Statistical Conservation Laws in Turbulent Transport

Itai Arad,[1] Luca Biferale,[2] Antonio Celani,[3] Itamar Procaccia,[1] and Massimo Vergassola[4]

[1]*Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel*
[2]*Department of Physics and INFM, University of Tor Vergata, Via della Ricerca Scientifica 1, I-00133 Roma, Italy*
[3]*CNRS, INLN, Route des Lucioles, 1361, 06560 Valbonne, France*
[4]*CNRS, Observatoire de la Côte d'Azur, B.P. 4229, 06304 Nice Cedex 4, France*

We address the statistical theory of fields that are transported by a turbulent velocity field, both in forced and in unforced (decaying) experiments. With very few provisos on the transporting velocity field, correlation functions of the transported field in the forced case are dominated by statistically preserved structures. In decaying experiments we identify infinitely many statistical constants of the motion, which are obtained by projecting the decaying correlation functions on the statistically preserved functions. We exemplify these ideas and provide numerical evidence using a simple model of turbulent transport. This example is chosen for its lack of Lagrangian structure, to stress the generality of the ideas.

Turbulent transport represents a dynamical system in which observable quantities (such as the concentration of a contaminant) change over time and space in a complex fashion [1]. Unless the transported agent is continuously renewed, irreversible mixing and dispersion result in a uniform distribution over the system [2,3]. It had been therefore quite surprising that in certain examples (in which the advecting velocity field is δ correlated in time) it was observed that there exists functions of space that remain invariant to the time evolution [4–6].

In this Letter, we propose that with very few provisos on the turbulent velocity field, the dynamics exhibits infinitely many statistically conserved structures. By "statistically conserved" we mean structures that are not conserved for every realization of the turbulent velocity field but are conserved after taking an average over all realizations of the turbulent velocity field. This observation is very general, pertaining to scalar fields (like contaminants) or vector fields (like magnetic fields), with pressure effects or without. Moreover, these conserved structures dominate the statistics of the transported fields in the forced case (in which a fresh supply is continuously furnished). Indeed, forced experiments are more frequent in the turbulence community since they provide much better statistical measurements than decaying experiments. The present ideas, if supported by further studies, provide a unifying approach to understanding such statistical properties in turbulent transport, including the much debated issue of anomalous scaling [7].

Consider a turbulent velocity field $\boldsymbol{u}(\boldsymbol{r}, t)$ whose statistics is assumed stationary, but without any further restrictions. We consider the wide class of problems in which another field is transported passively by that velocity. The field may be an advected scalar $\theta(\boldsymbol{r}, t)$, with the equation of motion

$$\frac{\partial \theta}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla}\theta = \kappa\nabla^2\theta \,, \tag{1}$$

or a vector, like a magnetic field $\boldsymbol{B}(\boldsymbol{r}, t)$ satisfying [8]

$$\frac{\partial \boldsymbol{B}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{B} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{u} + \kappa\nabla^2\boldsymbol{B} \,. \tag{2}$$

We may also consider advection, as in [9], of a vector field $\boldsymbol{w}$ whose divergence vanishes, $\nabla \cdot \boldsymbol{w} = 0$:

$$\frac{\partial \boldsymbol{w}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{w} = -\boldsymbol{\nabla}p + \kappa\nabla^2\boldsymbol{w} \,. \tag{3}$$

Other examples can be considered, as long as the equation for the transported field $\phi(\boldsymbol{r}, t)$ (scalar or vector) has the form

$$\partial_t\phi = \mathcal{L}\phi \,. \tag{4}$$

Here, $\mathcal{L}$ is a stochastic operator that is built out of the turbulent velocity field. The key point is that the equation of motion is *linear* in the field ϕ. Statistical averages, denoted as $\langle\cdots\rangle$, are performed on the initial conditions and on the realizations of $\boldsymbol{u}$. A fundamental consequence of the linearity of the equations of motion is that the correlation functions may be expressed as

$$\langle\phi(\boldsymbol{r}_1, t)\cdots\phi(\boldsymbol{r}_N, t)\rangle =$$
$$\int \mathcal{P}^{(N)}_{\underline{\boldsymbol{r}}, \underline{\boldsymbol{\rho}}}(t)\langle\phi(\boldsymbol{\rho}_1, 0)\cdots\phi(\boldsymbol{\rho}_N, 0)\rangle_0 \, d\underline{\boldsymbol{\rho}} \,, \tag{5}$$

where $\langle\cdots\rangle_0$ on the right-hand side is an average over the statistics of the initial conditions. The notation $\underline{\boldsymbol{r}} = (\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$ is used for simplicity. Note that we have used the passive nature of the transported field, i.e., the fact that the velocity is independent of the *initial* distribution of ϕ, to separate the averages over the initial conditions and the velocity. The linear operator $\mathcal{P}^{(N)}_{\underline{\boldsymbol{r}}, \underline{\boldsymbol{\rho}}}(t)$ propagates the Nth-order correlation function from time zero to time t.

