Fluctuation-response relation in turbulent systems

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(Received 29 May 2001; published 14 December 2001)

We address the problem of measuring time properties of response functions (Green functions) in Gaussian models (Orszag-McLaughin) and strongly non-Gaussian models (shell models for turbulence). We introduce the concept of *halving-time statistics* to have a statistically stable tool to quantify the time decay of response functions and generalized response functions of high order. We show numerically that in shell models for three-dimensional turbulence response functions are inertial range quantities. This is a strong indication that the invariant measure describing the shell-velocity fluctuations is characterized by short range interactions between neighboring shells.

DOI: 10.1103/PhysRevE.65.016302 PACS number(s): 47.27.−i

I. INTRODUCTION

The fluctuation-response (F/R) relation plays an important role in statistical mechanics and, more generally, in systems with chaotic dynamics. With the term F/R relation we indicate the connection between the relaxation properties of a system and its response to an external perturbation. The relevance of this relation is evident: it allows us to connect "nonequilibrium" features (i.e., response and relaxation) to "equilibrium" [1] properties (correlation functions). As an important example, we mention the Green-Kubo [2] formulas in the linear response theory which links the response to an external field with correlations computed at equilibrium.

Consider a system whose state is given by a finite dimension vector $\mathbf{x}=(x_1,\ldots,x_N)$, the average linear response $G_i^j(t)\equiv\langle R_i^j(t)\rangle$ is the average response after a time t of the variable x_i to a small perturbation of the variable x_j at time $t=0$. Under rather general conditions (basically one has to assume that the system is mixing) it is possible to show that a generalized F/R relation holds [3,4]:

$$\langle R_j^i(t)\rangle=\left\langle\frac{\delta x_i(t)}{\delta x_j(0)}\right\rangle=\langle x_i(t)f_j[\mathbf{x}(0)]\rangle, \tag{1}$$

where the functions f_j depend on the invariant probability distribution $\rho(\mathbf{x})$,

$$f_j[\mathbf{x}]=-\frac{\partial\ln\rho(\mathbf{x})}{\partial x_j}. \tag{2}$$

The physical meaning of Eq. (1) is the following: consider a small perturbation $\delta\mathbf{x}(0)=(\delta x_1(0),\ldots,\delta x_N(0))$ at $t=0$; the average distance $\langle\delta x_i(t)\rangle$ from the unperturbed values $x_i(t)$ is

$$\langle\delta x_i(t)\rangle=\sum_j\langle R_j^i(t)\rangle\delta x_j(0). \tag{3}$$

For Hamiltonian systems one realizes that Eq. (1) is the usual linear response theory. If $\rho(\mathbf{x})$ is Gaussian, one has a simple relationship between the response and the correlation function

$$\langle R_j^i(t)\rangle=\frac{\langle x_i(t)x_j(0)\rangle-\langle x_i\rangle\langle x_j\rangle}{\langle x_ix_j\rangle-\langle x_i\rangle\langle x_j\rangle}. \tag{4}$$

In the general case of non-Gaussian statistics, formula (1) just gives a qualitative information, i.e., the existence of a link between response and the general correlation function $\langle x_i(t)f_j[\mathbf{x}(0)]\rangle$. In particular, in the most interesting cases in which $\rho(\mathbf{x})$ is unknown, it is extremely important that the F/R relation (1) exists because it allows us to control some properties of the invariant measure $\rho(\mathbf{x})$ in terms of the response-functions behavior. In the past, this has not always been clear, e.g., some authors claim (with qualitative arguments) that in fully developed turbulence there is no relation between equilibrium fluctuations and relaxation to equilibrium [5], while a proper statement would limit to the nonexistence of the usual "Gaussian-like" F/R relation (4).

Response functions have a clear phenomenological importance in many applied problems where one needs to control and/or predict the system reaction as a function of external spatial and/or temporal perturbations. Moreover response functions, also known as Green functions, play a very important role in many nonequilibrium problems. In particular, in many analytical approaches to hydrodynamical problems described by Navier-Stokes equations, or models of them, Greens functions naturally show up both in perturbative [6] and closure schemes like the direct interaction approximation (DIA) [7].

We stress that the F/R, in the form (1), is a rather general relation which does not depend too much on the details of the measure of the systems, e.g., both in the presence or absence of an energy flux. For example, in the field of disordered systems the F/R had been widely studied in order to highlight nontrivial relaxation aspects, e.g., aging phenomena. In this paper we want to address the problem of the F/R relation for the case of dynamical models with many degrees of freedom and many characteristic times. We are also interested in exploiting the F/R relation in models which exhibit a strong departure from Gaussian statistics. We introduce a suitable numerical method for measuring the characteristic times involved in the response functions.

