Intermittency and eddy viscosities in dynamical models of turbulence

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The dependence of intermittent inertial properties on ultraviolet eddy viscosity closures is examined within the framework of shell models of turbulent flows. Inertial intermittent exponents turn out to be fairly independent on the way energy is dissipated at small scales. © *1999 American Institute of Physics.* [S1070-6631(99)00905-8]

I. INTRODUCTION

One of the most challenging open problems in three-dimensional fully developed turbulence is the assessment of the statistical properties of the energy transfer mechanism. In stationary turbulent flows, a net flux of energy establishes in the inertial range, i.e., from forced scales, L, down to the dissipative scale, r_d. Energy is transferred through a statistically scale-invariant process, characterized by a strongly non-Gaussian (intermittent) activity.

Intermittency is usually described by looking at the statistical properties of longitudinal velocity differences, $\delta_r v(x)=v(x)-v(x+r)$ (vector notation is relaxed for simplicity). In particular, the last 20 years[1] have witnessed a substantial focus of experimental and theoretical activity on structure functions: $S_p(r)=\langle(\delta_r v(x))^p\rangle$. A wide consensus exists on the fact that structure functions show a scaling behavior in the limit of very high Reynolds numbers, i.e., in the presence of a large separation between the integral and dissipative scales, $L/r_d\to\infty$:

$$S_p(r)\sim\left(\frac{r}{L}\right)^{\zeta(p)}. \tag{1}$$

The velocity fluctuations are anomalous in the sense that the $\zeta(p)$ exponents do not follow the celebrated dimensional Kolmogorov's prediction $\zeta(p)=p/3$. In fact, $\zeta(p)$ is observed to be a nonlinear function of its argument p, which is interpreted as the most important signature of the intermittent transfer of fluctuations from large to small scales.

As it is known, the dissipative structure of the Navier–Stokes equations (NS) is not dictated by compelling constraints on the inertial terms. This raises the question of whether or not the statistical properties of fully developed three-dimensional turbulent flows exhibit a strong dependency on the energy dissipation mechanism.

Kolmogorov theory suggests a strong universality assumption: strong independency of pure inertial range quantities on any dissipative mechanism. The theoretical implications of such an assumption are obvious. For instance, some of the most recent analytical attacks to the intermittency of structure functions assume that the phenomenon is fully captured by looking only at the nonlinear terms in the NS equations, at least in the limit of large Reynolds number.[2] However, because of intermittency, one can question the conceptual framework of the Kolmogorov theory and consequently the strong universality assumption.

Moreover, numerical investigations of turbulent flows are necessarily restricted to low (moderate) Reynolds numbers. Therefore, it is of primary importance to develop some controllable procedure which minimizes viscous effects (whenever possible). In the past, hyperviscosity (high powers of the Laplacian) has often been employed in order to extend the inertial range as much as possible. Contradictory claims have been reported on the influence of the energy dissipation mechanism on the inertial range dynamics.[3–7]

An important tool, heavily in use to perform reliable high Reynolds simulation, is based on the concept of eddy viscosity.[1,8,9]

In this paper we investigate robustness of the intermittent inertial properties in the context of simple dynamical eddy viscosity models. In particular, we present a detailed numerical investigation of such an issue in a class of dynamical models of turbulence (shell models) both in the case where the dynamics is resolved in real and Fourier space (tree model)[10,11] and in the case where only the Fourier space is taken into account (standard chain-models[12–16]).

In either case, we find strong independence of the inertial range statistics from the ultraviolet dynamical closure,

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indicating that most (eventually not all) eddy viscosity models do not destroy the quantitative and qualitative features of the inertial range dynamics.

The paper is organized as follows. In Sec. II we introduce the main ideas behind eddy viscosity models. In Sec. III we introduce the dynamical models we have used in order to test the dependency of intermittency on eddy viscosities. In Sec. IV we discuss the numerical results. Conclusions follow in Sec. V.

II. EDDY VISCOSITY MODELS

The idea of eddy viscosity was introduced over a century ago by Boussinesq and later developed further by G. Taylor and L. Prandtl[17] and it builds upon a direct analogy with the kinetic theory of gas. According to this analogy, the effect of short ''microscopic'' scales on large ''macroscopic'' scales can be likened to a sort of diffusion process characterized by an effective viscosity much larger than the molecular one. Strictly speaking, this is justified only when a sharp separation between fast and slow modes exists, but it turns out that the analogy proves useful in practice also in situations where, in principle, such an assumption would not hold.

