Probability distribution functions in turbulent flows and shell models

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In the multifractal framework some recent results [Phys. Rev. Lett. **67**, 2295 (1991)] on the probability distribution functions (pdf) in fully developed turbulence are reviewed, for the increments of the velocity field in the inertial range and for transversal gradients. New comparisons are also produced, in the same scenario, with clusters of isogradients and with pdf evaluated from a numerical simulation of a shell model. The Reynolds dependence of the flatness in the multifractal model is discussed.

I. INTRODUCTION

It is a common experience that viscous fluids driven by external forces can develop instabilities and show chaotic behavior.[1] The parameter that controls the degree of randomness is the Reynolds number, i.e., the ratio between nonlinear and linear terms in the Navier–Stokes (NS) equations. In the limit where the Reynolds number goes to infinity one speaks of the fully developed turbulence regime (FDT). A theory for FDT is still lacking. Only a few advances have been made since the seminal work of Kolmogorov published in 1941 (K41 hereafter).[2]

In K41, Kolmogorov made two main assumptions: the probability distribution function (pdf) of velocity increments at distances r—small compared to the integral scale L_0, $(\Delta_r v) \equiv |v(x)-v(x+r)|$ is isotropic, and the only two important parameters necessary to characterize the pdf are the viscosity ν and the mean energy dissipation $\bar{\epsilon}$. In the following we will consider K41 as our ground-state theory, without challenging its basis—as could be possible discussing the isotropy and universality assumptions (see the review[1] for a detailed exposition of the main experimental tests on these two issues).

From these two apparently harmless hypotheses Kolmogorov was able to predict one of the most striking experimental signatures of FDT: the appearance of an inertial range where moments of $\Delta_r v$ are ν independent with scaling:

$$\langle(\Delta_r v)^p\rangle \propto (\bar{\epsilon} r)^{\zeta(p)}, \quad \text{with } \zeta(p)=p/3. \tag{1}$$

This kind of reasoning has been challenged experimentally and theoretically.[1] It is highly unsatisfactory to assume that only the mean energy dissipation enters in (1): the whole probability distribution of ϵ should play an essential role in modeling turbulence features. In order to take into account ϵ fluctuations it is possible to generalize relation (1) to

$$\langle(\Delta_r v)^p\rangle \propto \langle\epsilon^{p/3}\rangle r^{p/3}. \tag{2}$$

Under this condition the function $\zeta(p)$ could display a nonlinear behavior, according to the r dependency of the ϵ probability distribution. This is exactly what is observed experimentally.[3,4] The bending of $\zeta(p)$ is commonly linked with the intermittency of the energy transfer rate from large scales to the dissipative scales, which are considered constant in the Kolmogorov theory. Looking at one typical configuration, one would see large laminar zones where v is regular, and narrow very active regions where most of energy is dissipated.

Up to now, although the experiments and numerical simulations are not precise enough to measure $\zeta(p)$ exponents larger than $p=10$ or 12, one can safely state that the K41 prediction $\zeta(p)=p/3$ is wrong. How to replace it is still an unsolved problem. As we shall show below there are a few proposals able to reproduce the correct p dependence of $\zeta(p)$, even if they remain at a phenomenological level.

Recently,[5–10] attention has been turned to the problem of pdf's.

Experimental pdf's[11] for the velocity increments in the inertial range show a tendency toward more and more non-Gaussian shapes at decreasing distances, while the gradient pdf's display the same behavior at increasing Reynolds number.

Kraichnan,[6] recently, advanced the possibility to extend for turbulent flows an analytical mapping closure, already applied to the Burgers equation. He proposed a heuristic form for the stretching function $J(s_0,t)$ which maps the initial gradient, s_0, at time $t=0$ to that at time t. The time evolution of J is given by a nonlinear equation that presents a competition between the straining term and viscous dissipation. Its stationary solution produces a pure exponential tail for the pdf in the limit of infinite Reynolds number. However, this prediction is not consistent with some experimental data[12] which suggest a powerlike behavior of flatness and skewness as a function of Reynolds number.

Starting from Kraichnan ideas, She,[7] and subsequently She and Orzsag,[8] introduced a more refined model by considering the different roles played by random eddies and small-scale coherent structures in FDT. They used an external parameter h to model the change of the typical velocity, v_l, at scale l, with respect to the Gaussian initial reference field v_g characterizing large-scale statistics:

$$v_l(t) \sim J^{-h}(t) v_g.$$

In their work suitable approximations of the eddy–structure (mean stretching), structure–structure (self-stretching), and eddy–eddy (mean dissipation) interaction terms are given. Using different h exponents as a function of l, they were able to reproduce the pdf for velocity increments in the inertial range, and the pdf for gradients for different Reynolds numbers.

