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Intermittency Correction to the Obukhov-Corrsin Theory of a Passive Scalar.

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Abstract. – We propose a simple argument to compute intermittency correction to the Obukhov-Corrsin inertial range scaling for the passive scalar. We find that the intermittency correction is of the opposite sign with respect to the intermittency correction of the Kolmogorov energy spectrum of fully developed turbulence. Our result is in qualitative and quantitative agreement with experimental data.

The dynamics of a passive scalar Q is described by the equation

$$\partial_t Q + V\partial Q = \chi \Delta Q, \tag{1}$$

where χ is the molecular diffusivity of Q and V is the three-dimensional fluid velocity satisfying the Navier-Stokes equation

$$\partial_t V + (V\partial) V = -\frac{1}{\rho}\partial p + \nu \Delta V, \tag{2}$$

ν being the kinematic viscosity. It has been pointed out long time ago that when the fluid is fully turbulent the small-scale statistics of Q display universal properties independent of χ, ν and the forcing mechanism for Q or V. In particular Obukhov and Corrsin [1] theory predicts that the correlation function $C_2(r) = \langle (Q(x+r) - Q(x))\rangle^2$ is given by

$$C_2(r) \approx r^{2/3}, \tag{3}$$

similarly to Kolmogorov [2] scaling of the two-point correlation function of the velocity field. The argument which justifies (3) is based on the «Q-cascade» from large to small scale of Q^2. If we indicate by

$$\Delta Q(l) = |Q(x+l) - Q(x)|, \qquad \Delta V(l) = |V(x+l) - V(x)|,$$

then the rate of the Q-cascade at scale l can be estimated to be [3]

$$\frac{\Delta Q(l)^2 \Delta V(l)}{l} = N(l).$$

By assuming $N(l)$ to be constant we simply obtain $\Delta Q(l) \approx \Delta V(l)^{-1} l \approx l^{2/3}$, where we have used the Kolmogorov scaling $\Delta V(l) \approx l^{1/3}$ for the fully turbulent velocity field. However, it turns out that (2) is not true, namely it has been found experimentally [4, 5] that

$$C_2(r) \approx r^b, \quad b = \frac{2}{3} - \Delta a, \quad \Delta a > 0. \tag{4}$$

It has been claimed that deviations from the scaling law (3) is due to intermittency both in the Q-cascade and in the energy cascade of the velocity field. Indeed, intermittency is a well-known effect in fully developed turbulence [3]. A popular way to measure intermittency in fully developed turbulence is to compute, from numerical or experimental data, the structure functions $S_m(l)$ defined as

$$S_m(l) = \langle |V(x+l) - V(x)|^m \rangle. \tag{5}$$

The Kolmogorov theory predicts $S_m(l) \approx l^{m/3}$ independent of m, while it has been found [4] $S_m(l) \approx l^{a(m)}$ with $a(m)$ a nonlinear convex function of m. In particular $a(2) \approx 2/3 + \delta_2$ with $\delta_2 > 0$. Thus, in fully developed turbulence the intermittency correction of the two-point correlation function has opposite sign to the one observed for the passive scalar.

Many authors [5] have presented theoretical arguments to explain the intermittency effect in Q. In this letter we present a very simple argument to explain the behaviour of $C_2(r)$.

First of all let us describe a naive approach to the problem which illustrates the difficulty to develop a theoretical framework for the intermittency effect in the passive scalar dynamics. As for the energy cascade in three-dimensional fully developed turbulence, intermittency effect can be taken into account by saying the rate of Q-cascade in the inertial range is not constant as assumed by Obukhov and Corrsin. For simplicity, let us introduce the scales $l_n = 2^{-n} l_0$ and let us define $\Delta Q_n = \Delta Q(l_n)$, $\Delta V_n = \Delta V(l_n)$. Then we can think that $N(l_n)$ and $N(l_{n+1})$ are related by the equation

$$N(l_n) = \beta(n+1) N(l_n), \tag{6}$$

where $\beta(n+1)$ can be considered as the fraction of space where the Q-cascade is taking place from scale l_n to scale l_{n+1}. Equation (6) is in complete analogy to what it has been proposed for the energy cascade of fully developed turbulence [6]. From (6) and the estimate of $N(l_n)$ previously given we obtain

$$\frac{\Delta Q_n^2 \Delta V_n}{l_n} = \beta(n+1) \frac{\Delta Q_{n+1}^2 \Delta V_{n+1}}{l_{n+1}}. \tag{7}$$