This method is based on the idea of characterizing the response behavior as a function of its *halving time statistics* (HTS), i.e., the time τ necessary for the response from a typical infinitesimal perturbation to reach, say, one-half of its initial value.

The plan of the paper is as follows. First we investigate a dynamical system with many degrees of freedom and many different characteristic times where still a classical Gaussian set of F/R relations holds. The model is the so-called "Orszag-McLaughlin" model which is used to probe the effective improvement of halving time statistics with respect to the usual direct measurement of time decaying properties. Then, we attack the much less trivial case of characterizing response behavior in models for three-dimensional turbulent energy cascade, i.e., shell models [8]. Also in the latter case, the halving time statistics will allow us to measure with good accuracy the nontrivial time properties of Green functions. We show that the response function (which probes linear features of the dynamical evolution) is strongly affected by the *nonlinear* inertial range physics. As a consequence, short range interactions between neighboring shells are thought to characterize the invariant measure describing shell-velocity fluctuations.

II. NUMERICAL SIMULATIONS

Before entering the detailed description of the results, we want to discuss a practical problem for the numerical computation of $G^i_j(t)=\langle R^i_j(t)\rangle$. In numerical simulations, $\langle R^i_j(t)\rangle$ is computed perturbing the variable x_i at time $t=t_1$ with an "infinitesimal" kick of amplitude $\delta x_j(t_1|t_1)=\tilde{x}_j(t=t_1)-x_j(t=t_1)=\epsilon$, for $\epsilon\to 0$, and the deviation $\delta\mathbf{x}(t|t_1)=\tilde{\mathbf{x}}(t)-\mathbf{x}(t)$ is computed integrating the two trajectories $\mathbf{x}$ and $\tilde{\mathbf{x}}$ up to a prescribed time $t_2=t_1+\Delta t$. At time $t=t_2$ the variable x_i is again perturbed with another kick, and a new sample $\delta\mathbf{x}(t)$ is computed and so forth. The procedure is repeated $M\ll 1$ times and the mean response is then evaluated as

$$G^j_i(t)=\frac{1}{M}\sum_{k=1}^{M}\frac{\delta x_i(t_k+t|t_k)}{\delta x_j(t_k|t_k)}. \tag{5}$$

In the presence of chaos, the absolute value of the deviation, $|\delta x_i(t_k+t,|t_k)|$, typically grows exponentially with t. Therefore, the mean response, $\langle R^i_j(t)\rangle$, is the result of a delicate balance of terms with nonfixed sign. As a result, we have that the error $\sigma(t)$ on $\langle R^i_j(t)\rangle$ increases exponentially with t,

$$\sigma(t)\sim\frac{\exp(\gamma t)}{\sqrt{M}}, \tag{6}$$

where γ is the generalized Lyapunov exponent of second order (greater than or equal to the maximum Lyapunov exponent). One easily understands the main problem in trying to numerically compute any response function for large times: one needs to control an observable which is rapidly decaying to zero with exponentially large fluctuations. In practice, it turns out to be impossible to have a reliable control on the asymptotic behavior of Green functions (see the following sections and figures therein).

In order to avoid this trouble we propose another approach. Let us first consider only *diagonal* responses, i.e., the response after a time t of the nth variable from a perturbation of the same nth variable at time $t=0$, $G^n_n(t)\equiv\langle R^n_n(t)\rangle$.

In this case, we claim that it is possible to have a good characterization of the main temporal properties by looking at the HTS, that is at the probability density functions $\mathcal{P}(\tau)$ of the time τ necessary to see an appreciable decay of the response function: $R^n_n(\tau=t)=\lambda R^n_n(0)$, with the threshold λ fixed to a macroscopic value, say $\lambda=1/2$. In practice, one performs many response experiments by collecting the statistics of the times necessary to see the response become one-half of its initial value. The advantage of this HTS with respect to the more standard way of characterizing the mean response $G^n_n(t)$ with some typical time is that one does not need to know any functional behavior for the averaged response and, moreover, one has also a control on the fluctuations of the characteristic times, i.e., the HTS integrates all times corresponding to *halving events*. In the following, we show that the HTS is at least able to reproduce with good accuracy the same results of the direct fitting procedure of the averaged response in cases when the classical F/R relation (4) holds (the Orszag-McLaughlin model, i.e., Gaussian statistics) and, more interesting, it is also able to give new hints on the F/R relations when time intermittency and a strong departure from Gaussianity are present (shell models). In the following we will also discuss the cases of nondiagonal responses, $\langle R^n_m(t)\rangle$, with $n\neq m$ and the cases of generalized higher order responses

$$\langle R^{n_1,n_2,\ldots,n_r}_{m_1,m_2,\ldots,m_r}(t_1,t'_1;t_2,t'_2;\ldots;t_r,t'_r)\rangle =\left\langle \frac{\delta x_{n_1}(t_1)}{\delta x_{m_1}(t'_1)}\frac{\delta x_{n_2}(t_2)}{\delta x_{m_2}(t'_2)}\cdots\frac{\delta x_{n_r}(t_r)}{\delta x_{m_r}(t'_r)}\right\rangle.$$

