Nonperturbative spectrum of anomalous scaling exponents in the anisotropic sectors of passively advected magnetic fields

Itai Arad,[1] Luca Biferale,[2] and Itamar Procaccia[1]
[1]*Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*
[2]*Department of Physics and INFM, University of Rome "Tor Vergata," Via della Ricerca Scientifica 1, I-00133 Rome, Italy*
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We address the scaling behavior of the covariance of the magnetic field in the three-dimensional kinematic dynamo problem when the boundary conditions and/or the external forcing are not isotropic. The velocity field is Gaussian, space homogeneous, and δ correlated in time, and its structure function scales with a positive exponent ξ. The covariance of the magnetic field is naturally computed as a sum of contributions proportional to the irreducible representations of the SO(3) symmetry group. The amplitudes are nonuniversal, determined by boundary conditions. The scaling exponents are universal, forming a discrete, strictly increasing, spectrum indexed by the sectors of the symmetry group. When the initial mean magnetic field is zero, no dynamo effect is found, irrespective of the anisotropy of the forcing. The rate of isotropization with decreasing scales is fully understood from these results.

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I. INTRODUCTION

The aims of this paper are twofold. First, we are interested in the statistical properties of magnetic fields advected by turbulent velocity fields. Such magnetic fields possess a "self-stretching" term that is absent in the context of advected passive scalars (for a general introduction, see Ref. [1]). Thus a dynamo effect may exist, and its relation to intermittency and anomalous scaling needs to be addressed. Second, we want to focus on the anisotropic nature of turbulence: generically turbulence is forced by agents that are neither isotropic nor homogeneous, but most of the fundamental theories regarding universal scaling properties consider an ideal model of isotropic turbulence. In the case of a magnetic field advected by a Gaussian, space homogeneous, δ-correlated velocity field with nontrivial spatial scaling we can present an exact (nonperturbative) solution of the full spectrum of anomalous scaling exponents of all the anisotropic contributions to the covariance of the magnetic field. We can thus offer a precise picture of the rate of isotropization upon diminishing scales, assess the importance of anisotropy for "inertial range" scaling, etc.

The equation of motion of a magnetic field $\boldsymbol{B}(\boldsymbol{r},t)$ reads

$$\begin{aligned}\partial_t \boldsymbol{B}(\boldsymbol{r},t)+\boldsymbol{u}(\boldsymbol{r},t)\cdot\boldsymbol{\nabla}\boldsymbol{B}(\boldsymbol{r},t)\\=\boldsymbol{B}(\boldsymbol{r},t)\cdot\boldsymbol{\nabla}\boldsymbol{u}(\boldsymbol{r},t)+\kappa\nabla^2\boldsymbol{B}(\boldsymbol{r},t)+\boldsymbol{f}(\boldsymbol{r},t),\end{aligned}\quad(1.1)$$

where $\boldsymbol{u}$ is the advecting velocity field, $\boldsymbol{f}$ is the external forcing, and κ is the magnetic diffusivity. We address a model in which the velocity is taken Gaussian, space homogeneous, isotropic, δ correlated in time, and its correlation function is

$$\begin{aligned}\langle u^\alpha(\boldsymbol{r},t)u^\beta(\boldsymbol{r}',t')\rangle&=\delta(t-t')D^{\alpha\beta}(\boldsymbol{r}-\boldsymbol{r}')\\&=\delta(t-t')[D^{\alpha\beta}(0)-S^{\alpha\beta}(\boldsymbol{r})].\end{aligned}\quad(1.2)$$

The structure function $\boldsymbol{S}$ scales with exponent ξ, $0\leqslant\xi\leqslant2$:

$$S^{\alpha\beta}(\boldsymbol{R})=DR^{\xi}\left[(\xi+2)\delta^{\alpha\beta}-\xi\frac{R^\alpha R^\beta}{R^2}\right],\quad \lambda\ll R\ll\Lambda.\quad(1.3)$$

On the other hand, the forcing $\boldsymbol{f}$ is taken here to be Gaussian, space homogeneous, δ correlated in time, but *nonisotropic*. The correlation function of the forcing has compact support in $\boldsymbol{k}$ space in an interval $0\leqslant k\leqslant 1/L$, where L is the outer scale of the forcing $\boldsymbol{f}$. We denote $F^{\alpha\beta}(\boldsymbol{R})\equiv\langle f^\alpha(\boldsymbol{R})f^\beta(0)\rangle$.

We are interested in the properties of the covariance of $\boldsymbol{B}$, $C^{\alpha\beta}(\boldsymbol{R},t)$,

$$C^{\alpha\beta}(\boldsymbol{R},t)\equiv\langle B^\alpha(\boldsymbol{R},t)B^\beta(0,t)\rangle,\quad(1.4)$$

and eventually in the stationary quantity $C^{\alpha\beta}(\boldsymbol{R})$ which is obtained in the stationary state if the forcing is balanced by dissipation. The calculation of this object in an *isotropic* ensemble was presented by Vergassola [2]. The anisotropic problem was addressed recently by Lanotte and Mazzino [3]. In the latter study, the covariance [Eq. (1.4)] was not properly expanded in terms of irreducible representation of the SO(3) symmetry group, and therefore an apparent mixing of the different sectors was found. As a result the authors had to tackle an infinite set of equations for all the sectors of the symmetry group. We show below that this mixup is spurious, originating from an improper expansion. In order to solve the infinite linear system the authors were forced to assume the existence of a hierarchy between exponents belonging to different sectors, and then only *a posteriori* to check the correctness of their assumption. In this way the calculation ends up with one correct set of exponents, as shown below by using the proper expansion. We compute additional exponents that were not considered in Ref. [3] because of their choice of forcing. We will also concern ourselves with the issues of the dynamo effect and the attainment of a stationary solution for Eq. (1.4).

The structure of this paper is as follows: in Sec. II, after presenting the equations of motion of the covariance, we expand the solutions in terms of basis functions of the SO(3) symmetry group. In Sec. III the above expansion is used to obtain the matrix representation of the linear operator which determines the dynamics of the covariance. In Sec. IV we use this matrix representation to show the absence of a dynamo effect in the anisotropic sectors of the covariance. Section V is devoted to a calculation of the anomalous scaling exponents in the anisotropic sectors, and Sec. VI offers a summary and a discussion.