The evolution operator $\mathcal{L}$ generally includes dissipative terms, and without fresh input (forcing) the statistics of the field ϕ is time dependent; this is the *decaying* case (4). A related problem of much experimental and theoretical interest is *forced* turbulent transport where an input term f is added to Eq. (4). Typically in turbulence the forcing

acts only at large scales of order L. The objects of major interest are the "forced" correlation functions of the transported field at the stationary state,

$$F^{(N)}(\boldsymbol{r}_1 \cdots \boldsymbol{r}_N) \equiv \langle \phi(\boldsymbol{r}_1, t) \cdots \phi(\boldsymbol{r}_N, t) \rangle_f, \qquad (6)$$

where the subscript f refers to the forced rather than the decaying ensemble. In the scaling limit that is obtained when $\kappa \rightarrow 0$, the correlation functions are expected to contain anomalous contributions behaving as

$$F^{(N)}(\lambda \boldsymbol{r}_1 \cdots \lambda \boldsymbol{r}_N) = \lambda^{\zeta_N} F^{(N)}(\boldsymbol{r}_1 \cdots \boldsymbol{r}_N), \qquad (7)$$

with scaling exponents ζ_N which cannot be inferred from dimensional analysis.

The aim of this Letter is to draw attention to two crucial ideas, without further provisos on the velocity field. We do not prove these ideas here; we offer them as conjectures and will exemplify them below using a simple example that strongly differs from the known previous cases. Specifically, the model discussed below lacks the Lagrangian structure heavily relied upon in previous works. This offers evidence for the generality of the following two conjectures:

(i) In the decaying cases, despite the nonstationarity of the statistics, there exist functions $Z^{(N)}(\underline{\boldsymbol{r}})$ such that

$$I^{(N)}(t) = \int Z^{(N)}(\underline{\boldsymbol{r}}) \langle \phi(\boldsymbol{r}_1, t) \cdots \phi(\boldsymbol{r}_N, t) \rangle \, d\underline{\boldsymbol{r}} \qquad (8)$$

are constants of the motion, $I^{(N)}(t) = I^{(N)}(0)$. In the limit of an infinitely large system it does not change with time. In a finite system, and see Fig. 1 as an example, it is constant in time until some outer time scale T_L is reached. It follows from (5) and the conservation of $I^{(N)}(t)$ that in the infinite size limit the $Z^{(N)}$'s are statistically conserved structures, being *left* eigenfunctions of eigenvalue 1 of the linear propagator:

$$Z^{(N)}(\underline{\boldsymbol{r}}) = \int \mathcal{P}^{(N)}_{\underline{\boldsymbol{\rho}}, \underline{\boldsymbol{r}}}(t) Z^{(N)}(\underline{\boldsymbol{\rho}}) \, d\underline{\boldsymbol{\rho}}. \qquad (9)$$

(ii) In general there can be a number of preserved functions for each value of N. We denote by $Z^{(N)}$ the one with the leading exponent. The second conjecture is that the anomalous part of the stationary correlation functions in the forced problem is dominated by the leading preserved function of the decaying problem, i.e., $F^{(N)}(\underline{\boldsymbol{r}}) \sim Z^{(N)}(\underline{\boldsymbol{r}})$. A direct consequence is that the small-scale statistics of the field ϕ in the forced case rests on the understanding of the decaying problem. A by-product is that the scaling exponents ζ_N are universal, i.e., independent of the forcing mechanisms for any forcing that is statistically independent of the velocity field.