A. The Orszag-McLaughlin model

Let us consider the following model [9]:

$$\frac{dx_n}{dt}=x_{n+1}x_{n+2}+x_{n-1}x_{n-2}-2x_{n+1}x_{n-1}, \tag{7}$$

with $n=(1,2,\ldots,N)$, $N=20$, and the periodic condition $x_{n+N}=x_n$. This model contains some of the main features of inviscid hydrodynamics: (a) there are quadratic interactions; (b) a quadratic invariant exists ($E=\Sigma_{n=1}^{N}x_n^2$); (c) the Liouville theorem holds. For sufficiently large N the distribution of each variable x_n is Gaussian. In this situation, a classical F/R relationship exists for each of the n variables: self-

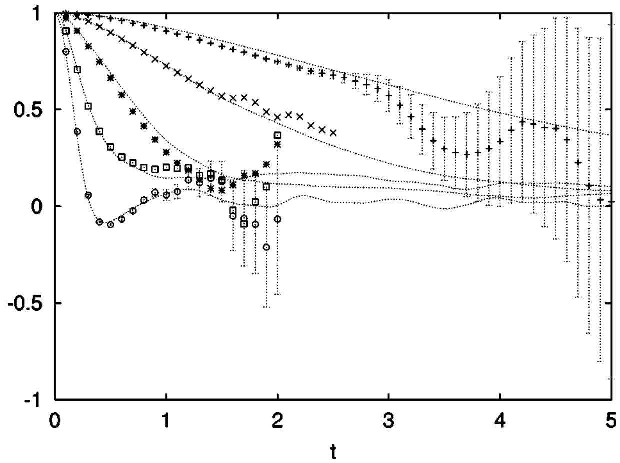

FIG. 1. Plot of the averaged response functions $G_n^n(t)$ and correlation functions $C_{n,n}(t)$ for five fast variables of the modified Orszag-McLaughlin model, $n=6$ (+), $n=7$ (×), $n=8$ (*), $n=9$ (□), and $n=10$ (○). Statistical error bars are shown only for response functions corresponding to $n=6$ and $n=10$. Thin lines represent correlation functions. The statistics is over 10^5 events.

response functions to infinitesimal perturbations are indistinguishable from the corresponding self-correlation functions [3].

We have slightly modified the system (7) in order to have variables with different characteristic times. This can be done, for instance, by rescaling the evolution time of each variable,

$$\frac{dx_n}{dt}=k_n(x_{n+1}x_{n+2}+x_{n-1}x_{n-2}-2x_{n+1}x_{n-1}), \tag{8}$$

where the factor k_n is a function of the ''number of identification'' (e.g., a site in the chain) of the variables defined as $k_n=\alpha\beta^n$, with $\alpha=5\times10^{-3}$ and $\beta=1.7$, for $n=1,2,\ldots,N/2$, with the ''mirror'' property $k_{n+N/2}=k_{N/2+1-n}$. An immediate consequence is that the quadratic observable E is no longer invariant during the time evolution of the system (8). The mean energy per mode, $E_n=\langle x_n^2\rangle$ (not shown), follows a linear law with $k=k_n$. It can be demonstrated that a new quadratic integral of motion exists, and this has the form

$$I=\sum_{n=1}^{N}\frac{x_n^2}{k_n}. \tag{9}$$

Moreover, the x_n variables are shown to preserve the Gaussian statistics to a good extent. Therefore, the only effect of the change in the original Orszag-McLaughlin system is that each variable now has its own characteristic time.

Let us see how correlation and response functions behave for the system (8). In Fig. 1 the self-correlation functions

$$C_{n,n}(t)=\frac{\langle x_n(t)x_n(0)\rangle-\langle x_n\rangle^2}{\langle x_n^2\rangle-\langle x_n\rangle^2} \tag{10}$$

and the self-response functions

$$R_n^n(t)=\left\langle\frac{\delta x_n(t)}{\delta x_n(0)}\right\rangle \tag{11}$$

are shown. As a consequence of the preserved Gaussian statistics, the F/R relation of the form (4) holds for each of the variables, at least over time delays not too long. The (linear) response functions are computed as decay functions of single variable perturbations to infinitesimal instantaneous ''kicks,'' averaged over a large number of simulations. If we conventionally define the correlation time of a variable x_n as the time delay $\tau_C(n)$ after which the correlation function becomes lower than the value 1/2, we find that

$$\tau_C(n)\sim k_n^{-3/2}. \tag{12}$$

The exponent of the scaling law (12) can be explained with a dimensional argument by noticing that from the mean energy per variable we get $x_n^2\sim k_n$, so from Eqs. (8) and (9) the characteristic time results to be $\tau_C(n)\sim k_n^{-3/2}$. We notice that the scaling (12) is robust with respect to the choice of the threshold value, λ, i.e., it is observed even if the decay factor λ is chosen slightly different from 1/2.