II. BASIC EQUATIONS AND THE DECOMPOSITION IN TERMS OF BASIS FUNCTIONS

The equation of motion of the covariance were derived by the authors of Ref. [3] with the final result

$$\begin{aligned}\partial_t C^{\alpha\beta} &= S^{\mu\nu}\partial_\mu\partial_\nu C^{\alpha\beta} - [(\partial_\nu S^{\mu\beta})\partial_\mu C^{\alpha\nu} + (\partial_\nu S^{\alpha\mu})\partial_\mu C^{\nu\beta}] \\ &\quad + (\partial_\mu\partial_\nu S^{\alpha\beta})C^{\mu\nu} + 2\kappa\nabla^2 C^{\alpha\beta} + F^{\alpha\beta} \\ &\equiv \hat{T}^{\alpha\beta}_{\sigma\rho}C^{\sigma\rho} + F^{\alpha\beta},\end{aligned} \tag{2.1}$$

$$\partial_\alpha C^{\alpha\beta} = 0, \tag{2.2}$$

where the last equation follows from the solenoidal condition for the magnetic field. It is advantageous to decompose the covariance $C^{\alpha\beta}$ in terms of basis functions that block diagonalize the angular part of the operator $\hat{\boldsymbol{T}}$. These basis functions are implied by the symmetries of $\hat{\boldsymbol{T}}$. Since this operator contains only isotropic differential operators and contractions with either $\delta^{\alpha\beta}$ or $R^\alpha R^\beta$, it is invariant to all rotations [4]. Accordingly, the natural basis functions should belong to irreducible representation of the SO(3) symmetry group, and can be indexed by pairs of indices j,m, where $j = 0,1,2,\ldots$ and $-j \leqslant m \leqslant j$. We are going to refer to solutions of Eq. (2.1) that belong to irreducible representation with a definite j,m as the "j,m sector." The operator $\hat{\boldsymbol{T}}$ leaves such sectors invariant. In addition, $\hat{\boldsymbol{T}}$ is invariant to the parity transformation $\boldsymbol{R}\to-\boldsymbol{R}$, and to the index permutation $(\alpha,\mu)\Leftrightarrow(\beta,\nu)$. Accordingly, $\hat{\boldsymbol{T}}$ can be further block diagonalized into blocks with definite parity and symmetry under permutations.

In light of these consideration, we seek solutions of the form

$$C^{\alpha\beta}(\boldsymbol{R},t) = \sum_{q,j,m} a_{q,jm}(|\boldsymbol{R}|,t)B^{\alpha\beta}_{q,jm}(\hat{\boldsymbol{R}}), \tag{2.3}$$

where $\hat{\boldsymbol{R}}\equiv\boldsymbol{R}/R$ and $\boldsymbol{B}^{\alpha\beta}_{q,jm}(\hat{\boldsymbol{R}})$ are tensor functions on the unit sphere, which belong to the sector j,m of the SO(3) symmetry group. The index q enumerates different tensor functions belonging to the same sector. While for scalar functions on the sphere there exist only one spherical harmonic Y_{jm} in each sector, for the second rank tensor functions on the sphere there exist nine different tensors [4]. The additional symmetries under parity and index permutation group into four subgroups with four tensors, two tensors, two tensors, and one tensor, respectively. With $\Phi_{jm}(\boldsymbol{R})\equiv R^j Y_{jm}(\hat{\boldsymbol{R}})$, in the notation of Ref. [4], the 4-group (denoted below as subset I) is

$$\begin{aligned} B^{\alpha\beta}_{9,jm}(\hat{\boldsymbol{R}}) &\equiv R^{-j-2}R^\alpha R^\beta\Phi_{jm}(\boldsymbol{R}), \\ B^{\alpha\beta}_{7,jm}(\hat{\boldsymbol{R}}) &\equiv R^{-j}(R^\alpha\partial^\beta + R^\beta\partial^\alpha)\Phi_{jm}(\boldsymbol{R}), \\ B^{\alpha\beta}_{1,jm}(\hat{\boldsymbol{R}}) &\equiv R^{-j}\delta^{\alpha\beta}\Phi_{jm}(\boldsymbol{R}), \\ B^{\alpha\beta}_{5,jm}(\hat{\boldsymbol{R}}) &\equiv R^{-j+2}\partial^\alpha\partial^\beta\Phi_{jm}(\boldsymbol{R}). \end{aligned} \tag{2.4}$$

These are all symmetric in α,β, and have a parity of $(-1)^j$. The 2-groups are denoted, respectively, as subsets II and III:

$$B^{\alpha\beta}_{8,jm}(\hat{\boldsymbol{R}}) \equiv R^{-j-1}[R^\alpha\epsilon^{\beta\mu\nu}R_\mu\partial_\nu + R^\beta\epsilon^{\alpha\mu\nu}R_\mu\partial_\nu]\Phi_{jm}(\boldsymbol{R}), \tag{2.5}$$

$$B^{\alpha\beta}_{6,jm}(\hat{\boldsymbol{R}}) \equiv R^{-j+1}[\epsilon^{\beta\mu\nu}R_\mu\partial_\nu\partial^\alpha + \epsilon^{\alpha\mu\nu}R_\mu\partial_\nu\partial^\beta]\Phi_{jm}(\boldsymbol{R}), \tag{2.6}$$

$$B^{\alpha\beta}_{4,jm}(\hat{\boldsymbol{R}}) \equiv R^{-j-1}\epsilon^{\alpha\beta\mu}R_\mu\Phi_{jm}(\boldsymbol{R}), \tag{2.7}$$

$$B^{\alpha\beta}_{2,jm}(\hat{\boldsymbol{R}}) \equiv R^{-j+1}\epsilon^{\alpha\beta\mu}\partial_\mu\Phi_{jm}(\boldsymbol{R}). \tag{2.8}$$

The first pair is symmetric to α,β exchange, and has a parity $(-1)^{j+1}$. The second has the same parity but is antisymmetric to α,β exchange. The remaining basis function is $B^{\alpha\beta}_{3,jm}(\hat{\boldsymbol{R}})\equiv R^{-j}(R^\alpha\partial^\beta - R^\beta\partial^\alpha)\Phi_{jm}(\boldsymbol{R})$, which is antisymmetric to α,β exchange, with parity $(-1)^j$. This will be denoted as subset IV. In Ref. [4] it was proven that this basis is complete, and indeed transforms under rotations as required for a j,m sector.