Another method has been published by Castaing *et al.*,[10] who proposed a comprehensive theory to predict longitudinal gradients (with skewness $\neq 0$) and the $\zeta(p)$ function. According to their studies, it should be possible to obtain a satisfactory agreement with experimental data assuming a lognormal distribution for ϵ_r, the energy transfer rate at scale r toward smaller scales. In this way—as in the She–Orszag theory—one must abandon the self-similar structure of $\Delta_r v$ moments, producing a weak r dependence in $\zeta(p)$.

Frisch and She[9] proposed a simpler theory, closer to the original Kolmogorov idea. They use dimensional arguments to relate fluctuations on small scales to the large-scale Gaussian statistic, in both K41 and the beta model.[13]

In this paper, we review the results on the pdf for transversal gradients (S) and velocity increments in the multifractal scenario produced by an appropriate random beta model (RBM).[5] We extend to this general case the ideas presented in Ref. 9. We show that it is possible to obtain a good agreement with two independent sets of numerical data obtained from direct integration of the Navier–Stokes equations,[14,15] and with pdf's obtained from a shell model of three-dimensional (3-D) turbulence.[16–18]

It is not possible to apply the same reasoning to the statistics of longitudinal gradients because the latter shows some geometrical (incompressibility) and dynamical (nonzero skewness) features which are not reproducible by any phenomenological model as the RBM.

We give a Reynolds-dependent expression for the gradient pdf's $P(S)$ (normalized to have $\langle S^2\rangle = 1$) and for the pdf's of velocity differences, $P(\Delta_r v)$, in the inertial range. Both of them show the correct dependence, respectively, on the Reynolds number and on the inertial range distance.

The paper is organized as follows: in Secs. II and III we review the main ideas on this subject as discussed in Ref. 5 and we give an explicit expression for the pdf as a function of the Reynolds number; in Sec. IV we show a new comparison with numerical data from Ref. 15 for the cluster of isogradients; in Sec. V we discuss a shell model for the energy cascade in 3D[16–18] and compare our prediction with numerical data; in Sec. VI are the conclusions.

II. THE pdf's IN FRACTALS AND MULTIFRACTALS

Neglecting the dissipative term and the external forcing, NS equations are invariant for a simultaneous rescaling of time and space, in the limit of high Reynolds number:

$$x \to \lambda x_0, \quad t \to \lambda^{1-h} t_0, \quad \Delta_r v \to \lambda^h \Delta_0 v,$$

where λ is a free parameter and h is the Holder exponent of the velocity field. In K41, h is assumed to be constant and equal to 1/3 in order to satisfy automatically the exact relation $\zeta(3) = 1$.[19]

In order to take into account fluctuations, Parisi and Frisch[20] proposed a local scaling invariance. In their approach h becomes x *dependent* and characterizes the strength of singularity of the velocity gradient at point x. Without entering in subtle mathematical definitions, one can relate several quantities in the inertial range to the large-scale fluctuating field v_0 through the dimensional relation

$$\Delta_r v(x) \propto v_0 (r/L_0)^{h(x)}, \tag{3}$$

where now both sides of Eq. (3) are local. For simplicity hereafter the integral scale, L_0, will be taken equal to 1.

According to some commonly assumed arguments,[1,21] but without any solid theoretical ground, the self-similar inertial range is produced by a cascade of energy from the largest scale, in which the fluid develops instabilities, to the smallest scale, where dissipation produces an overall smoothing. This energy cascade is assumed to stop when the effective Reynolds number $\mathrm{Re}(r)$, defined in terms of quantities at scale r, becomes of order 1:

$$\mathrm{Re}(r_d) = \Delta_{r_d} v r_d / \nu \sim O(1), \tag{4}$$

where r_d is by construction the smallest significant scale of the system. It is customary to use this phenomenological definition to obtain an explicit expression for the gradient S:[9]

$$S = \Delta_{r_d} v / r_d .$$

The length r_d is itself, in the multifractal scenario, a fluctuating quantity.[22,23] From relations (4) and (3) one obtains

$$r_d(h) = (\nu / v_0)^{1/(1+h)}; \tag{5}$$

this means that the strongest singularity in the velocity gradients corresponds to extremely small dissipative scales, r_d. Generally h is expected to run in the interval $0 < h < 1/3$: for $h < 0$, in the limit of $\mathrm{Re} \to \infty$,[24] incompressibility is broken, while for $h > 1/3$, due to the relation $\epsilon_r \propto r^{3h-1}$, there is no cascade of energy toward small scales.