The response time $\tau_R(n)$ is defined as the time interval after which the averaged response function becomes lower than 1/2. We must observe that the computation of the mean response function is practically impossible after a certain time delay because of exponentially growing errors.

Last, halving times $\tau(n)$ have been computed always for the variables of the system (8), using the same procedure as before (i.e., infinitesimal kicks). The halving-time probability distribution functions (PDF's) decay exponentially and, as can be shown, they all can be ''collapsed'' to the same renormalized PDF for a proper rescaling of the halving time (see below). A comparative plot of $\langle\tau(n)\rangle$, $\tau_C(n)$, and $\tau_R(n)$ is shown in Fig. 2. Halving times and correlation times follow the same scaling law with k_n and, for each variable, have values very close to each other. Typical time decaying of the averaged response, $\tau_R(n)$ are very difficult to estimate due to high errors for slow variables. In fact, only a few points are shown in Fig. 2, the ones for which the mean response drops down to 1/2 fast enough, before the statistical error becomes too large (say larger than 100%). The advantage of the HTS with respect to the mean response function is that, with the same statistics, halving times can be computed for all variables within reasonable uncertainty, while response times $\tau_R(n)$ are generally affected by exponentially growing errors and are practically not defined when the typical relaxation time scale is longer than the error growth time scale.

The numerically computed PDF's of the halving time τ can be rescaled as follows:

$$\tau\to\frac{\tau}{\langle\tau\rangle},\quad P(\tau)\to\langle\tau\rangle P(\tau). \tag{13}$$

We show in Fig. 3 the overlap of some rescaled PDF's of the halving times. As a consequence, all moments of the $\tau(n)$ PDF's have a simple scaling

$$\langle\tau(n)^p\rangle\sim k_n^{-(3/2)\cdot p}.$$

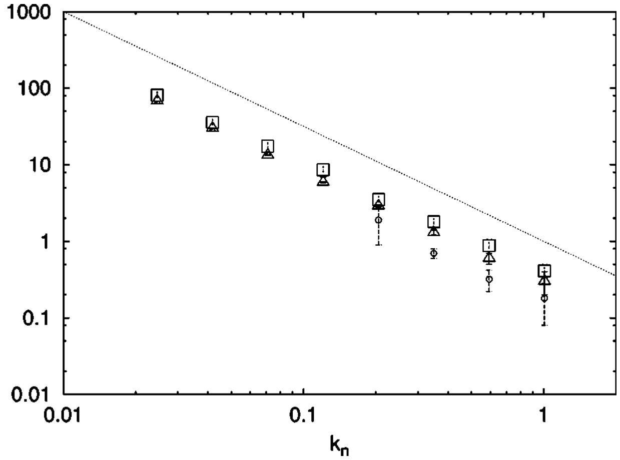

FIG. 2. Log-log plot of correlation times $\tau_C(n)$ ($\triangle$), mean halving times $\langle\tau(n)\rangle$ ($\square$), and response times $\tau_R(n)$ ($\bigcirc$) as a function of k_n for the modified Orszag-McLaughlin model. Notice the much larger errors found when measuring the characteristic response times, $\tau_R(n)$. Errors on $\tau_C(n)$ and $\langle\tau(n)\rangle$ are of the same size as the representative symbols. The statistics is over 10^5 realizations. All these characteristic times follow the same scaling law with k_n. The exponent $-3/2$ of the scaling law follows from dimensional arguments.

We have found that in a Gaussian case, the response to infinitesimal perturbations can be characterized both with the classic mean response function and with the mean halving-time technique. It is worth stressing that the HTS could be the only technique usable for studying relaxation to nonlinear perturbation in complex systems.