By mere dimensional arguments, the effective eddy viscosity at scale r reads as follows,

$$\nu_E(r) \sim r \cdot \delta v(r), \tag{2}$$

where $\delta v(r)$ is the velocity fluctuation across a distance r (vector indices are relaxed for simplicity).

Equation (2) can be also deduced by using the refined Kolmogorov hypothesis (RKH) as follows. According to Kolmogorov,[1] a simple way to take into account the intermittent fluctuations in the inertial range is to define a coarse-grained energy dissipation $\epsilon_r(x)$:

$$\epsilon_r(x) = \frac{1}{r^3} \int_{\Lambda_r(x)} \epsilon(y)\, d^3y, \tag{3}$$

where $\Lambda(x)$ denotes a volume of fluid centered in x.

In terms of ϵ_r one can generalize the Kolmogorov ''4/5'' equation by assuming that $(\delta_r v(x))^3 \sim r\epsilon_r$.

Now, let us define Δ the scale at which we want to compute the eddy viscosity. At such a scale, one expects $\epsilon_\Delta = \nu_E(\Delta)(\delta v(\Delta)/\Delta)^2$. By combining these two expressions, (2) is readily obtained.

The eddy viscosity is much larger than the molecular one, which reflects the enhanced mass and momentum transport observed in turbulent flows.

As it is well known, for most turbulent flows of practical interest, the dissipative scale η is far too short to be resolved by any foreseeable computer. In fact, η scales like $L \cdot \mathrm{Re}^{-3/4}$, L being the outer scale of the flow, and consequently the scale separation L/η can easily span three to six orders of magnitude in practical applications.

Given this state of affairs, subgrid models and large-eddy simulations (LES) are mandatory. Generally speaking, the common aim of these models is to incorporate the effects of unresolved scales ($r < \Delta$, Δ being a typical mesh size) on the resolved ones, $r > \Delta$.

One of the simplest and most popular subgrid models is due to Smagorinski,[18] which can be derived by (2).

The idea is to replace r with the mesh size Δ in Eq. (2) and subsequently replace $\delta v(\Delta) \sim S\Delta$, where (we dispense from tensor indices for the sake of the argument) S is the strain tensor $S \sim \delta v/r$ evaluated at $r = \Delta$.

The result is

$$\nu_{\mathrm{SGS}} \sim \Delta^2 S. \tag{4}$$

This expression is less transparent than it looks. In fact, it is based on the assumption that the velocity field at the scale Δ is smooth enough to allow the definition of the space derivative S.

This flies in the face of the fact that, if Δ belongs to the inertial range (as it should for the whole LES procedure to make sense), the velocity field is *known* not to be differentiable since δv scales like $r^{1/3}$. On account of this, one expects $\delta v(\Delta)/\Delta$ to be much larger than the corresponding ratio evaluated at $r = \eta$ (the only scale where this operation is conceptually allowed). This ''inconsistency'' is usually acknowledged by prefactoring the right-hand side of the equation with an empirical coefficient C_S smaller than 1, typically $C_S \sim 0.12$.

Putting all this together, and restoring tensorial indices, the full Smagorinski eddy viscosity reads as

$$\nu_{\mathrm{SMG}}(x) = C_S \Delta^2 |S|, \tag{5}$$

where $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ is the large-scale stress tensor and $|S| = (2S_{ij}S_{ij})^{1/2}$. The Smagorinski model is widely used in practical engineering applications in spite of its several weaknesses. Among these, worth mentioning are (i) the over damping of resolved scales, and (ii) the, at least partial, assumption of isotropy of the turbulent flows. The former flaw may seriously hinder the development of genuine instabilities,[19] while the latter casts doubts on the applicability of the model in the vicinity of walls and solid boundaries where the dynamics of turbulence is dominated by directional effects.

Another recent development in the area of Smagorinski models is the so-called structure-function eddy viscosity by Lesieur[20] and co-workers. The idea is to account for intermittency by estimating $\delta v(r)$ with the square root of the second-order local structure function

$$\delta v(r) \sim S_2(r,x)^{1/2} \equiv \langle \delta v(r,x)^2 \rangle^{1/2}, \tag{6}$$

where the local average is computed with the local energy spectrum $E(k,x,t)$ according to the Batchelor relation

$$S_2(r,x) = \int E(k,x) \frac{\sin(kr)}{r}\, dk. \tag{7}$$

The relation (6) implies a certain degree of arbitrariness. Why not choose $\delta v \sim S_3^{1/3}$ or more generically $S_p^{1/p}$ with p any integer? In the absence of intermittency all p's are equivalent, but when intermittency is on, every value of p would provide a different, yet equally acceptable, answer. At this stage, the specific choice of p becomes a matter of taste, or, better said, of how much emphasis is to be placed on the most-singular structures (those sampled by highest p's). The correct recipe is probably a weighted average of all possible

p's, the weighting factor (most likely a space–time-dependent function) being basically unknown.