It should be noted that not all subsets contribute for every value of j. Space homogeneity implies the obvious symmetry of the covariance:

$$C^{\alpha\beta}(\boldsymbol{R},t) = C^{\beta\alpha}(-\boldsymbol{R},t). \tag{2.9}$$

Therefore, representations symmetric to α,β exchange must also have even parity, while antisymmetric representations must have odd parity. Accordingly, even j's are associated with subsets I and III, and odd j's are associated with subset II. We show below that subset IV cannot contribute to this theory due to the solenoidal constraint.

III. MATRIX REPRESENTATION OF THE OPERATOR $\hat{T}$

Having the angular basis functions we seek the representation of the operator $\hat{\boldsymbol{T}}$ in this basis. In such a representation $\hat{\boldsymbol{T}}$ is a differential operator with respect to $|\boldsymbol{R}|$ only. In Appendix A we demonstrate how $\hat{\boldsymbol{T}}$ mixes basis functions within a given subset, but not between the subsets — as is expected in Sec. II. In finding the matrix representation of $\hat{\boldsymbol{T}}$ we are aided by the incompressibility constraint. Consider first subset I with four basis functions [Eqs. (2.4)] in a given j,m sector. To simplify the notation we will denote the amplitude $|\boldsymbol{R}|$ simply as R, and redenote the a coefficients according to $a(R)\equiv a_{9,jm}(R)$, $b(R)\equiv a_{7,jm}(R)$, $c(R)$

$\equiv a_{1,jm}(R)$, and $d(R)\equiv a_{5,jm}(R)$. Primes will denote differentiation with respect to R.

In this basis the operator $\hat{\boldsymbol{T}}$ takes on the form

$$\hat{\boldsymbol{T}}\left[\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}\right]=\boldsymbol{T}_1\begin{pmatrix} a''\\ b''\\ c''\\ d''\end{pmatrix}+\boldsymbol{T}_2\begin{pmatrix} a'\\ b'\\ c'\\ d'\end{pmatrix}+\boldsymbol{T}_3\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}. \tag{3.1}$$

On the right-hand side we have matrix products. In addition, the solenoidal condition implies the following two constraints on a, b, c, and d (cf. the Appendix of Ref. [4]):

$$\begin{aligned} 0&=a'+2\frac{a}{x}+jb'-j^2\frac{b}{x}+c'-j\frac{c}{x},\\ 0&=b'+3\frac{b}{x}+\frac{c}{x}+(j-1)d'-(j-1)(j-2)\frac{d}{x}. \end{aligned} \tag{3.2}$$

Using these conditions one can bring $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ to diagonal forms:

$$\boldsymbol{T}_1=2(DR^{\xi}+\kappa)\begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1\end{pmatrix},$$

$$\boldsymbol{T}_2=\frac{4}{R}[(DR^{\xi}+\kappa)+\xi DR^{\xi}]\begin{pmatrix} 1&&&\\ &1&&\\ &&1&\\ &&&1\end{pmatrix}. \tag{3.3}$$

$\boldsymbol{T}_3$ can be written in the form

$$\boldsymbol{T}_3=DR^{\xi-2}\boldsymbol{Q}(j,\xi)+\kappa R^{-2}\boldsymbol{Q}(j,0) \tag{3.4}$$

where the four columns of $\boldsymbol{Q}(j,\xi)$ are

$$\begin{pmatrix} -(2+\xi)(j+2)(j+3)+2\xi[(j+1)(2+\xi)+8]+\xi^2(1-\xi)\\ (2+\xi)(2-\xi)\\ (2+\xi)(2-\xi)(1-\xi)\\ 0\end{pmatrix},\quad \begin{pmatrix} -2j(j+1-\xi)\xi(2-\xi)\\ -j(2+\xi)(j+1)+2\xi(7-\xi)\\ -2j\xi(2+\xi)(2-\xi)\\ 2(2+\xi)(2-\xi)\end{pmatrix},$$
$$\begin{pmatrix} -\xi(2-\xi)(2j-3-\xi)\\ \xi(2-\xi)\\ -j(2+\xi)(j+1)+\xi^2(3+\xi)\\ 0\end{pmatrix},\quad \begin{pmatrix} -j(j-1)(2-\xi)(4-\xi)\xi\\ -\xi(j-1)(2-\xi)(j-4)\\ -j(j-1)(2-\xi)(2+\xi)\xi\\ -(2+\xi)(j-2)(j-1+2\xi)-2\xi\end{pmatrix}. \tag{3.5}$$

In Appendix B we present the two remaining blocks (subsets II and III), in the matrix representation of $\hat{\boldsymbol{T}}$ as a function of j. The single basis $B_{3,jm}$ (subset IV) cannot appear in the theory since $a_{3,jm}=0$ by the solenoidal condition (cf. the Appendix of Ref. [4]):

$$\begin{aligned} a'_{3,jm}-jR^{-1}a_{3,jm}&=0,\\ a'_{3,jm}+R^{-1}a_{3,jm}&=0. \end{aligned} \tag{3.6}$$

Finally, there are no solutions belonging to the $j=1$ sector. This is due to the fact that such solutions correspond to subset II. In this subset the $j=1$ solenoidal condition implies the equation

$$a'_{8,1m}+\frac{3a_{8,1m}}{R}=0, \tag{3.7}$$

or $a_{8,1m}\propto R^{-3}$ which is not an admissible solution.

IV. ABSENCE OF DYNAMO EFFECT

The first issue to clarify is the existence of a stationary solution for $t\to\infty$. A dynamo effect may cause the covariance to grow unboundedly. Vergassola [2] showed that this is not the case in the isotropic sector as long as $\xi<1$. We demonstrate that for these values of ξ, the dynamo effect is absent also in the anisotropic sectors.