B. Shell model

Shell models for a turbulent energy cascade have proved to share many statistical properties with turbulent three-dimensional velocity fields [10,8]. Let us introduces a set of wave numbers $k_n = 2^n k_0$ with $n=0,\dots,N$. The shell-velocity variables $u_n(t)$ must be understood as the velocity fluctuation over a distance $l_n = k_n^{-1}$. It is possible to write down many different sets of coupled ordinary differential equations possessing the same kinematical features necessary to mimic Navier-Stokes nonlinear evolution. In the following we will present numerical results for a particular choice, the so-called Sabra model [11], namely,

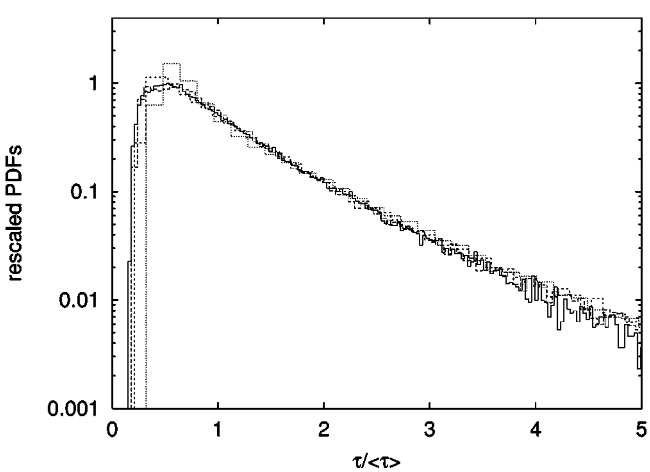

FIG. 3. Collapse of the rescaled PDFs of the halving times for the modified Orszag-McLaughlin model. For simplicity, only the PDFs relative to the fastest four variables are shown, $n=7,\dots,10$. The statistics is over 10^5 impulsive infinitesimal perturbations as in Fig. 2.

$$\left(\frac{d}{dt}+\nu k_n^2\right)u_n = i[k_n u_{n+1}^* u_{n+2} + b k_{n-1} u_{n+1} u_{n-1}^* + (1+b) k_{n-2} u_{n-2} u_{n-1}] + f_n, \tag{14}$$

where b is a free parameter, ν is the molecular viscosity, and f_n is an external forcing acting only at large scales, necessary to maintain a stationary temporal evolution. The main, strong, difference with the model discussed in Sec. II A consists of the existence of a mean energy flux from large to small scales which drives the system toward a strongly non-Gaussian stationary temporal evolution [12]. Shell models discussed here present exactly the same qualitative difficulties of the original Navier-Stokes equations: strong nonlinearity and far from equilibrium statistical fluctuations. The most striking quantitative feature of the non-Gaussian statistics is summarized in the existence of anomalous scaling laws of velocity moments:

$$\langle |u_n|^p\rangle \sim k_n^{-\zeta(p)}, \tag{15}$$

with $\zeta(p) \neq p/2\zeta(2)$. Anomalous scaling, also known as intermittency, is the quantitative way to state that velocity PDF's at different scales cannot be rescaled by any changing of variables.

Let us now discuss two subtle points. Using some general arguments from the dynamical systems theory, one has that all the (typical) correlation functions at large time delay have to relax to zero with the same characteristic time, related to spectral properties of the Perron-Frobenius operator. If one uses this argument in a blind way, the apparently paradoxical result is that all correlation functions, $C_{n,n}(t) = \langle u_n(t) u_n(0)\rangle$, must go to zero with the same characteristic times. On the contrary one expects a whole hierarchy of characteristic times distinguishing the behavior of the correlation functions at different scales [13]. In particular, the self-correlation function, $C_{n,n}(t)$, decays with a characteristic time decreasing with n. The paradox is only apparent since the dynamical systems argument is valid at very long times, i.e., much longer than the longest characteristic time, and therefore in systems with many different time fluctuations it is not helpful. In fact, it is well established numerically and well understood theoretically [13–15] that general multiscale multitime correlation functions of the kind $C_{n,m}^{p,q}(t) = \langle |u_n(0)|^p |u_m(t)|^q\rangle$ are described by the cascade formalism. In particular, most of the statistical properties in the inertial range can be well parametrized by the multifractal formalism.

On the other hand, the response properties are related to infinitesimal perturbations. Therefore, it is not obvious that the response depends on inertial range properties. The exis-

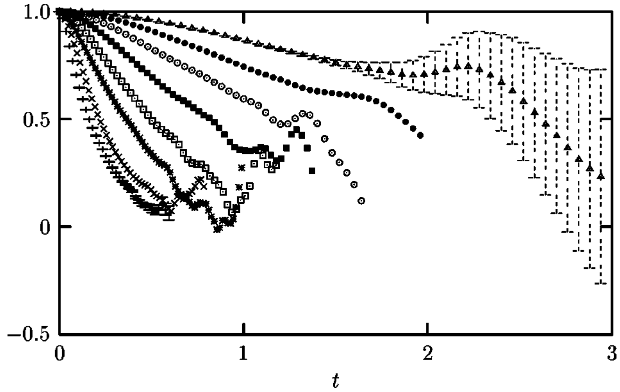

FIG. 4. Modulus of the average response functions, $G_n^n(t) = \langle R_n^n(t) \rangle$, for shells $n=7, \ldots, 14$ (from top to bottom). Error bars are shown only for the smallest and the largest scales. The number of independent kicks used to perform the averages is around 2×10^5. Notice the extremely large error bars measured for the slowest shell variables. The parameters entering in the equations of motion (14) are $b=0.4$, $\nu=5\times10^{-7}$ for $N=25$ shells.