Another scenario is to assume that intermittency ignores the details of the dissipation mechanisms, in which case the idea of including intermittency effects on LES models dissolves on its own.

The discussion of the sophisticated developments of LES modeling is beyond the scope of this paper. Here we shall focus exclusively on the specific question of the interrelation between dissipation and intermittency. Tackling this question within the true three-dimensional Navier–Stokes context is a very daunting task, in view of the enormous amount of data to be produced and carefully analyzed.

It therefore makes sense to attack the problem within the context of simplified dynamical models sharing as much physics as possible with Navier–Stokes equations while giving away most of its computational complexity.

III. DYNAMICAL MODELS OF TURBULENCE

In the recent years, an interesting vehicle for this kind of investigations has emerged in the form of the so-called ''shell models.''[12–16]

Shell models work on the principle of collapsing the whole set of degrees of freedom lying in a finite shell $k_n < k < k_{n+1}$, with $k_n = 2^n k_0$, into a handful (one or two) of representative modes.

The dynamics of such a low-dimensional representation is subsequently arranged in such a way as to preserve the nonlinear structure of the NS equations; of course all genuinely three-dimensional effects are lost in the process.

The most popular shell model is the Gledzer–Ohkitani–Yamada (GOY) model where only one (complex) mode per shell is used. Recently, a new class of model has been introduced in which, by allowing two complex modes per shell, a second invariant with a close connection to NS helicity can be defined.

The statistical properties of such a helical shell model have been recently explored in depth,[16,21] the major finding being that it possesses a rich physics and it exhibits a striking similarity (in a statistical sense) to NS intermittency. Shell models do nonetheless miss all spatial effects, since they can be regarded as zero-dimensional field models based on space-filling coherent planar waves.

The dynamics of our helical shell model is governed by the following evolution equation:

$$\dot{u}_n^{\pm} = ik_n(a u_{n+1}^{\pm} u_{n+2}^{\mp} + b u_{n-1}^{\pm} u_{n+1}^{\mp} + c u_{n-1}^{\mp} u_{n-2}^{\mp}) - D_n^{\pm} u_n^{\pm} + \delta_{n,n_0} f^{\pm}, \tag{8}$$

where $u_n^{\pm}$ represent the positive/negative helicity carriers respectively and $f^{\pm}$ is a large scale forcing. In the previous equations the term $D_n^{\pm}$ is a function which reproduces the effects of viscous damping at scale n. In the usual case where only molecular viscosity, ν, is acting we have

$$D_n^{\pm} = \nu k_n^2.$$

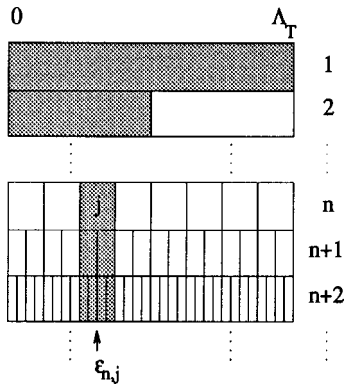

FIG. 1. A picture of the hierarchical system, covering the one-dimensional interval $[0,\Lambda_T]$.

Upon choosing $a=1$, $b=-\frac{5}{12}$ and $c=-\frac{1}{24}$, the above equations are readily shown to conserve the following (energy, helicity) invariants:

$$E=\sum_{n=0}^{N} (|u_n^+|^2+|u_n^-|^2); \quad H=\sum_{n=0}^{N} k_n(|u_n^+|^2-|u_n^-|^2). \tag{9}$$

Real turbulence consists of localized eddies of all sizes that interact, merge, and break up locally:[22] the physical picture is that of a large eddy which decays into smaller eddies. The number of degrees of freedom in such a field problem in d dimensions grows with the wave number as $N(k)\sim k^d$ ($d=0$ in shell models). The first step in reproducing this kind of hierarchical structure is to transform a *chain*-model into a *tree*-model with $d=1$.[10] This is achieved by letting the number of degrees of freedom grow with the shell index n as 2^n.

