LETTERS

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Isotropy vs anisotropy in small-scale turbulence

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The decay of large-scale anisotropies in small-scale turbulent flow is investigated. By introducing two different kinds of estimators we discuss the relation between the presence of a hierarchy for the isotropic and the anisotropic scaling exponents and the persistence of anisotropies. By direct measurements on a channel flow numerical simulation we show that the presence of a hierarchy for the isotropic and the anisotropic scaling exponent is not in contradiction with the persistence of anisotropies at small scales. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1381019]

One of the main assumptions made by Kolmogorov in his 1941 theory is the restoring of universality and isotropy at small scales in turbulent flows. The idea is that the effects of a large-scale anisotropic forcing and/or boundary conditions are rapidly lost during the process of energy transfer toward small scales. The overall result is that both isotropy and universality of turbulent fluctuations should be locally restored at small enough scales and large enough Reynolds numbers. The rate of convergence toward isotropy can be quantitatively predicted within the K41 theory both as a function of the scale, e.g., for the structure functions, and as a function of the Reynolds numbers, e.g., for the single-point moments of velocity gradients. Experiments[1,2] and numerical simulations[3–5] do not confirm those predictions. The skewness of the transversal gradients, $S_3 = \langle (\partial_y u_x)^3 \rangle / \langle (\partial_y u_x)^2 \rangle^{3/2}$ is, for example, found to have a very slow decay with Re_λ. The effect is even stronger for the fifth-order skewness $S_5 = \langle (\partial_y u_x)^5 \rangle / \langle (\partial_y u_x)^2 \rangle^{5/2}$, observed to remain $O(1)$ for all available Re_λ. Similar results were recently reported on a series of hydrodynamical problems. The most striking ones were obtained analytically in passive scalar/vector models advected by isotropic, Gaussian and white-in-time velocity fields (the so-called Kraichnan model[6]) with a large scale anisotropic forcing.[7,8] Numerical[9,10] and experimental (see, e.g., Refs. 11 and 12) evidences of persistence of anisotropies in real passive scalars have also been reported.

On the one hand, there are then strong indications in favor of a persistent memory of the large-scale anisotropies even at the smallest scales of a turbulent flow. On the other hand, there are theoretical arguments[13] going in the opposite direction, i.e., that anisotropic fluctuations are subdominant with respect to the isotropic ones (see below). This short note is meant to clarify the relation between the previous results and support the arguments by numerical simulations on channel flow turbulence.

The analysis in Ref. 13 is based on the invariance under rotations of the unforced Navier–Stokes equations. We shall specifically restrict here to the structure functions and refer to the original paper for more complex tensorial objects. Since the Navier–Stokes equations are invariant under rotations, the correlations are conveniently decomposed in terms of the irreducible representations of the rotation group. For the nth order longitudinal structure function we have

$$S_n(\mathbf{r}) = \langle [(\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x}+\mathbf{r})) \cdot \mathbf{r}]^n \rangle = \sum_{jm} S_n^{jm}(|r|) Y_{jm}(\hat{\mathbf{r}}),$$

(1)

where we have used the fact that the basis of the rotation group for scalar functions is the set of spherical harmonics Y_{jm}. The coefficients $S_n^{jm}(|r|)$ are expected to behave as power laws $r^{\xi_n^j}$ and the scaling exponents, ξ_n^j, are expected to depend only on the index j (the exponents should not depend on the m-index since it does not appear in the equations of motion, see Ref. 13 for more details). The previous strong assumption is motivated by the idea of universality, i.e., that inertial-range scaling behaviors are independent of the large-scale boundary and forcing effects. Furthermore, it is natural to suppose a hierarchical organization of the different sectors in the inertial range, i.e., the existence of a hierarchy among the scaling exponents characterizing different sectors,

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$$\xi_n^{j=0} < \xi_n^{j=1} < \xi_n^{j=2} < \cdots. \tag{2}$$

This statement, even if not proved for the Navier–Stokes equations, is verified analytically in various Kraichnan models of passive fields.[14,15] The existence of the hierarchy (2) implies that the anisotropic fluctuations become more and more subdominant at the small scales as their degree of anisotropy increases.