Consider the forceless case of Eq. (2.1) with $F^{\alpha\beta}=0$. In addition, assume initial conditions such that $\langle B\rangle=0$. It is easy to see that no mean magnetic field can appear in time. Accordingly our covariance $C^{\alpha\beta}(\boldsymbol{R},t)$ tends to zero when $R\to L$ since $C^{\alpha\beta}(\boldsymbol{R},t)\to\langle B\rangle^2$. We note that for $\xi=0$, $\hat{T}^{\alpha\beta}_{\mu\nu}=2\kappa\Delta\,\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}$. In the space of functions $C^{\alpha\beta}(\boldsymbol{R},t)$, which vanish outside the domain $|\boldsymbol{R}|\leqslant L$, this operator is diagonalizable due to its Hermiticity, with negative discrete spectrum $\{-E_\lambda\}$ due to the compactness of the domain. Thus the general solution in this case is

$$C^{\alpha\beta}(\boldsymbol{R},t)=\sum_{\lambda} e^{-E_\lambda t}C^{\alpha\beta}_{\lambda}(\boldsymbol{R}). \tag{4.1}$$

In a spherical domain the index λ contains the indices j,m and an index specifying one of the three subsets discussed above. We will assume that for $\xi\neq 0$ $\hat{\boldsymbol{T}}$ remains diagonalizable. We will demonstrate that the eigenvalues E_λ remain positive for $0<\xi<1$. This will imply that $C^{\alpha\beta}(\boldsymbol{R},t)$ and in particular, $\langle B^2(t)\rangle = \delta_{\alpha\beta}C^{\alpha\beta}(\mathbf{0},t)$, is a monotone decreasing function of time, and hence will imply the absence of a dynamo effect.

To this end, we define the inner product

$$(\boldsymbol{C}_1,\boldsymbol{C}_2)\equiv\int_{R\le L}\frac{(C_1^{\alpha\beta})^* C_2^{\alpha\beta}}{2(DR^{\xi}+\kappa)}d^3R, \tag{4.2}$$

and demonstrate that

$$-E_\lambda(\boldsymbol{C}_\lambda,\boldsymbol{C}_\lambda)=(\boldsymbol{C}_\lambda,\hat{\boldsymbol{T}}\boldsymbol{C}_\lambda)<0, \tag{4.3}$$

indicating that $E_\lambda>0$. We first consider the 4×4 block with a given j,m. In this case $\boldsymbol{C}_\lambda$ is given by

$$\boldsymbol{C}_\lambda(\boldsymbol{R})=a_\lambda(R)\boldsymbol{B}_{9,jm}(\hat{\boldsymbol{R}})+b_\lambda(R)\boldsymbol{B}_{7,jm}(\hat{\boldsymbol{R}})+c_\lambda(R)\boldsymbol{B}_{1,jm}(\hat{\boldsymbol{R}})+d_\lambda(R)\boldsymbol{B}_{5,jm}(\hat{\boldsymbol{R}}). \tag{4.4}$$

Using Eq. (3.1), we obtain

$$(\boldsymbol{C}_\lambda,\hat{\boldsymbol{T}}\boldsymbol{C}_\lambda)=\int_0^L dR\frac{R^2}{2(DR^{\xi}+\kappa)}(a_\lambda^* b_\lambda^* c_\lambda^* d_\lambda^*)\boldsymbol{M}(j)\times\left[\boldsymbol{T}_1\begin{pmatrix}a''_\lambda\\ b''_\lambda\\ c''_\lambda\\ d''_\lambda\end{pmatrix}+\boldsymbol{T}_2\begin{pmatrix}a'_\lambda\\ b'_\lambda\\ c'_\lambda\\ d'_\lambda\end{pmatrix}+\boldsymbol{T}_3\begin{pmatrix}a_\lambda\\ b_\lambda\\ c_\lambda\\ d_\lambda\end{pmatrix}\right], \tag{4.5}$$

where the matrix $\boldsymbol{M}(j)$ arises from the angular integration over the spherical tensors $\boldsymbol{B}_{q,jm}$. This matrix is obtained by a direct calculation. For example $M_{1,1}(j)\equiv\int d\hat{\boldsymbol{R}}\boldsymbol{B}^*_{9,jm}(\hat{\boldsymbol{R}})\boldsymbol{B}_{9,jm}(\hat{\boldsymbol{R}})$. The full matrix reads

$$\boldsymbol{M}(j)=\begin{pmatrix}1 & 2j & 1 & j(j-1)\\ 2j & 2j(3j+1) & 2j & 2j(j-1)(2j+1)\\ 1 & 2j & 3 & 0\\ j(j-1) & 2j(j-1)(2j+1) & 0 & j(j-1)(2j-1)(2j+1)\end{pmatrix}. \tag{4.6}$$

We note that $\boldsymbol{M}(j)$ is symmetric and positive definite. By integration by parts, using the fact that our covariances vanish for $R=L$, we demonstrate in Appendix C that Eq. (4.3) is true.

One important conclusion of this calculation is the relative rate of decay of the various anisotropic contributions. We see that upon increasing j the inner product (4.3) becomes more negative. Thus any anisotropic initial conditions results in a rapid decay of the higher j contributions. Without anisotropic forcing the covariance of the magnetic field becomes isotropic in time. We will show below that in the (anisotropic) stationary state maintained by anisotropic forcing, the covariance also isotropizes on the smaller scales. The scaling exponents governing the R dependence are also strictly increasing with increasing j. Thus, invariably, for small enough scales and for long times one restores local isotropy.

V. CALCULATION OF THE SCALING EXPONENTS

In the absence of a dynamo effect, we can consider a stationary state of the system, maintained by the forcing term $\boldsymbol{f}(\boldsymbol{r},t)$. The covariance in such a case will obey the following equation:

$$0=\hat{T}^{\alpha\beta}_{\sigma\rho}C^{\sigma\rho}+F^{\alpha\beta}. \tag{5.1}$$

Deep in the inertial range we look for scale invariant solutions, obtained as zero modes of Eq. (5.1). Indeed, when $\xi>0$ and well within the inertial range we can take the magnetic dissipation to zero, and as a result, the homogeneous part of Eq. (5.1) (without $F^{\alpha\beta}$) will be scale invariant, leading to scale invariant solutions. We will need to match these zero modes to the appropriate zero modes computed in the dissipative range at the end. This will necessitate the discussion of zero modes when $\xi=0$, and see below.