To appreciate the generality of those statements it is desirable to review briefly the development of the ideas leading to them. The first instance [4–6] in which the role of statistically conserved structures emerged was passive scalar advection by a velocity field with a short correlation time and self-similar spatial correlations. This is the well known Kraichnan model [10], which is nongeneric, but it

has the advantage that it lends itself to analytic calculations. In the Kraichnan model the simultaneous Nth-order correlation function satisfies a linear differential equation, which is inhomogeneous in the presence of the forcing. The general solution is the sum of the inhomogeneous and homogeneous parts. It turns out that both exhibit scaling properties, but the scaling exponent of the homogeneous part (the zero modes) is leading (smaller) compared to the other. Moreover, the subleading inhomogeneous exponent is predictable by dimensional analysis à la Kolmogorov, whereas the leading exponent is anomalous and calls for an explicit calculation. In the case of the Kraichnan model it is also clear that the homogeneous part of the linear operator governs the rate of change of the correlation function in the decaying case, and thus the zero modes are statistically conserved structures for the decaying problem.

Next, the understanding of the nature of the conserved structures followed from the analysis of the dynamics of *groups* of Lagrangian trajectories of tracer particles [11–14]. The remark is that for the passive scalar equation (1) the transported field is conserved along the trajectories of the tracer particles $d\boldsymbol{R}(t) = \boldsymbol{u}[\boldsymbol{R}(t), t] \, dt + \sqrt{2\kappa} \, d\boldsymbol{\beta}(t)$, where $\boldsymbol{\beta}(t)$ is a Brownian process. To know the scalar field at position $\boldsymbol{r}$ and time t it is enough to track the corresponding tracer particle back to the initial position $\boldsymbol{\rho}$. The evolution operator $\mathcal{P}^{(N)}_{\underline{\boldsymbol{r}}, \underline{\boldsymbol{\rho}}}(t)$ in (5) coincides then with the probability density that N tracer particles reach the positions $\underline{\boldsymbol{r}}$ at time t given their initial positions $\underline{\boldsymbol{\rho}}$. For example, to understand the exponent ζ_3 one needs to focus on the dynamics of three tracer particles. Obviously, three particles define at any moment of time a triangle, which in its turn is fully characterized by one length scale R (say the sum of the lengths of its sides), two of its internal angles, and all the angles that specify the orientation of the triangle in space. When the particles are advected by the turbulent velocity field, the scale R of the triangle and its shape (angles) change continuously. The statement that can be made is that *there exist distributions on the space of the triangle configurations, that are statistically invariant to the turbulent dynamics* [11,12,15]. In other words, if we release trios of Lagrangian tracers many times into the turbulent fluid, and we choose the distribution of their shapes and sizes correctly, it will remain invariant to the turbulent advection. Such statistically conserved structures are the aforementioned zero modes and they come to dominate the statistics of the scalar field at small scales. The anomalous exponents of the zero modes, such as ζ_3, can be understood as the rescaling exponents characterizing precisely such special distributions. Of course, the same ideas apply to any order correlation function with the appropriate shape dynamics. The relevance of Lagrangian trajectories can also be demonstrated for the magnetic field case (2), by adding a tangent vector to the tracer particle; see [16] for more details.

All this was established for the nongeneric Kraichnan model of passive scalar advection. Generalizing the zero-mode ideas to generic cases has a clear importance, but it calls for coping with two different problems. First, if the velocity field is not δ correlated in time, we have no time-independent operators anymore, and the idea of zero modes as functions annihilated by a fixed operator cannot be carried over. The correlation functions still obey linear equations of motion, but the nonvanishing correlation time of the velocity field statistically couples different times. It is therefore not obvious that single-time (simultaneous) objects might still be statistically conserved. The first indication in favor of the generality of the picture came in [17]. The velocity field is generated by the two-dimensional Navier-Stokes equations (in the inverse cascade regime) and has a finite correlation time. It is clear from the previous discussion that breaking the nongeneric approximation of δ correlation in time is the crucial point. The numerical evidence is that statistically conserved structures are still present. Two parallel things have been done: on the one hand, the third-order correlation function $\langle \phi(r_1)\phi(r_2)\phi(r_3) \rangle$ was measured in forced simulations. This yielded the distribution expected to be invariant. On the other hand, trios of Lagrangian trajectories were released into the flow, and it was demonstrated that their evolution agreed with the notion of statistical invariance.

