scales and therefore as a direct consequence that the field  $\lambda_n$  is passively advected by  $w_n$ . Of course the phenomenology here discussed will be affected by intermittency, i.e. the above exponents may show some weak/strong deviations from the predicted values, without, however having the possibility to change the rôle of dominant and sub-dominant contribution in expression (21) and (22).

In order to test this prediction we plot in Figure 2 the energy and helicity spectrum respectively  $E(k_n)/k_n \sim w_n^2 \sim k_n^{-2/3}$  and  $H(k_n)/k_n \sim w_n \lambda_n \sim k_n^{-5/3}$ . As one can see, a part from the strong bottleneck effect showed by the helicity spectrum, the agreement is perfect.

In conclusion we have investigated the Helicity transfer statistics in a shell model of turbulence. In particular, we have found that the strong intermittent properties of Helicity flux -measured numerically- can be explained in terms of the phenomenological evidence that Helicity becomes a passive quantity advected -at small enough scale- by the energy flux.

## References

- [1] C. Meneveau, K. R. Sreenivasan, J. Fluid Mech. **224**, 429 (1991).
- [2] R. Benzi, S. Ciliberto, R. Tripiccion, C. Baudet, C. Massaioli, S. Succi, Phys. Rev. E **48**, R29 (1993).
- [3] V. S. L'Vov, E. Podivilov and I. Procaccia, *Exact Results for the 3rd Order Correlations of Velocity in Turbulence with Helicity*. Preprint in `chao-dyn@xyz.lanl.gov # 9705016`.
- [4] R. H. Kraichnan, J. Fluid Mech. **59**, 745 (1973).
- [5] Levich, Physica A **170**, 241 (1991).
- [6] L. Biferale, D. Pierotti, F. Toschi, Phys. Rev. E **57**, 3, (1998).
- [7] V. Borue, S.A. Orszag, Phys. Rev. E **55**, 7005 (1997).
- [8] F. Waleffe, Phys. Fluids A **4**, 350 (1992).
- [9] L. Biferale, R. Kerr, Phys. Rev. E **52**, 6113 (1995).
- [10] O. Chkhetiani, JETP Lett. **63**, 808 (1996).
- [11] L. Kadanoff, Physics Today, September **11** (1995).
- [12] R. Benzi, L. Biferale, R. Kerr, E. Trovatore, Phys. Rev. E **53**, 3541 (1996).
- [13] E.B. Gledzer, Sov. Phys. Dokl. **18**, 216 (1973).
- [14] M. Yamada, K. Ohkitani, Prog. Theor. Phys. **81**, 329 (1989); M. Yamada, K. Ohkitani, J. Phys. Soc. Jpn. **56**, 4210 (1987); Phys. Rev. Lett **60**, 983 (1988); Prog. Theor. Phys. **79**, 1265 (1988).
- [15] M.H. Jensen, G. Paladin, A. Vulpiani, Phys. Rev. A **A43**, 798 (1991).
- [16] L. Biferale, A. Lambert, R. Lima, G. Paladin, Physica D **80** 105 (1995).
- [17] D. Pisarenko, L. Biferale, D. Courvoisier, U. Frisch, M. Vergassola, Phys. Fluids **A65** 2533 (1993).
- [18] L. Kadanoff, D. Lohse, J. Wang, R. Benzi, Phys. Fluids **7** 617 (1995).
- [19] E. Leveque, Z.-S. She, Phys. Rev. E **55**, 2789 (1997).
- [20] R. Benzi, L. Biferale, E. Trovatore, Phys. Rev. Lett. **77**, 3114 (1996).