tence of the F/R relation and the fact that $C_{n,n}(t)$ (and other similar correlation functions) are determined by the inertial range properties suggest that also the response features are ruled by the inertial range behavior if the invariant measure is dominated by local interactions among shells.

Let us now examine the numerical results concerning response functions in the shell model. Figure 4 shows the diagonal mean response $G_n^n(t)$ for a range of inertial shells, $n \in [7–14]$. The most striking property is the impossibility to follow the response behavior at large scales (small shells) for large times, i.e., the explicit evidence that errors grow exponentially. In order to compare the different behavior (and different error propagation) between the response and correlation function, we plot in Fig. 5 both the average response and the self-correlation, $C_{n,n}(t)$, for the shell $n=10$. As it is clear from the previous figures, only a response at the smallest scales (fast scales) in the inertial range can be computed with enough accuracy to follow an asymptotic decay. Still, also for this response the clear departure between the response and the self-correlation shows another indication that

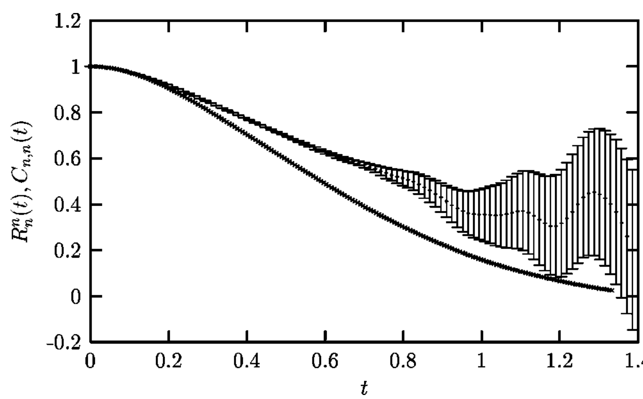

FIG. 5. Comparison between the averaged response function $G_n^n(t)$ (top) and the self-correlation $C_{n,n}(t)$ (bottom) for the shell $n=10$. Notice the different order of magnitude of error bars.

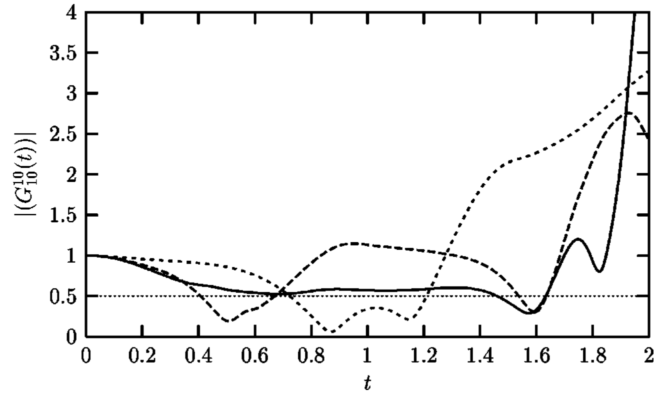

FIG. 6. Plot of three different diagonal instantaneous responses, $R_n^n(t)$, for the shell $n=10$, versus time. Halving time is fixed by the first time when the curve touches the threshold at $\lambda=1/2$.

the inertial-range statistics is far from Gaussian. By using HTS we can get information on the temporal dependence of the response function in the whole range of inertial shells. In Fig. 6 we report some realizations of the instantaneous response function, which shows typical halving time experiments.

In Fig. 7 we summarize the results we obtain by comparing the mean halving time, $\langle \tau(n) \rangle$, with the characteristic times one extracts from the decay properties of both the mean response $\tau_R(n)$ and correlation functions $\tau_C(n)$ for those shells where such a behavior can be safely extracted. It is worth noticing how the mean halving time allows a full characterization of time properties also for those shells where the mean response $G_n^n(t)$ cannot be measured for large time legs t. Also, the dependence from the scale of the mean halving time is given as a best fit $\langle \tau(n) \rangle \sim k_n^{-\chi}$, with $\chi=0.53 \pm 0.03$. The value $\chi=0.53\pm0.03$ can be seen as an intermittent correction to the dimensional inertial-range prediction 2/3. On the other hand, the dependency from the scale of $\tau_R(n)$ is difficult to extract due to the small number of points available.