As in the original shell models, this tree model must be in some sense reminiscent of the NS equations. It can be regarded as describing the evolution of the coefficients of an orthonormal wavelets expansion of a one-dimensional projection of the velocity field $v(x,t)$:

$$v(x,t)=\sum_{n,j} \hat{v}_{n,j}(t)\psi_{n,j}(x). \tag{10}$$

Here $\psi_{n,j}(x)$ is a complete orthonormal set of wavelets generated from an analyzing wavelet $\psi_{0,0}(x)$ by discrete translations and dilatation:

$$\psi_{n,j}(x)=2^{n/2}\psi_{0,0}(2^n x-j). \tag{11}$$

Each dynamical variable $\hat{v}_{n,j}$ describes fluctuations in a box of length $l_n=2^{-n}$, centered in the interval ranging from $(j-1)l_n$ to jl_n. At each scale n there are 2^{n-1} boxes, covering a total length $\Lambda_T=2^{n-1}l_n=\frac{1}{2}$ (see Fig. 1).

For the sake of convenience we define the tree model in terms of *density* variables, $u_{n,j}$, which would correspond to $\hat{u}_{n,j}=2^{n/2}\hat{v}_{n,j}$ in a wavelets expansion. In this notation, $|u_{n,j}|^2$ represents the energy density in a flow structure of length $l_n=2^{-n}$ and spatially labeled by the index j.

In this tree structure, each variable $u_{n,j}$ continues to interact with the nearest and next-nearest levels, as in Eq. (8);

however, many possibilities are now opened by the presence of many horizontal degrees of freedom localized on each shell.

The simplest choice is depicted in Fig. 2, where a portion of the tree structure is shown and the evolving in time variable, $u_{n,j}$, is represented by a black ball. In the figure, solid lines connect interacting balls (variables).

The dynamical tree equations are as follows:

$$\dot{u}^{+}_{n,j} = -D^{+}_{n} u^{+}_{n,j} + \delta_{n,n_0} F^{+} + ik_n \left\{ \frac{a}{4} [u^{+}_{n+1,2j-1}(u^{-}_{n+2,4j-3} + u^{-}_{n+2,4j-2}) + u^{+}_{n+1,2j}(u^{-}_{n+2,4j-1} + u^{-}_{n+2,4j})] + \frac{b}{2} [u^{+}_{n-1,\bar{j}} \times (u^{-}_{n+1,2j-1} + u^{-}_{n+1,2j})] + c[u^{-}_{n-2,\bar{\bar{j}}} u^{-}_{n-1,\bar{j}}] \right\}^{*}, \tag{12}$$

where, in the indexes, $\bar{\bar{j}}$ is the integer part of $((j+3)/4)$ and $\bar{j}$ is the integer part of $((j+1)/2)$.

Again, in the standard case with only molecular viscosity we have $D^{\pm}_{n} = \nu k_n^2$. The interaction terms with coefficients $a/4$, $b/2$, and c are depicted in Figs. 2 respectively. The same equation holds, with all helicities reversed, for $\dot{u}^{-}_{n,j}$. The numerical values of a, b, and c are the same of the original helical shell.

To make contact with the issue of intermittency–dissipation interrelation, we shall replace the viscous coefficients $D^{\pm}_{n}$ of Eqs. (8) and (12) with an ''effective viscosity'' term, $\mathcal{D}^{\pm}_{n}$, which now acquires both nontrivial dependencies on time and shell indexes. It reads for the two cases

$$\mathcal{D}^{\pm}_{n}(t) \equiv \nu_S (\delta_{n,N} + \delta_{n,N-1}) \frac{|u^{\pm}_{n}|}{k_n} k_n^2; \qquad \mathcal{D}^{\pm}_{n,j}(t) \equiv \nu_S (\delta_{n,N} + \delta_{n,N-1}) \frac{|u^{\pm}_{n,j}|}{k_n} k_n^2, \tag{13}$$

where ν_S has been empirically chosen of the order $\nu_s \sim 1$ mainly on account of conceptual simplicity. Again, an attempt to match as closely as possible Navier–Stokes practice would have suggested $\nu_S \sim 0.1$, but since this prescription is mostly empirical (as is well known from direct simulation of turbulent shear flows, the ''constant'' C_S is not a constant at all but must be heavily adjusted close to the walls as well as in the central region of the flow), we conclude that the plain value $\nu_S = 1$ meets optimally the criteria of simplicity and conceptual transparency.

This ''subgrid scale'' term is clearly patterned after the simplest NS effective viscosity model. The only difference is that due to the short-range interactions of our shell models, the subgrid modeling is applied only to the last and last-but-one shells k_N, and k_{N-1}.

Our sub-grid closure combines features of the classical Smagorinski large eddy simulation model and the so-called hyperviscosity models used in the direct spectral simulation of incompressible turbulence. This is consistent with the double-locality in real and momentum space of the wavelet basis functions.