The calculation of the scale-invariant solutions becomes rather immediate once we know the functional form of the operator $\hat{\boldsymbol{T}}$ in the basis of the angular tensors $\boldsymbol{B}_{q,jm}$. Using expansion (2.3), and the fact that $\hat{\boldsymbol{T}}$ is block diagonalized by such an expansion, we obtain a set of second order coupled ODE's for each block. To demonstrate this point, consider the four dimensional block of $\hat{\boldsymbol{T}}$, created by the four basis tensors $\boldsymbol{B}_{q,jm}$ of subset I. According to the notation of the last section, we denote the coefficients of these angular tensors in Eq. (2.3), by the four functions $a(R)$, $b(R)$, $c(R)$, and $d(R)$,

$$C^{\alpha\beta}(\boldsymbol{R})\equiv a(R)B^{\alpha\beta}_{9,jm}+b(R)B^{\alpha\beta}_{7,jm}+c(R)B^{\alpha\beta}_{1,jm}+d(R)B^{\alpha\beta}_{5,jm}+\cdots, \tag{5.2}$$

where $(\cdots)$ stand for terms with other j,m and other symmetries with the same j,m. Let us first consider the case where $\xi>0$. According to Eq. (3.1), well within the inertial range, these functions obey

$$\boldsymbol{T}_1(\kappa=0)\begin{pmatrix}a''\\ b''\\ c''\\ d''\end{pmatrix}+\boldsymbol{T}_2(\kappa=0)\begin{pmatrix}a'\\ b'\\ c'\\ d'\end{pmatrix}+\boldsymbol{T}_3(\kappa=0)\begin{pmatrix}a\\ b\\ c\\ d\end{pmatrix}=0. \tag{5.3}$$

Due to the scale invariance of these equations, we look for scale invariant solutions in the form

$$a(R)=aR^{\zeta},\quad b(R)=bR^{\zeta},\quad d(R)=cR^{\zeta},\quad d(R)=dR^{\zeta}, \tag{5.4}$$

where a, b, c, and d are complex constants. Substituting Eq. (5.4) into Eq. (5.3) results in a set of four linear homogeneous equations for the unknowns a, b, c, and d:

$$[\zeta(\zeta-1)\boldsymbol{T}_1(\kappa=0)+\zeta\boldsymbol{T}_2(\kappa=0)+\boldsymbol{T}_3(\kappa=0)]\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix}=0. \tag{5.5}$$

The last equation admits nontrivial solutions only when

$$\det[\zeta(\zeta-1)\boldsymbol{T}_1(\kappa=0)+\zeta\boldsymbol{T}_2(\kappa=0)+\boldsymbol{T}_3(\kappa=0)]=0. \tag{5.6}$$

This solvability condition allows us to express ζ as a function of j and ξ. Using MATHEMATICA we find eight possible values of ζ, out of which only four are in agreement with the solenoidal condition:

$$\zeta_i^{(j)}=-\frac{1}{2}\xi-\frac{3}{2}\pm\frac{1}{2}\sqrt{H(\xi,j)\pm 2\sqrt{K(\xi,j)}},$$
$$K(\xi,j)\equiv\xi^4-2\xi^3+2\xi^3 j+2\xi^3 j^2-4\xi^2 j-3\xi^2-4\xi^2 j^2 -8\xi j^2-8\xi j+4\xi+16j+16j^2+4, \tag{5.7}$$
$$H(\xi,j)\equiv-\xi^2-8\xi+2\xi j^2+2\xi j+4j^2+4j+5.$$

Not all of these solutions are physically acceptable, because not all of them can be matched to the zero-mode solutions in the dissipative regime. To see why this is so, consider the zero-mode equation for $\xi=0$:

$$(2\kappa+2D)\nabla^2\boldsymbol{C}=0. \tag{5.8}$$

The main difference between the $\xi=0$ case and the $\xi>0$ case is that in the former the same scale invariant equation holds *both* for the inertial range and the dissipative range. As a result, for $\xi=0$, the zero modes scale with the same exponents in the two regimes. These exponents are given simply by Eq. (5.7) with $\xi=0$, because for $\xi=0$ the zero-mode equation with $\kappa=0$ is the same as Eq. (5.8) up to the overall factor $D/(D+\kappa)$ which does not change the exponent. For $\xi=0$ our solutions should be valid for the dissipative regime as well as for the inertial regime, ruling out the two solutions with negative exponents in Eq. (5.7), for they will give a nonphysical divergence as $R\to 0$. Assuming now that the solutions (including the exponents) are continuous in ξ (and not necessarily analytic), we find that also for finite ξ only the positive exponents appear in the inertial range (an exception to that is the $j=0$, to be discussed below). Finally there exist two branches of solutions corresponding to the $(-)$ and $(+)$ in the square root:

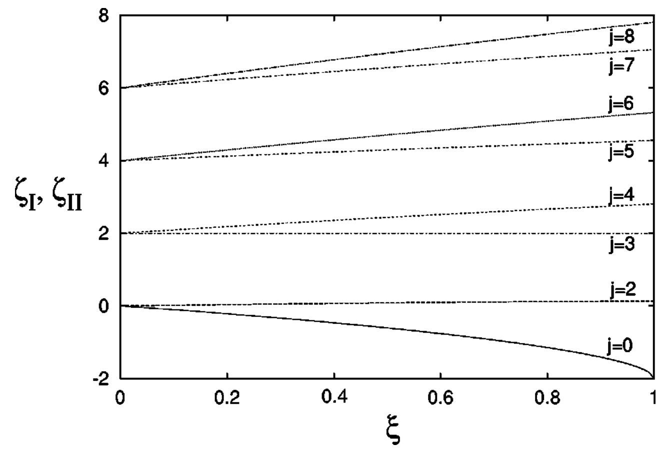

FIG. 1. The leading exponents of the symmetric parts of the zero modes of the magnetic covariance.

$$\zeta_{I\pm}^{(j)}=-\frac{3}{2}-\frac{1}{2}\xi+\frac{1}{2}\sqrt{H(\xi,j)\pm 2\sqrt{K(\xi,j)}},\quad \text{subset I.} \tag{5.9}$$

These exponents are in agreement with Refs. [3,2]. Note that for $j=0$, only $\zeta_{I+}^{(0)}$ exists since the other exponent is not admissible, being negative for $\xi\to 0$, and therefore excluded by continuity. However, $\zeta_{I+}^{(0)}$ becomes negative as ξ increases (see Fig. 1). For $j\geqslant 2$ both solutions are admissible, and the leading one is $\zeta_{I-}^{(0)}$, which is smaller.