Let us now analyze in a quantitative way the relative importance of isotropic and anisotropic fluctuations. In the following we shall concentrate for simplicity on the structure functions, but the same arguments could be generalized to other correlations. Isotropic flows are characterized by having only the sector $j=0$, $m=0$ excited. One is therefore naturally led to introduce two different tests to quantify the degree of isotropy/anisotropy. First [case (A)], one can analyze fluctuations of comparable intensity, i.e., fixing the order n of the structure function and measuring the scaling in different sectors. We can introduce the ratio between the projection on the anisotropic sector with the nonvanishing indices j, m and the projection on the isotropic sector $j=m=0$,

$$T_n^{jm}(r) \equiv \frac{S_n^{jm}(r)}{S_n^{00}(r)}. \tag{3}$$

We thus have the possibility to disentangle different degrees of anisotropy depending on the typical intensity of the velocity fluctuations. Looking at the structure functions of low order (small n's) gives a test on the isotropy of the weak fluctuations, while looking at high orders (large n's) gives a test on the statistics of strong turbulent fluctuations. A second possible estimator [case (B)] consists of first normalizing the field and then taking moments of it. As it is done for the skewness and the kurtosis, we can for example normalize by the isotropic component of the second order longitudinal structure function, $S_2^{00}(|r|) = \langle [((\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x}+\mathbf{r})) \cdot \mathbf{r}]^2 \rangle_{j=0,m=0}$. The resulting dimensionless stochastic variable can then be studied by looking at its decomposition in different j, m sectors,

$$\hat{S}_n^{jm}(|r|) \equiv \frac{S_n^{jm}(|r|)}{(S_2^{00}(|r|))^{n/2}}. \tag{4}$$

If the hierarchy (2) holds, all the observables of case (A) tend to zero as the scale is decreased. The decay rates possibly differ from the dimensional predictions due to intermittency, but they are guaranteed to be positive. There is no experimental or numerical evidence that the hierarchy (2) is violated. The situation with observable of case (B) is quite different. The dimensionless quantities are indeed formed by comparing anisotropic and isotropic fluctuations of different intensity [in the numerator and denominator of (4) structure functions of different orders are involved]. The hierarchy (2) does not give any constraint in this case and it is well possible that $\xi_n^j < (n/2)\xi_2^{j=0}$. The corresponding observable of case (B) $\hat{S}_n^{jm}$ defined in (4) would then diverge going toward the small scales, even in the presence of the hierarchy (2). That divergence is the effect of persistence of anisotropies reported in experiments and numerical simulations both for the passive scalars and Navier–Stokes turbulence (see Refs.
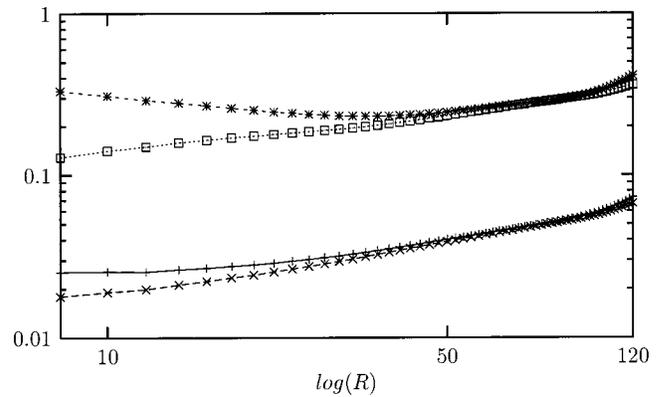
FIG. 1. Analysis of the persistence of anisotropies with observable belonging to cases (A) and (B) for the projection of structure functions in the sector $j=2$, $m=2$. Bottom: projections of fourth moment, $T_4^{2,2}(|r|)$ ($\times$) and $\hat{S}_4^{2,2}(|r|)$ ($+$). Top: projection of sixth moment, $T_6^{2,2}(|r|)$ ($\square$) and $\hat{S}_6^{2,2}(|r|)$ ($*$). Notice how for the moment of order 6 we have a clear tendency toward increasing of $\hat{S}_6^{2,2}(|r|)$ at small scales. Scales are dropped at $R \sim 10$ which corresponds to the onset of the viscous scale in the simulation. For details on how to compute numerically the projections, $S_n^{jm}(|r|)$, see Ref. 16.

7 and 12). It is important to notice that, in presence of hierarchy (2), the persistence of anisotropies is only possible thanks to the existence of intermittent corrections in the anisotropic sectors. If hierarchy (2) is valid and there is not intermittency in all sectors, i.e., $\xi_n^j = (n/2)\xi_2^j$, then the observable of case (B) with positive j would vanish at small scales.