The two methods are quite different in scope and formulation: Smagorinski works in real space as a local, dynamic, effective viscosity responding to the local stress so as to mimic the effects of unresolved scales on the resolved ones. Hyperviscosity is local in k-space, static, and does *not* aim at representing the effects of unresolved scales, but simply at reducing the size of the dissipative region so as to take full advantage of the grid resolution.

As far as connections with the NS physics is involved, we remind the reader that it is well known from pseudospectral simulations that the effective viscosity $\nu_E \equiv T_{>}(k)/k^2 E(k)$, where $T_{>}(k)$ is the energy transferred from wave number k beyond the cutoff $k_{>}$, is an increasing function of the wave number k. This dependence is reasonably fitted by a constant plus a power law in k. This suggests that a Smagorinski-like model (roughly constant in k space) plus an hyperviscous term k^{2p} $p>2$ might be a good choice for a sub-grid scale model.[23] Our dissipative term displays both aforementioned features: hyperviscous behavior reflects into the presence of dissipation only in the last and last-but-one cell (''cusp-like'' shape), whereas a Smagorinski-like argument is manifestly behind the specific prescription of the dissipative terms, namely $\Delta^2 S \to k^{-2} k u(k)$. The analogy stops here, though. At variance with real-space Smagorinski, we do not have *any* dissipation below k_{N-1}, and differently from usual hyperviscous operators, we do not have any high power k^{2p}. Trying to push this qualitative analogy further, in an attempt to make full contact with Navier–Stokes, would be a purposeless, possibly obscuring, exercise since we are well aware that the present shell model does not reproduce the physics of fluids as described by the Navier–Stokes equations.

On the other hand, this qualitative analogy serves well

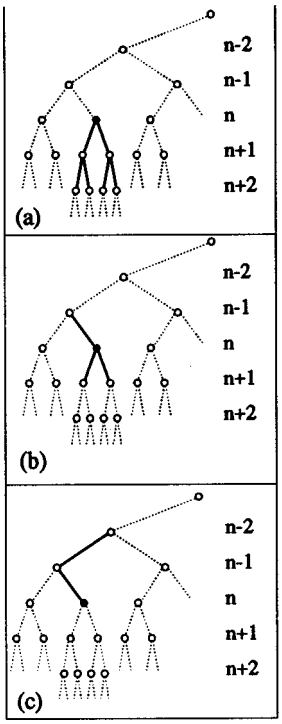

FIG. 2. Type of interaction [(a)–(c)] for the tree model.

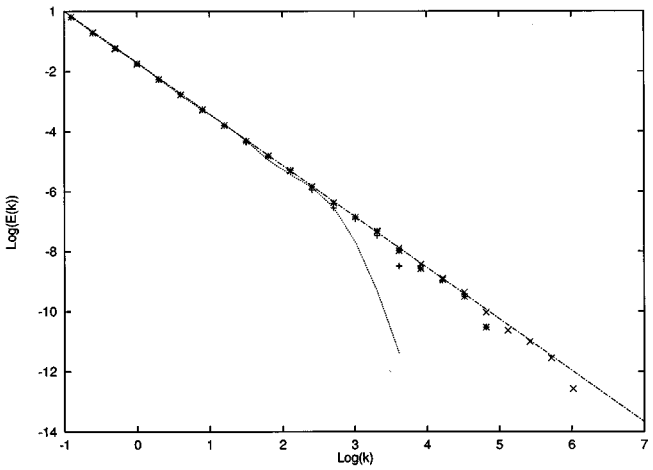

FIG. 3. Log–log plot of the energy spectra versus the wave number for the chain model with eddy-viscosity at three different resolutions $N=16$ (pluses), $N=20$ (stars), and $N=24$ (crosses). For the sake of comparison the case with normal viscosity is also reported for $N=16$ (dotted line). The straight line has slope $-1-\zeta_2$.

our main purpose, namely asking the question of whether or not intermittency ''feels'' the way that energy is dissipated, in the general context of nonlinear dynamical systems with many degrees of freedom.

IV. RESULTS

As previously observed, the common aim of any turbulence model or large-eddy simulation is to capture the effects of unresolved scales on the resolved ones. In practice, this means that once the subgrid model is appropriately tuned, the resolved scales should be basically unaffected by grid resolution.[24]

This is indeed the case for our subgrid model. In Fig. 3 we show the energy spectra for the chain model with eddy viscosity at three different resolutions $N=16,20,24$. For the sake of comparison the case with normal viscosity is also reported for $N=16$. As a first remark, we note that the presence of the eddy viscosity considerably widens the inertial regime which extends deep down to the last-but-one shell. Moreover, the slope of the spectrum is basically the same independent of the number of shells used, which is exactly the property we were looking for.