Let us find the behavior of the zero modes in the dissipative regime for $\xi>0$. Here the dissipation terms become dominant and we can neglect all other terms in $\hat{\boldsymbol{T}}$. The zero-mode equation in this regime becomes $2\kappa\nabla^2 C^{\alpha\beta}=0$, which is again, up to an overall factor, identical to the zero-mode equation with $\kappa=0$ and $\xi=0$. The solutions in this region are once again scale invariant with scaling exponents $\zeta_{I\pm}^{(j)}|_{\xi=0}=j,j-2$. As expected, the correlation function $C^{\alpha\beta}(\boldsymbol{R})$ becomes smooth in the dissipative regime.

In addition to subset I, one needs to compute the exponents corresponding to subsets II and III. The computation in the other two blocks follows the same lines. Since these are 2×2 they furnish two solutions for the exponents, one of which is negative. We end up finding

$$\zeta_{II}^{(j)}=-\frac{3}{2}-\frac{1}{2}\xi+\frac{1}{2}\sqrt{1-10\xi+\xi^2+2j^2\xi+2j\xi+4j+4j^2},\quad \text{subset II,} \tag{5.10}$$

$$\zeta_{III}^{(j)}=-\frac{3}{2}-\frac{1}{2}\xi+\frac{1}{2}\sqrt{\xi^2+2\xi+1+4j^2+2j^2\xi+4j+2\xi j},\quad \text{subset III.} \tag{5.11}$$

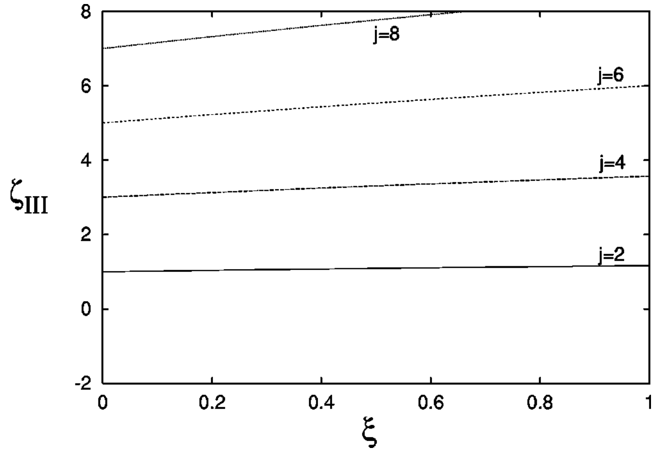

FIG. 2. The leading exponents of the antisymmetric parts of the zero modes of the magnetic covariance.

For $j=0$ there is no contribution from this subset, as the exponent is negative. The dependence of the admissible leading exponents on ξ is displayed in Figs. 1 and 2. In Table I we summarize which are the leading exponents in each sector.

After matching the zero modes to the dissipative range, one has to guarantee matching at the outer scale L. The condition to be fulfilled is that the sum of the zero-modes with the inhomogeneous solutions (whose exponents are 2-ξ) must give $\boldsymbol{C}(\boldsymbol{R})\to 0$ as $|\boldsymbol{R}|\to L$. Obviously this means that the forcing must have a projection on any sector $\boldsymbol{B}_{q,jm}$ for which $a_{q,jm}$ is nonzero.

VI. SUMMARY AND CONCLUSIONS

The results of this paper should be examined in light of the recent progress in understanding the effects of anisotropy on the statistics of fully developed turbulence [4,8–10]. Whereas in the Navier-Stokes case one cannot present exact results, the present study affords exact calculations of the whole spectrum of scaling exponents that determine the covariance of a vector field in the presence of anisotropy. We have presented a detailed and systematic investigation of scaling properties of the covariance of a magnetic field advected by a Gaussian and δ correlated in time velocity field. We have extended the nonperturbative analysis presented by Vergassola in Ref. [2] for the isotropic sector to all the sectors of the SO(3) symmetry group. Our analysis leads to the conclusions that the scaling exponents are strictly increasing with the index of j of the sector, meaning that there is a tendency toward isotropization upon decreasing the scales of observation. We also showed that as far as the dynamo problem is concerned, anisotropic sectors are less unstable than the isotropic sector: in the absence of an external forcing anisotropies decay in time faster then isotropic fluctuations. In distinction with the expansion presented in Ref. [3], our results are free of any assumptions about the hierarchy of scaling exponents belonging to different SO(3) sectors. This is due to the employment of a proper basis set. The equations for the magnetic covariance foliate into independent closed equations for each set of irreducible representations of the SO(3) group.

TABLE I. The leading exponents in the various sectors.

	Symmetric	Antisymmetric
$j=0$	ζ_{I+}	–
Even $j>0$	ζ_{I-}	ζ_{III}
Odd $j>1$	ζ_{II}	–

In summary, we have shown that the covariance of the magnetic field is naturally computed as a sum of contributions proportional to the irreducible representations of the SO(3) symmetry group. The amplitudes are nonuniversal, determined by boundary conditions. The scaling exponents are universal, forming a discrete, strictly increasing spectrum indexed by the sectors of the symmetry group. Similar results were presented for passive scalar fluctuations in Ref. [5], and for Navier-Stokes fluctuations in Refs. [4,8–10]. In the present case anomalous scaling laws are found as the zero modes of the inertial operator governing the stationary equation for the magnetic covariance [6,7]. Matching with the UV boundary conditions selects the physically acceptable solutions. It now appears quite clear that the issue of anomalous, universal scaling exponents in turbulence has ramifications on the multitude of sectors of the appropriate symmetry groups.