The second problem is that the formulations proposed up to now heavily exploit the Lagrangian structure of the dynamics. In this Letter we take the ideas further and demonstrate that their generality transcends the applicability of Lagrangian trajectories. The moment that we consider other models of transport, e.g., with pressure, the properties of Nth-order correlation functions can no longer be connected to N Lagrangian tracer particles. A new formulation of the statistical conservation laws is thus needed. That is what (8) provides for. It is clear from it that the key point is not the Lagrangian structure of the correlation function propagation but that the linear operator is associated with (statistically) preserved functions [cf. Eq. (9)]. The preserved functions then define the integrals of motion. Of course, this still does not necessarily mean that they also dominate the field correlations in the forced case. To prove it in full generality is difficult and certainly beyond the scope of this Letter. Rather, we demonstrate the point with a nontrivial example.

To address the issue in a model as different as possible from those considered so far, we analyze shell models, where the discretization of the phase space destroys any Lagrangian structure. A possible discretization of the field equation (4) reads [18]

$$\frac{d\phi_m}{dt} = i(k_{m+1}\phi_{m+1}u_{m+1} + k_m\phi_{m-1}u_m^*) - \kappa k_m^2 \phi_m,$$

$$\equiv \mathcal{L}_{m,m'}\phi_{m'}, \tag{10}$$

where the u_n variables are generated by the "Sabra" shell

model [19]

$$\frac{du_n}{dt} = i(ak_{n+1}u_{n+2}u_{n+1}^* + bk_nu_{n+1}u_{n-1}^*$$

$$+ ck_{n-1}u_{n-1}u_{n-2}) - \nu k_n^2 u_n + f_n, \tag{11}$$

where the coefficients a, b, and c are real. In our simulations $\kappa = \nu = 5 \times 10^{-7}$, $a = 1$, $b = -0.4$, and $c = a + b$. The wave vectors are $k_n = k_0 2^m$ with $n = 0,\ldots,N$. The smallest wave vector is given by $k_0 = 0.05$, while N defines the ultraviolet cutoff. The velocity forcing f_n is limited to the first shell $n = 0$. For $\kappa = \nu = 0$ the energies $\sum_n |u_n|^2$ and $\sum_n |\phi_n|^2$ are *dynamically conserved,* i.e., realization by realization. The operator $\mathcal{P}^{(N)}$ of Eq. (5) takes here the explicit form $\langle R(t\,|\,t_0)\cdots R(t\,|\,t_0)\rangle$, where $R(t\,|\,t_0) \equiv T^+ \exp[\int_{t_0}^t ds\,\mathcal{L}(s)]$, with T^+ being the time ordering operator.

To demonstrate the *statistical* conservation laws, two things were done. First, we considered the forced problem, adding random forcing to the first shell of (10). We have measured all the available correlation functions of second, fourth, and sixth order. Because of phase symmetry constraints, these are (we put a subscript f to stress that these are statistical averages in the stationary forced ensemble):

$$F_2(n) \equiv \langle |\phi_n|^2 \rangle_f, \tag{12}$$

$$F_4^{(1)}(n,m) \equiv \langle |\phi_n|^2|\phi_m|^2 \rangle_f, \tag{13}$$

$$F_4^{(2)}(n) \equiv \langle \phi_{n+2}\phi_{n+1}^*\phi_{n+1}^*\phi_{n-1} \rangle_f, \tag{14}$$

$$F_6(n,m,k) \equiv \langle |\phi_n|^2|\phi_m|^2|\phi_k|^2 \rangle_f. \tag{15}$$

Second, we studied the decaying problem, preparing initial states $\phi_n(t = 0)$ and following their evolution. As initial states we took distributions of $\phi_n = 0$ except for $n = 14, 15$, where we initialized the field with a constant modulus and random phases. In light of Eq. (8) and conjecture (ii) above we then computed the following objects:

$$I^{(2)}(t) \equiv \sum_n \langle |\phi_n|^2 \rangle (t) F_2(n), \tag{16}$$

$$I^{(4)}(t) \equiv \sum_{n,m} \langle |\phi_n|^2|\phi_m|^2 \rangle (t) F_4^{(1)}(n,m)$$

$$+ \sum_n \langle \phi_{n+2}^*\phi_{n+1}\phi_{n+1}\phi_{n-1}^* \rangle (t) F_4^{(2)}(n), \tag{17}$$

$$I^{(6)}(t) \equiv \sum_{n,m,k} \langle |\phi_n|^2|\phi_m|^2|\phi_k|^2 \rangle (t) F_6(n,m,k). \tag{18}$$