We note that is not the case with normal viscosity, where, in order to widen the inertial range, it is necessary to lower the value of the viscosity so as to increase the Reynolds number. Of course resolution must be increased accordingly so as to resolve the dissipative region in order to prevent numerical problems. In order to gain a more quantitative assessment on the grid independence of our results, we shall evaluate the scaling exponents ζ_p up to $p=8$. In Table I we show the scaling exponents for the chain model with (ζ_p^S) and without (ζ_p^D) subgrid eddy viscosity (''*S*'' stands for subgrid and ''*D*'' for direct). The simulation was run with 16 shells for about 10^5 eddy turnover time of the largest scale.

The first remark is that in both cases a significant departure from Kolmogorov K41 law is observed, i.e., the subgrid model does *not* destroy intermittency.

More precisely, ζ_p^S and ζ_p^D coincide within statistical error, which means that intermittency survives and it is basically insensitive to eddy viscosity. The scaling exponents reported in Table I have been computed as a direct fit on structure functions in k space. Statistical accuracy is generally good due to the large number of shells available.

It is nonetheless interesting to note that such an estimate is even more accurate using extended self-similarity (ESS), namely by representing the pth-order structure function versus the third-order one. In Fig. 4 we show S_6 as a function of S_3 for the case *with* and *without* eddy viscosity. As we see, the case of normal viscosity displays two distinct slopes in the inertial and dissipative regimes, whereas with eddy viscosity this slope is everywhere the ''inertial'' one.

This suggests that the combined use of LES models and ESS analysis might prove useful for the analysis of scaling exponents in more complex simulations.

We now move on to the discussion of the results with the tree model.

Before analyzing the results it is worthwhile to point out that the tree formulation makes more contact with the usual Navier–Stokes Smagorinski eddy viscosity in that it intro-

TABLE I. Scaling exponents for the chain model with eddy viscosity, $\zeta^S(p)$, for $N=16,20,24$ and with normal viscosity, $\zeta^D(p)$, with $N=16$.

p	$\zeta^D(p)$ $N=16$	$\zeta^S(p)$ $N=16$	$\zeta^S(p)$ $N=20$	$\zeta^S(p)$ $N=24$
1	0.368±0.007	0.367±0.002	0.367±0.002	0.367±0.001
2	0.700±0.005	0.699±0.002	0.699±0.002	0.699±0.001
3	1.0±0.0	1.0±0.0	1.0±0.0	1.0±0.0
4	1.271±0.007	1.273±0.004	1.268±0.007	1.272±0.003
5	1.52±0.01	1.522±0.007	1.50±0.02	1.518±0.008
6	1.74±0.02	1.75±0.01	1.71±0.04	1.74±0.02
7	1.94±0.04	1.97±0.01	1.90±0.07	1.96±0.02
8	2.12±0.05	2.17±0.02	2.08±0.09	2.16±0.03
9	2.29±0.08	2.37±0.02	2.3±0.1	2.36±0.04
10	2.5±0.1	2.57±0.03	2.4±0.1	2.56±0.04
11	2.6±0.1	2.76±0.04	2.6±0.1	2.76±0.05
12	2.8±0.2	2.96±0.06	2.8±0.2	2.96±0.06

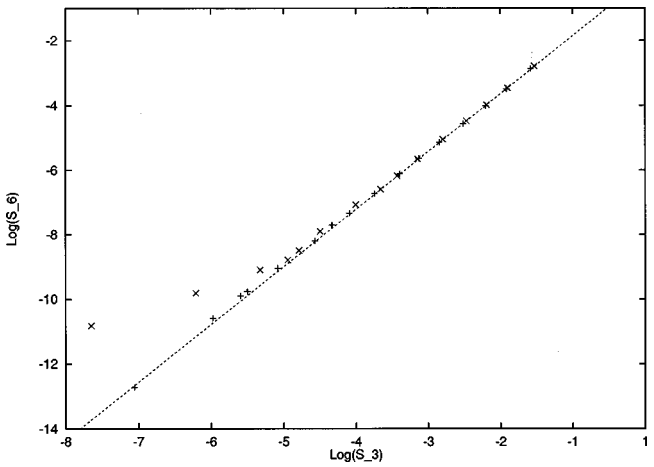

FIG. 4. Log–log plot of S_6 versus S_3 for the chain model with $N=16$ *with* eddy-viscosity (pluses) and *without* eddy viscosity (crosses). The straight line has slope ζ_6.

duces a spatial dependence in the model. It is therefore of interest to investigate how this spatial dependence is going to affect the physical picture described so far.