Let us now focus on the whole PDF of the halving time

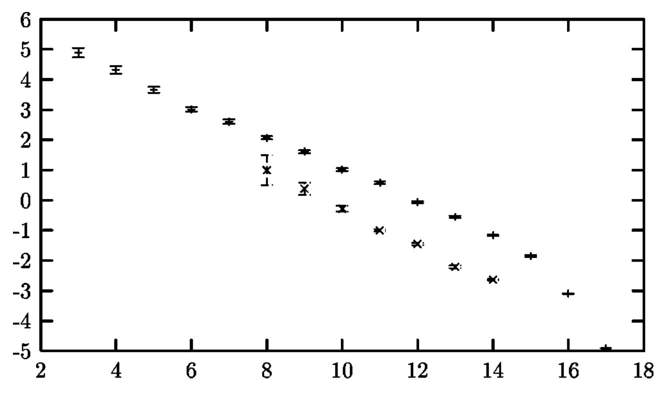

FIG. 7. Log-log plot of the mean halving times, $\langle \tau(n) \rangle$ $(+)$ and of decaying times of the mean diagonal response, $\tau_R(n)$ $(\times)$, versus k_n. We have checked that a different choice of the threshold $\lambda=1/2$ used to compute halving time does not affect the slope of the graph.

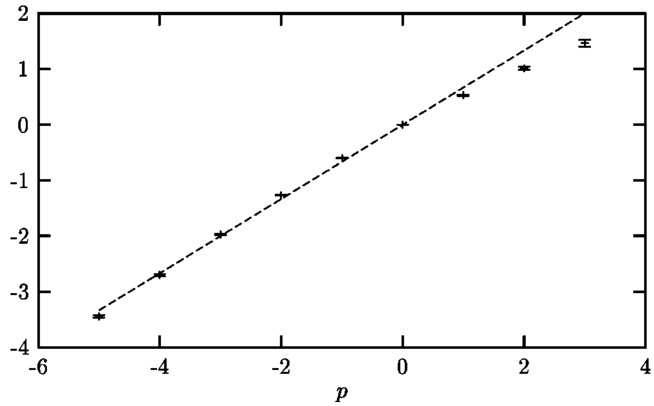

FIG. 8. ψ_p exponents of the pth moment of halving times. The straight line corresponds to the dimensional prediction : $\psi_p = 2/3p$.

statistics. We first analyze the positive and negative moments of the halving times,

$$T^p(n) = \langle (\tau^p(n)\rangle \sim k_n^{-\psi(p)}, \tag{16}$$

with $p = -5, \ldots, 3$. Dimensional, nonintermittent, scaling would predict the linear behavior for the scaling exponents: $\psi(p) = 2/3p$. In Fig. 8 we plot the results for the halving times scaling exponents $\psi(p)$ for all moments from $p = -5, \ldots, 3$ and the straight line corresponding to the dimensional inertial range estimate. We notice that intermittent corrections are much stronger for the positive moments than for the negative moments. This must be related to the fact that positive moments of the halving times are dominated by rare events where the response has a very long decaying. We interpret the fact that $\psi(p) \simeq (2/3)p$ as an indication that linear response functions are inertial range quantities. The latter results lead to the important conclusion that the invariant measure is well approximated by short range interaction among shells in the inertial range.

As for the nondiagonal response function and for the generalized response function of higher order the numerical problems to measure them are even more pronounced. First, let us examine the off-diagonal response function $R_n^m(t) = \delta u_n(t)/\delta u_m(0)$. Of course these responses start from zero at time zero instead of starting from one as in the diagonal case. Measuring them by a direct average is strictly impossible because of very large errors. We still decided to measure their characteristic time by using the time the response reaches a ''macroscopic'' fraction, say 1/2, of the typical fluctuations on the scale where we are measuring the response, i.e., we collect the statistics of the first times τ such that $R_n^m(t=\tau) = 1/2\langle |u_n|^2 \rangle$. We expect a strong asymmetry of the characteristic times depending whether the perturbation is done at smaller $(m > n)$ or larger $(n > m)$ scales. Indeed, by using the usual inertial range arguments we expect that the response reacts always with a typical time given by the time of the largest between the two shells involved n,m. By fixing, therefore, the shell where we perturb, say m, we expect that the typical time of $R_n^m(t)$, $\tau_n^{(m)}$ is constant when $n > m$ and scales as $k_n^{-2/3}$ when $n < m$. In Fig. 9 we show that indeed this behavior is well reproduced numerically.