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APPENDIX A: DEMONSTRATION OF THE ACTION OF $\hat{\boldsymbol{T}}^{\alpha\beta}_{\mu\nu}$

As an example of the operation of $\hat{\boldsymbol{T}}$ on the basis function, consider an explicit calculation of $\partial^2 C^{\alpha\beta}$. Such a term appears as a part of $S^{\mu\nu}\partial^\mu\partial^\nu$ which is a part of $\hat{\boldsymbol{T}}$, and also in the magnetic dissipation term. Considering explicitly the part $a_{9,jm}(R,t)B^{\alpha\beta}_{9,jm}(\hat{\boldsymbol{R}})$:

$$\begin{aligned}\partial^2 a_{9,jm}R^{-j-2}R^\alpha R^\beta\Phi_{jm}&=\partial^\mu\partial_\mu a_{9,jm}R^{-j-2}R^\alpha R^\beta\Phi_{jm}\\&=\partial^\mu[a'_{9,jm}R^{-j-3}-(j+2)a_{9,jm}R^{-j-4}]R_\mu R^\alpha R^\beta\Phi_{jm}+\partial^\alpha a_{9,jm}R^{-j-2}R^\beta\Phi_{jm}\\&\quad+\partial^\beta a_{9,jm}R^{-j-2}R^\alpha\Phi_{jm}+\partial^\mu a_{9,jm}R^{-j-2}R^\alpha R^\beta\partial_\mu\Phi_{jm}\\&=\left[a''_{9,jm}-(j+3)\frac{a'_{9,jm}}{R}-(j+2)\frac{a'_{9,jm}}{R}+(j+2)(j+4)\frac{a_{9,jm}}{R^2}\right]B^{\alpha\beta}_{9,jm}\\&\quad+(j+5)\left[\frac{a'_{9,jm}}{R}-(j+2)\frac{a_{9,jm}}{R^2}\right]B^{\alpha\beta}_{9,jm}+2\left[\frac{a'_{9,jm}}{R}-(j+2)\frac{a_{9,jm}}{R^2}\right]B^{\alpha\beta}_{9,jm}+2\frac{a_{9,jm}}{R^2}B^{\alpha\beta}_{1,jm}\\&\quad+\frac{a_{9,jm}}{R^2}B^{\alpha\beta}_{7,jm}+j\left[\frac{a'_{9,jm}}{R}-(j+2)\frac{a_{9,jm}}{R^2}\right]B^{\alpha\beta}_{9,jm}+\frac{a_{9,jm}}{R^2}B^{\alpha\beta}_{7,jm}\\&=\left[a''_{9,jm}+2\frac{a'_{9,jm}}{R}-(j+2)(j+3)\frac{a_{9,jm}}{R^2}\right]B^{\alpha\beta}_{9,jm}+2\frac{a_{9,jm}}{R^2}B^{\alpha\beta}_{7,jm}+2\frac{a}{R^2}B^{\alpha\beta}_{1,jm}.\end{aligned}\tag{A1}$$

In performing the computation, we make use of the following basic identities that are employed repeatedly in all our calculations:

$$\partial^\mu\partial_\mu\Phi_{jm}=0,\tag{A2}$$

$$R^\mu\partial_\mu\Phi_{jm}=j\Phi_{jm}.\tag{A3}$$

The first identity follows from $\partial^2 Y_{jm}=-j(j+1)R^{-2}Y_{jm}$. The second from the fact that Φ_{jm} are homogeneous polynomials of degree j. As expected, the result remains in a j,m sector, and mixes only basis functions with the same symmetry properties.

APPENDIX B: $\hat{T}$ AND THE SOLENOIDAL CONDITION IN THE TWO REMAINING SUBSETS

In this appendix we present the two blocks pertaining to the $(-1)^{j+1}$ parity. The part denoted in Eq. (3.1) as $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ remain unchanged except that the identity matrix is now two dimensional. For the case of invariance under α,β interchange (subset II) we find the 2×2 matrix $\boldsymbol{Q}(j,\xi)$:

$$\begin{pmatrix}-(j+1)(2+\xi)(j+2-\xi)+2\xi(7-\xi) & -\xi(j-1)^2(2-\xi)\\(2-\xi)(2+\xi) & j(j-1)(2+\xi)+\xi(j-3)(2+\xi)+2\xi\end{pmatrix}.\tag{B1}$$

The solenoidal condition reads in this case (cf. the Appendix of Ref. [4]):

$$a'_{8,jm}+3R^{-1}a_{8,jm}+(j-1)a'_{6,jm}-(j-1)^2R^{-1}a_{6,jm}=0.\tag{B2}$$

From this equation we learn that a contribution pertaining to $j=1$ cannot appear in this theory, since for this value of j $a'_{8,jm}$ must have a negative scaling exponent which is not admissible.

For the case of antisymmetry under α,β interchange (subset III) we find the 2×2 matrix $\boldsymbol{Q}(j,\xi)$:

$$\begin{pmatrix}\xi(4+2\xi+4j)-(j+1)(j+2)(2+\xi) & \xi j(j-1)(2-\xi)\\4-2\xi & -(j-1)[j(2+\xi)+4\xi]\end{pmatrix}\tag{B3}$$

with the solenoidal condition (cf. Appendix of [4])

$$R^{-1}a_{4,jm}-a'_{2,jm}+(j-1)R^{-1}a_{2,jm}=0.\tag{B4}$$

APPENDIX C: PROOF OF EQ. (4.3)

To demonstrate Eq. (4.3) we note $\hat{\boldsymbol{T}}$ as well as $\boldsymbol{M}(j)$ are m independent. We can therefore consider the $m=0$ case without loss of generality. In this case the basis functions as well as the coefficients a, b, c, and d are real. For nonzero m the imaginary components have to cancel with the imaginary components of $-m$ since the covariance is real. We treat separately the contributions associated with $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, and $\boldsymbol{T}_3$. showing that they are all negative definite.

1. Integrals of T_1 and T_2

For the evaluation of these integrals it is convenient to work in the basis that diagonalizes $\boldsymbol{M}(j)$. Since $\boldsymbol{M}(j)$ is a real and symmetric matrix, it is diagonalizable, and as it is nonnegative its eigenvalues μ_i, $i=1$, 2, 3, and 4, are nonnegative. T_1 and T_2 are proportional to the unit matrix, and therefore they remain so in any basis, and in particular in the diagonal basis of M. In that basis, $a,\ b,\ c$, and d are replaced by $a_1,\ a_2,\ a_3$, and a_4, and the contributions of T_1, and T_2 is

$$\sum_{i=1}^{4}\mu_i\int_0^L dx\frac{x^2}{Dx^{\xi}+\kappa}a_i\times\left[2(Dx^{\xi}+\kappa)a_i''+4(Dx^{\xi}+\kappa)\frac{a_i'}{x}+4\xi Dx^{\xi}\frac{a_i'}{x}\right].\tag{C1}$$