The physical picture as it comes from the analysis of intermittency in the inertial range is pretty much the same as with the chain model: in particular, intermittency survives and shows no dependency on whether a subgrid closure is used or not (see Table II).

The actual numerical values of the scaling exponents are slightly higher than in the chain case, and this is hardly surprising since the tree model allows for spatial redistribution of the energy flow so that spotty events are somehow smeared out.

Refined Kolmogorov hypothesis (RKH)

In a tree structure we may also test the robustness of the RKH. As previously discussed, the RKH links statistical properties of the energy dissipations, $\epsilon(r)$ averaged on a box of size r, to the inertial range fluctuations, $\delta v(r)$:

$$\epsilon_r(x)=\frac{1}{r^3}\int_{\Lambda_r(x)}\epsilon(y)d^3y\sim\frac{(\delta v(r))^3}{r}. \tag{14}$$

In particular one may therefore write

$$\langle\epsilon_r(x)^{p/3}\rangle\sim S_p(r)/r^{p/3}. \tag{15}$$

TABLE II. Scaling exponents for the tree model *with* eddy viscosity $\zeta^S(p)$, and *without* eddy-viscosity, $\zeta^D(p)$.

p	$\zeta^D(p)$	$\zeta^S(p)$
1	0.348±0.005	0.347±0.005
2	0.682±0.005	0.681±0.005
3	1.00	1.00
4	1.303±0.006	1.302±0.006
5	1.59±0.01	1.59±0.01
6	1.86±0.02	1.85±0.02
7	2.12±0.03	2.10±0.03
8	2.35±0.03	2.32±0.03

The first step in constructing the energy dissipation field in any tree model[10] is to consider the energy dissipation *density*, $\eta_{n,j}$, in the structure covering the region $\Lambda_j(n)$ of length 2^{-n}, centered in the spatial site labeled by j. These structures are represented by boxes in Fig. 1.

In the case with eddy viscosity we have

$$\eta_{n,j}=\mathcal{D}^{\pm}_{n,j}(|u^+_{n,j}|^2+|u^-_{n,j}|^2). \tag{16}$$

Let us notice that in the above expression only the last and the last-but-one shells give nonzero contribution, different from the case when a molecular viscosity acting on all scales is considered [the latter would correspond to the choice of D instead of $\mathcal{D}$ in Eq. (16)].

The total energy dissipation density, $\epsilon=(1/\Lambda_T)\times\int_{\Lambda_T}\epsilon(x)\,dx$, where Λ_T is the total space length, is, by definition, the sum of all these contributions (sum over boxes at all scales in Fig. 1):

$$\epsilon=\sum_{n,j}2^{-n}\eta_{n,j}. \tag{17}$$

On the other hand, in order to study the scaling properties of the energy dissipation field, one has to disentangle in ϵ the contributions coming from the coarse-grained energy dissipation field ϵ_r.

In our formulation, we can then rewrite

$$\epsilon=\frac{1}{\Lambda_T}\int_{\Lambda_T}\epsilon(x)dx=\frac{1}{2^{n-1}}\sum_{j=1}^{2^{n-1}}\left(\frac{1}{2^{-n}}\int_{\Lambda_{j(n)}}\epsilon(x)\,dx\right)$$
$$=\frac{1}{2^{n-1}}\sum_{j=1}^{2^{n-1}}\epsilon_{n,j}, \tag{18}$$

where the last expression is independent of n and the $\epsilon_{n,j}$'s are the coarse-grained energy dissipation densities, obtained as averages over spatial regions of length 2^{-n}. Note that the average density $\epsilon_{n,j}$ over $\Lambda_j(n)$ does not coincide simply with the density $\eta_{n,j}$ of the structure living in $\Lambda_j(n)$, namely,

$$\epsilon_{n,j}=\eta_{n,j}+\sum_{m<n}\eta_{m,k(m)}+\sum_{m>n}\langle\eta_{m,k(m)}\rangle_{I(m)}. \tag{19}$$

Here, in the second (third) term on the rhs we take into account density contributions coming from larger (smaller) scale structures (as an example, all regions contributing to the definition of $\epsilon_{n,j}$ are represented as shadowed boxes in Fig. 1). The index $k(m)$ in the second term on the rhs labels the location of larger scale structures containing the region $\Lambda_j(n)$ under consideration (shadowed boxes with $m<n$ in Fig. 1). In the third term, an average is performed over $k(m)\in I(m)$, where $I(m)$ labels the set of structures contained in $\Lambda_j(n)$, for any $m>n$ [in Fig. 1, $I(m)$ labels the two boxes at $n+1$, the four boxes at $n+2$, and so on].