By iterating eq. (7) we obtain

$$\Delta Q_n = l_n^{1/2} \Delta V_n^{-1/2} \prod_{i=1,\ldots,n} \beta(i)^{-1/2}, \tag{8}$$

where we have assumed for simplicity $l_0 = \Delta V_0 = \Delta Q_0 = 1$. From eq. (8) we can compute the

m-moments of ΔQ_n:

$$\langle(\Delta Q_n)^m\rangle = l_n^{m/2}\left\langle \Delta V_n^{-m/2} \prod_{i=1,\dots,n} \beta(i)^{1-m/2}\right\rangle, \tag{9}$$

where a volume factor $\beta(i)$ has been inserted in the definition of the average. Equation (9) shows that in order to compute the structure functions of ΔQ_n one has to develop a theory which takes into account the correlation of βq with ΔV_n. Let us remark that the structure functions of the velocity field $S_m(l_n)$ are usually well explained by the so-called multifractal hypothesis. Thus in order to compute the average of equation (9) one has to define a joint multifractal model which takes into account both the multifractal scaling of ΔV_n and ΔQ_n.

Recently Crisanti *et al.* [7] have proposed that the estimate of $N(l_n)$ given before is not justified by the passive scalar dynamics. Indeed this estimate is based on the assumption that the characteristic time of Q-cascade from scale l_n to scale l_{n+1} is given, as in fully developed turbulence, by $l_n/\Delta V_n$. In order to explain the sign of the intermittency correction to $C_2(l_n)$ they assumed that the characteristic time is equal to the average time for two particles to increase their distance up to scale l_n.

Here we take eq. (9) as starting point to compute $C_2(l_n)$. We first observe that

$$C_2(l_n) = \langle \Delta Q_n^2\rangle = l_n\langle \Delta V_n^{-1}\rangle. \tag{10}$$

Thus the intermittency effect on the Obukhov-Corrsin theory should be computed regardless of the fluctuations of $\beta(i)$ in eq. (9), *i.e.* no joint multifractal theory should be needed in order to compute $C_2(l_n)$.

At first sight, one could be tempted to compute $\langle \Delta V_n^{-1}\rangle$ by using one of the multifractal model proposed for fully developed turbulence [6], like for instance the β random model. However, such models have been proposed to understand the scaling properties of $\langle \Delta V_n^m\rangle$ for positive values of m: it is not clear that $\langle \Delta V_n^{-1}\rangle$ can be naively extrapolated from our knowledge of $\langle \Delta V_n^m\rangle$ for positive m. As an example of the above statement we have obtained, by using the random β model, $\langle \Delta V_n^{-1}\rangle = l_n^{0.22}$. Using this value in eq. (10) we get $C_2(l_n) = l_n^{0.78}$, *i.e.* a quite different result from what it has been experimentally observed, namely $C_2(l_n) = l_n^{0.61}$.