This integral is negative definite for all values of i, since it is the sum of two negative definite integrals I_1 and I_2:

$$\begin{aligned}I_1&=\int_0^L dx\frac{x^2}{Dx^{\xi}+\kappa}a_i\left[2(Dx^{\xi}+\kappa)a_i''+4(Dx^{\xi}+\kappa)\frac{a_i'}{x}\right]\\&=2\int_0^L dx(xa_i)\frac{d^2}{dx^2}(xa_i)\\&=-2\int_0^L dx\left[\frac{d}{dx}(xa_i)\right]^2<0,\end{aligned}\tag{C2}$$

$$\begin{aligned}I_2&=4D\int_0^L dx\frac{x^2}{Dx^{\xi}+\kappa}a_ix^{\xi}\frac{a_i'}{x}\\&=-4D\int_0^L dx\frac{d}{dx}\left[\frac{x^{\xi+1}}{Dx^{\xi}+\kappa}\right]a_i^2-I_2.\end{aligned}\tag{C3}$$

Accordingly,

$$\begin{aligned}I_2&=-2D\int_0^L dx\frac{d}{dx}\left[\frac{x^{\xi+1}}{Dx^{\xi}+\kappa}\right]a_i^2\\&=-2D\int_0^L dx\frac{Dx^{2\xi}+\kappa(1+\xi)x^{\xi}}{(Dx^{\xi}+\kappa)^2}a_i^2<0.\end{aligned}\tag{C4}$$

2. Integral of T_3

The contribution of $\boldsymbol{T}_3$ has two parts: One which is proportional to κ, and one which is proportional to D. We shall analyze each of them separately and show that $\boldsymbol{M}(j)\cdot\boldsymbol{T}_3$ is a nonpositive matrix for every $j\geqslant 2$ and every $0\leqslant\xi\leqslant 2$.

(1) The part involving κ is

$$I_3=\int_0^L dx\frac{x^2}{Dx^{\xi}+\kappa}(a\quad b\quad c\quad d)\boldsymbol{M}(j)\kappa x^{-2}\boldsymbol{Q}(j,0)\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\tag{C5}$$

$$=\kappa\int_0^L dx\frac{1}{Dx^{\xi}+\kappa}(a\quad b\quad c\quad d)\frac{1}{2}\times[\boldsymbol{M}(j)\boldsymbol{Q}(j,0)+(\boldsymbol{M}(j)\boldsymbol{Q}(j,0))^T]\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\tag{C6}$$

$$\equiv\kappa\int_0^L dx\frac{1}{Dx^{\xi}+\kappa}(a\quad b\quad c\quad d)\boldsymbol{X}(j,0)\begin{pmatrix}a\\b\\c\\d\end{pmatrix},\tag{C7}$$

where $\boldsymbol{X}(j,\xi)$ is the symmetric matrix

$$\boldsymbol{X}(j,\xi)\equiv\frac{\boldsymbol{M}(j)\boldsymbol{Q}(j,\xi)+(\boldsymbol{M}(j)\boldsymbol{Q}(j,\xi))^T}{2}.\tag{C8}$$

For $j=2$ and $\xi=0$, $\boldsymbol{X}(j,\xi)$ is given by

$$\boldsymbol{X}(2,0)=\begin{pmatrix}-20&-32&-12&0\\-32&-176&-48&0\\-12&-48&-36&0\\0&0&0&0\end{pmatrix},\tag{C9}$$

with the eigenvalues $(-12.97\ldots,-196.43\ldots,-21.59\ldots,0)$, so the expression is obviously nonpositive. For higher j's, we can look at the determinant of $X(j,0)$:

$$\det\boldsymbol{X}(j,0)=(j+3)(j+2)^2(j+1)^4j^4(j-1)^2(j-2);\tag{C10}$$

this function is positive for every $j>2$, which means that we have 4 negative eigenvalues when $j>2$.

(2) The part involving D is

$$\begin{aligned}I&=D\int_0^L dx\frac{x^2}{Dx^{\xi}+\kappa}\big(a\quad b\quad c\quad d\big)\boldsymbol{M}(j)x^{\xi-2}\boldsymbol{Q}(j,\xi)\begin{pmatrix}a\\b\\c\\d\end{pmatrix}\\&=D\int_0^L dx\frac{x^{\xi}}{Dx^{\xi}+\kappa}\big(a\quad b\quad c\quad d\big)\boldsymbol{X}(j,\xi)\begin{pmatrix}a\\b\\c\\d\end{pmatrix}.\end{aligned}\tag{C11}$$

The proof of the nonpositivity of this expression follows the same lines of the previous discussion. We know that for $\xi=0$ and $j=2$ $\boldsymbol{X}(j,\xi)$ has three negative eigenvalues and one zero. Therefore, It is sufficient to show that $\det\boldsymbol{X}(j,\xi)$ is positive for every $0<\xi<2$ and $j\geqslant 2$ to ensure that $\boldsymbol{X}(j,\xi)$ is indeed nonpositive. This is indeed the case, as can be verified explicitly using MATHEMATICA.

[1] S. Childress and A. Gilbert, *Twist Stretch Fold: the Fast Dynamo* (Springer, Berlin, 1995).

[2] M. Vergassola, Phys. Rev. E **53**, R3021 (1996).

[3] A. Lanotte and A. Mazzino, Phys. Rev. E **60**, R3483 (1999).

[4] I. Arad, V.S. L'vov, and I. Procaccia, Phys. Rev. E **59**, 6753 (1999).

[5] I. Arad, V.S. L'vov, E. Podivilov, and I. Procaccia, Phys Rev. E (to be published).

[6] K. Gawedzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3834 (1995).

[7] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedeev, Phys. Rev. E **52**, 4924 (1995).

[8] I. Arad, B. Dhruva, S. Kurien, V.S. L'vov, I. Procaccia, and K.R. Sreenivasan, Phys. Rev. Lett. **81**, 5330 (1998).

[9] I. Arad, L. Biferale, I. Mazzitelli, and I. Procaccia, Phys. Rev. Lett. **82**, 5040 (1999).

[10] S. Kurien, V.S. L'vov, I. Procaccia, and K.R. Sreenivasan, Phys. Rev. E **61**, 407 (2000).