The best spatially resolved energy dissipation field is for $n=N$:

$$\epsilon_{N,j}=\sum_{m\leqslant N}\eta_{m,k(m)};\quad j=1,...,2^{N-1}. \tag{20}$$

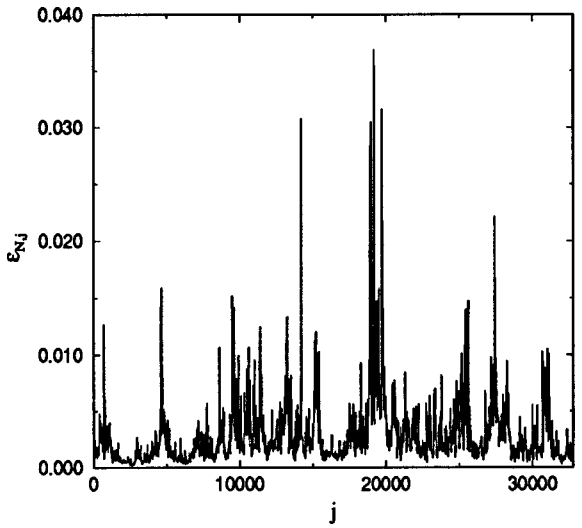

FIG. 5. Instantaneous configuration of the coarse-grained energy dissipation density field, ϵ_{Nj}, over the last level sites.

In Fig. 5, the instantaneous values assumed by $\epsilon_{N,j}$ in the $N_T/2=32\,768$ locations of the last level are shown. The chaotic, intermittent character of this spatial signal is evident.

In Table III we show that the RKH is still well verified also in the subgrid modeling picture, proving to be a robust and nontrivial property connecting small scales and inertial range scales in turbulent flows.

V. CONCLUSIONS

Summarizing, we have presented a detailed study of dynamical eddy viscosity models in chain and tree shell models of fluid turbulence.

The main goal was to check whether or not the inertial range properties are affected by the way the flow dissipates energy. We found a strong robustness of inertial range intermittency once the proposed eddy viscosity model is implemented in our shell models.

The eddy viscosity closure that we have adopted may also be regarded as a multiplicative closure of the small-scale equations of motion, i.e., it is tantamount to assuming that $u_{n+1}\sim a_{n+1,n}\cdot u_n$ with an appropriate multiplicative random coefficient $a_{n+1,n}$. The fact that intermittency is not affected by the details of the eddy-viscosity models indicates that fine tuning of the coefficients in front to the eddy-viscosity term is probably not demanded. Nevertheless, oversimplified eddy viscosity models based only on dimensional analysis would probably fail on the same goal, due to their inability to dissipate violent intermittent bursts.

TABLE III. Slope, $\chi(p)$, of the log–log plot of Eq. (15) for the tree model for $p=1,...,10$. Notice that when $\chi(p)=1$ the RKH is verified.

p	$\chi(p)$
1	1.00 ± 0.02
2	1.001 ± 0.008
4	1.000 ± 0.007
5	1.000 ± 0.01
6	1.00 ± 0.02
7	1.01 ± 0.03
8	1.02 ± 0.04
9	1.02 ± 0.06
10	1.02 ± 0.07

Moreover, the usual phenomenological RKH which links energy dissipation statistics with inertial range properties is also largely unaffected by this kind of modeling. Whether the same universality is present in real Navier–Stokes equations is still a matter of debate in the scientific community.[3,4,7] Certainly, in order to properly test this question it is always necessary to have a fine resolution of the smallest resolved scales and, more important, a detailed study of the dependence on finite Reynolds effects. Indeed, in many cases, bottleneck phenomena close to the subgrid modeling scales may arise.[25] These bottleneck effects may introduce a finite-Reynolds bias which could lead to erroneous conclusions on the dependence of inertial range statistics on eddy viscosity or hyperviscosity modeling.

The problem of coherent structure should be carefully addressed in NS equations, especially in cases of nonhomogeneous and nonisotropic turbulence. This kind of flow cannot be easily investigated by using surrogates as ''shell models.'' In these cases also nonlocal interactions may play some major role in determining overall statistical properties of the flow. Such questions are beyond the main purpose of this work.

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