Figure 1 summarizes the results. We show, for these three orders, (i) the time dependence of the nth-order decaying correlation functions themselves and (ii) the time dependence of $I^{(N)}(t)$. In panel (C) we show also for comparison the time dependence of $I^{(6)}(t)$ if we replace the measured forced F_6 by its dimensional shell dependence (i.e., the shell dependence if the Kolmogorov theory is right). We see that only the properly computed $I^{(n)}(t)$ are time independent for times smaller than the large
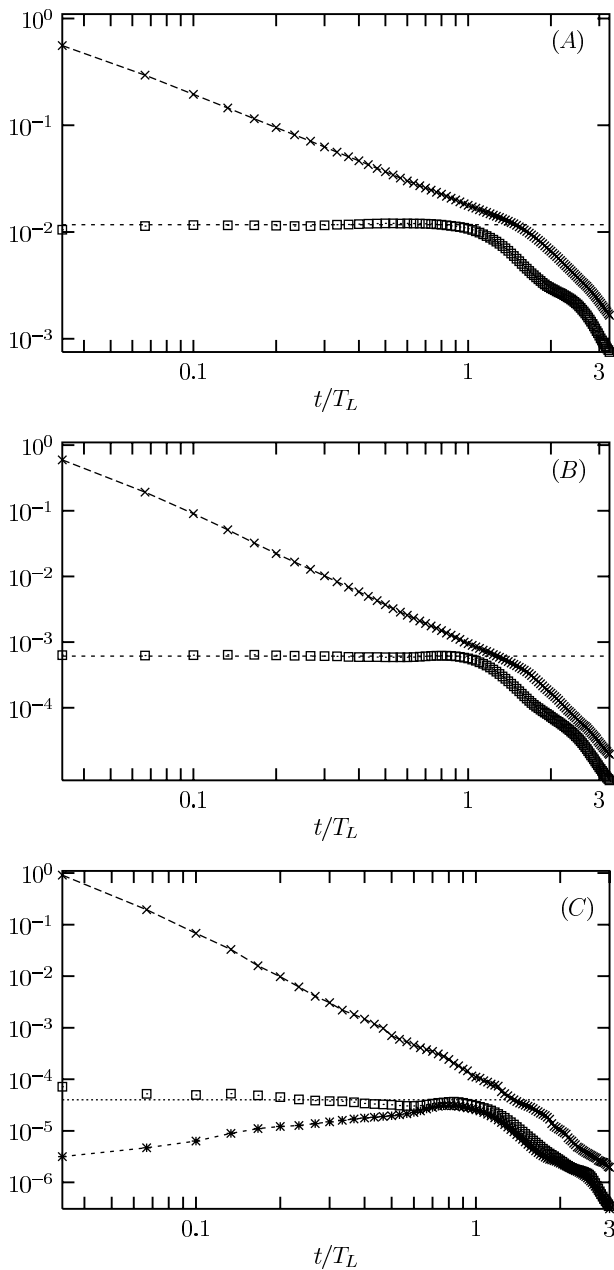
FIG. 1. Panel (A): time dependence of the decaying second-order correlation functions ($\times$), together with the time dependence of the statistically conserved quantities $I^{(2)}$ ($\square$). Equations (11) and (10) have been integrated with a total number of shells $N = 33$. Time in the horizontal axis is given in units of the eddy turnover time T_L. Panel (B): the same as panel (A) but for the fourth-order correlation function and with $N = 25$. Panel (C): the same as panel (B) but for the sixth-order correlation function. Here we also present $I^{(6)}$ when we replace the forced solution $F_6(n, m, k)$ with its dimensional prediction ($*$).

scale eddy turnover time, T_L. The decay observed for times larger than T_L is simply due to finite size effects intervening when the decaying field reaches the largest scales. Indeed, the correlation functions of the forced problem (that we used for $Z^{(N)}$) are not dominated by the inertial contributions at those scales.

In conclusion, the importance of statistical conservation laws for turbulent transport was demonstrated. We have exemplified both the existence of statistically conserved objects *and* the fact that they come to dominate the small-scale behavior of the forced stationary correlation functions. The only provisos are that the turbulent velocity field has the usual scaling properties observed in turbulence and that the transported field is passive. That offers a unifying picture to the statistical theory of turbulent transport. We propose that further research, both numeric and analytic, can contribute to the solidification of the generality of these ideas.

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