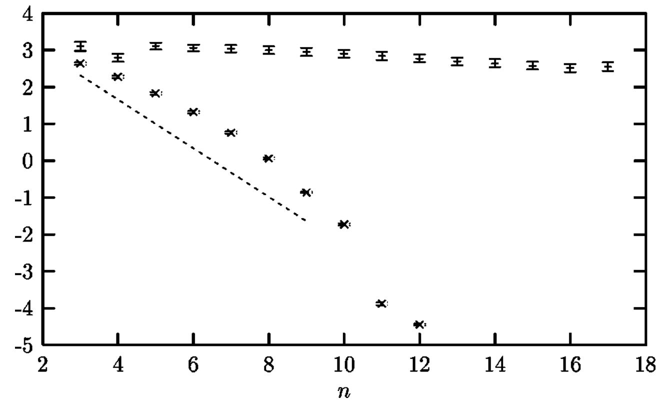

FIG. 9. Log-log plot of the off-diagonal response characteristic times $\tau_n^{(m)}$ versus k_n. We performed two experiments. First, we perturb at large scales, $m=2$, and we follow the response at small scales $n>2$ $(+)$; the expected independence of characteristic times from the scale is well reproduced. Second, we perturb at small scales, $m=13$, and we follow the response at larger scales, $n<13$, $\times$. In the latter case, for comparison we also plot the straight line (dashed) with the expected dimensional slope $-2/3$.

The strong intermittency shown by halving times in Fig. 8 is the clear signature of deviations from a simple Gaussian-like behavior of response functions. We must therefore also expect that generalized responses of higher order are not simply related to the linear response. For example, let us consider the third moments of the linear response [16]:

$$S_n^n(t) \equiv \langle [R_n^n(t)]^3 \rangle.$$

A simple nonintermittent behavior would suggest that $S_n^n(t) \propto [G_n^n(t)]^3$ while in Fig. 10 it is possible to see that this is definitely not the case for all t legs where we have a measurable signal. Unfortunately the already discussed statistical problems in measuring averaged response functions for long times are even more pronounced for generalized response functions. Therefore, we refrain from showing any results for the scaling behavior of typical times of the generalized response functions.

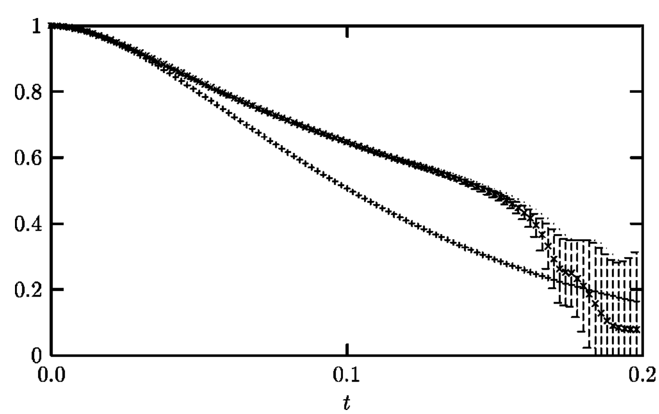

FIG. 10. Comparison between the third power of the mean diagonal response, $|G_n^n(t)|^3$ $(+)$ and the generalized third order response, $|S_n^n(t)|$ $(\times)$ computed for the shell $n=13$, with 5×10^5 kicks.

III. CONCLUSIONS

We have addressed the problem of measuring time properties of response functions in Gaussian models (Orszag-McLaughin) and strongly non-Gaussian models, like shell models for turbulence. We have introduced the concept of halving time statistics with the aim of having a statistically stable tool to quantify the time decaying of response functions and generalized response functions of high order. We have shown numerically that in shell models for three-dimensional turbulence response functions are inertial range quantities. This is a strong indication that the invariant measure describing the shell-velocity fluctuations is characterized by short range interactions between neighboring shells.

Response functions and generalized response functions play an important role in any diagrammatic approach for nonlinear out-of-equilibrium systems. In this work we have presented the first numerical attempt to measure some of their properties in a systematic way. More work is needed, both numerically and analytically, in order to better understand the detailed structure of the invariant measure governing F/R relations.

ACKNOWLEDGMENTS

We acknowledge useful discussions with G. Boffetta and V. L'Vov. This work has been partially supported by the EU under the Grant No. HPRN-CT 2000-00162 ''Non Ideal Turbulence.'' G.L. thanks the Department of Physics of L'Aquila University for the kind hospitality.

[1] With ''statistical equilibrium'' we mean the statistical properties described by the invariant probability measure of the dynamical system.

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[16] Even moments, being positive definite, cannot be considered a response function. They possess the typical exponential growth with a characteristic time given by the Lyapunov exponent and then, for times large enough, they saturate to the averaged maximal distance in the attractor.