We claim that the scaling behaviour of $\langle \Delta V_n^{-1}\rangle$ is quite different from what one can estimate from multifractal models tuned to compute the scaling of positive moments of ΔV_n. In order to show that this is true we have directly computed $\langle \Delta V_n^{-1}\rangle$ by using a quite simple numerical model of fully developed turbulence recently introduced by Yamada and Okhitani [8]. This model shows scaling properties of $\langle \Delta V_n^m\rangle$ for positive m in quite close agreement with what has been observed in experimental data (see Jensen *et al.* [9] for a complete detailed description of the model and the computation of the above-mentioned scaling properties of $\langle \Delta V_n^m\rangle$). Thus we can check if the scaling of $\langle \Delta V_n^{-1}\rangle$ can be extrapolated from the scaling of $\langle \Delta V_n^m\rangle$ with $m > 0$. A few comments on the computations of $\langle \Delta V_n^{-1}\rangle$ are needed. If there is a nonzero probability for ΔV_n to be zero, then any estimate of $\langle \Delta V_n^{-1}\rangle$ is meaningless. However ΔV_n has a real and imaginary part and ΔV_n^{-1} should be computed as $|\Delta V_n|^{-1}$. Thus it is possible that the real and imaginary parts of ΔV_n are zero with nonzero probability and $|\Delta V_n|^{-1}$ is zero with zero probability. For instance, if the probability of both the real and imaginary part of ΔV_n is approximately Gaussian near zero, then $\langle \Delta V_n^{-1}\rangle$ is well defined. This is indeed the case for the Yamada-Okhitani model here considered. In real turbulence $|\Delta V_n|$ should be computed as the square root of the energy fluctuations at scale l_n.

Figure 1 shows $\log[\langle \Delta V_n^{-1}\rangle]$ as a function of $\log(l_n)$. At variance with the prediction of the random β model, the scaling of $\langle \Delta V_n^{-1}\rangle$ is given by l_n^{Δ} with $\Delta \approx -0.45 \pm 0.05$. Using this scal-

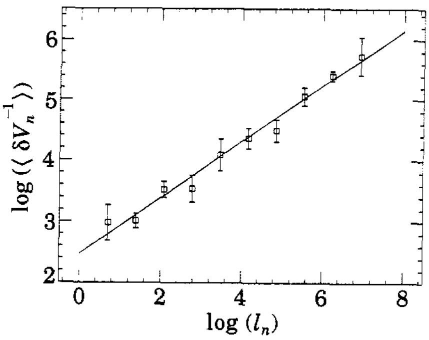

Fig. 1. – The figure illustrates the scaling behaviour of $\log[\langle|\Delta V_n|^{-1}\rangle]$ *vs.*$\log(l_n)$ computed from the shell model of Yamada and Okhitani [8]. In the inertial-range region a clear scaling law can be measured.

ing into eq. (10), we obtain

$$C_2(l_n) \approx l_n^{0.55 \pm 0.05}, \tag{11}$$

in reasonable agreement with experimental data and also with a generalization of the Yamada-Okhitani model, recently introduced by Jensen *et al.* [10], aimed to describe the statistical properties of a passive scalar.

From fig. 1 we argue that in fully developed turbulence

$$\langle \Delta V_n^{-1} \rangle = l_n^{-1/3 - \sigma} \tag{12}$$

with $\sigma > 0$. Thus the scaling of the Obukhov-Corrsin theory is $l_n^{2/3 - \sigma}$.

Although eq. (11) is based on a very simplified model of fully developed turbulence we think that experimental data will confirm the scaling (12).

It is possible to prove eq. (12) starting from eq. (10). Let us recall that $a(m)$ are the scaling exponents of the structure functions $S_m(l_n)$ and $a(m)$ is a convex function of m. In a similar way we can define $H(m)$ as the scaling exponents of the structure functions of Q:

$$\langle \Delta Q_n^m \rangle \approx l_n^{H(m)},$$

where $H(m)$ is also a convex function of m. From experimental data we know that $a(1) = 1/3 + \delta_1$ with $\delta_1 > 0$. Because $a(0) = 0$ and $a(m)$ is convex, we obtain $a(-1) \leqslant -a(1) = -1/3 - \delta_1$. Inserting the last inequality into eq. (10), we finally obtain

$$H(2) = 1 + a(-1) \leqslant \frac{2}{3} - \delta_1, \tag{13}$$

which proves eq. (12).

It is not difficult in principle to generalize the previously proposed multifractal model, like the random β model, in order to take into account the scaling properties of $\langle \Delta V_n^{-1} \rangle$. However, we think that at this stage it is not worthwhile. We certainly need more experimental and numerical data analysis in order to understand whether or not the scaling (12) always occurs. If this is the case, the possibility to develop a realistic joint multifractal model for the passive scalar may not be a difficult problem.

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