Let us now support the above arguments by presenting some results obtained in channel flow simulations. The simulations are performed on a grid of $128 \times 128 \times 256$ points with periodic boundary conditions in the streamwise and spanwise directions and no-slip boundary conditions at the top and bottom walls. At the center of the channel we have $\mathrm{Re}_\lambda \sim 70$. Due to the relatively moderate Reynolds number, no scaling laws are observed. Still, even in the absence of scaling laws, it is quite clear from the data that the two sets of observable defined in cases (A) and (B) behave in a very different way. In Fig. 1 we present the quantities defined in cases (A) and (B) for the structure functions of order 4 and 6 at the center of the channel for the sector $j=2$, $m=2$. In Fig. 2 the same is presented but for a higher sector, $j=4$, $m=2$.
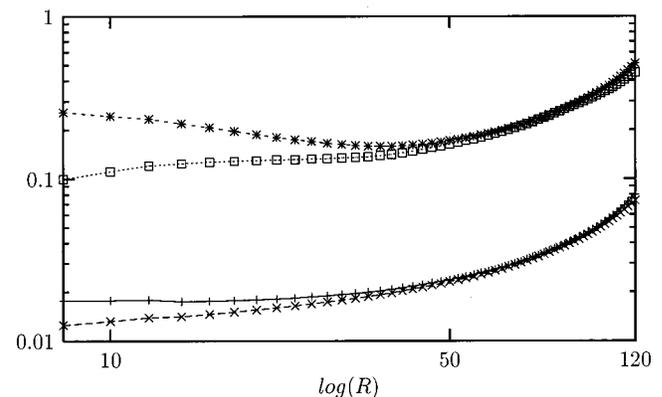
FIG. 2. The same as Fig. 1 but for the sector $j=4$, $m=2$. Bottom: projections of fourth moment, $T_4^{4,2}(|r|)$ ($\times$) and $\hat{S}_4^{4,2}(|r|)$ ($+$). Top: projection of the sixth moment, $T_6^{4,2}(|r|)$ ($\square$) and $\hat{S}_6^{4,2}(|r|)$ ($*$).

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While the observable of case (A) always monotonically decreases with the scale, the observable of case (B) for the sixth order moments shows a clean tendency to increase. That is the manifestation of persistence of anisotropies at small scales and gives further support to the observations first made in Ref. 4. Note that the scales shown in the figure go from the largest available one (the box size) to the beginning of the viscous scale. The decomposition in spherical harmonics at the very small scales (inside the viscous range) is hard to obtain because of interpolation errors of the cubic grid on the sphere. Details on the numerical procedure to compute the projections on different sectors can be found in Ref. 16. Let us stress again that by using the SO(3) decomposition we have an exact separation of isotropic and anisotropic fluctuations. Therefore the persistence of anisotropies observed at small scales cannot be due to spurious large scales contamination of the small scales statistics.

As for the intermittency in the anisotropic sectors, the situation is still moot. Experimentally, the lack of control on the whole velocity field does not allow us to perform an exact projection on each separate sector, only indirect fits of the superposition of many anisotropic contribution is possible.[17,18] There is only one attempt to directly measure the projections on each single sector in the same channel flow data set used here.[16,19] As stated previously, the Reynolds number is unfortunately not high enough and scaling exponents of the anisotropic sectors can be measured only via the ESS.[20] In the anisotropic sectors it is not even quite clear what would be the dimensional prediction for the ξ_n^j with $j > 0$. Different dimensionless quantities can indeed be built by using some anisotropic mean observable, e.g., the mean shear, and the usual energy dissipation. The dimensional predictions would then depend on the requirement that the anisotropic correction is (or is not) an analytical, smooth deviation from the isotropic sector. Furthermore, the comparison with the behavior observed in the Kraichnan models of scalar/vector fields[6,21] suggests that the anisotropic sectors may show intermittent corrections induced by the homogeneous (nonlinear, in the Navier–Stokes case) part of the equations for the correlation functions. If that is the case, the dimensional predictions might be very far from the observed behaviors.

In conclusion, we have discussed the decay of large-scale anisotropy memory in the small scales of turbulent flows. The analysis of numerical data from channel flow simulations indicate that the anisotropies persist at the small scales but still respecting the hierarchy (2) between the isotropic and anisotropic velocity components.

Recently, a further support to the hierarchy (2) has been obtained in a numerical simulation of homogeneous-anisotropic turbulence.[22]

ACKNOWLEDGMENTS

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