Noting that the following relation holds:

$$|S| = \Delta_{r_d} v / r_d = v_0 r_d^{h-1} = v_0^{2/(1+h)} \nu^{(h-1)/(h+1)} \tag{6}$$

it is possible to obtain a closed formula for $P(S|h)$, the gradient pdf conditioned to the particular value of h, as a function of the initial reference field v_0. Let us stress that in our derivation, like in any other one based on dimensional grounds, it is not possible to capture any dynamical or geometrical aspects of 3-D turbulence. Therefore we consider only the absolute value of various observables and neglect their (possible) vectorial origins.

From the previous formula it is easy to derive[5,9] an expression for $P(S|h)$ starting from the (supposed) known pdf of v_0, $\Pi(v_0)$. The latter is usually assumed to be Gaussian, both for experimental outcomes[10,11] and semi-theoretical central limit arguments.[1]

Since

$$P(S|h) = \Pi(v_0) \frac{dv_0}{dS}$$

and

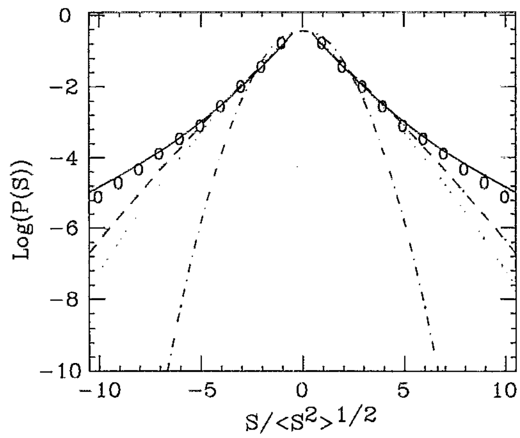

FIG. 1. Log-linear plot of the pdf for transversal gradients, $P(S)$, normalized to have $\langle S^2\rangle=1$. The circles are the numerical data taken from the direct integration of NS equations of Ref. 14, with Re=1000. The solid line is the multifractal prediction (16), while the dashed, the dotted, and the dash-dotted lines are, respectively, the Fractal prediction,[9] the K41 theory, and a Gaussian case.

$$\Pi(v_0)=\text{const}\ \exp-\frac{v_0^2}{2\langle v_0^2\rangle}$$

we obtain from (6)

$$P(S|h)=\left(\frac{\nu}{S}\right)^{(1-h)/2}\exp-\frac{\nu^{1-h}S^{1+h}}{2\langle v_0^2\rangle}. \tag{7}$$

In order to achieve a description for the small S region one has to add to (7) the contribution coming from the laminar zones. They can be modeled, for every practical purpose, by a delta function concentrated at $S=0$. Therefore one has

$$P_{\text{tot}}(S|h)=P(S|h)+\text{const}\ \delta(S) \tag{8}$$

where the const is chosen such that P_{tot} has an integral correctly normalized to one.

It is important to emphasize that in any approach described by one exponent, h, like in the K41 case $(h=1/3)$ or in the beta model $(h=h_F)$,[9,13] the expression (7) reduces asymptotically to a stretched exponential form $\exp-\text{const}\ |S|^t$ with $t>1$. In a log-linear plot this would be more intermittent than a Gaussian reference bell curve, but always with the same convexity behavior. This convexity seems to be in contradiction with the qualitative feature of the experimental and numerical data.[11,14]

On the other hand the multifractal scenario,[5] introducing in a natural way a nontrivial spectrum of local singularities, could escape this problem, producing a continuous changing in the local stretching exponent with an overall final result in quantitative and qualitative agreement with data (see Fig. 1). In this circumstance the unconditional pdf $P(S)$ is obtained by integrating over the whole range of h exponents times the probability, $P_{r_d}(h)$ of picking out that particular h value.[5] The final expression is given by a weighted integral over the allowed class of singularities:

$$P(S)=\int dh\, P_{r_d}(h)P(S|h)$$
$$\sim\int dh\,\rho(h)\left(\frac{\nu}{S}\right)^{\{4-[h+D(h)]\}/2}\exp-\frac{\nu^{1-h}S^{1+h}}{2\langle v_0^2\rangle}, \tag{9}$$

where $P_{r_d}(h)=r_d^{3-D(h)}\rho(h)$ is the probability to have an exponent h, expressed—in the multifractal scenario—by a set of fractal dimensions $D(h)$ and by a smooth function $\rho(h)$ independent of r. An analytic estimate of the above formula is not simple. It is not possible to apply a saddle point method even in the limit of $\nu\to 0$. This is due to the fact that according to relation (6) one has $S\sim\nu(h-1)/(h+1)$ and therefore the factor in the exponent of Eq. (9) is always of order one even in presence of very strong energy burst, i.e., $h\approx 0$.

Furthermore one has to build up a model which produces a closed expression for the unknown function $D(h)$, appearing on the left-hand side of (9). In order to do this we will use again the well-known random beta model that has been very efficient in fitting the experimental $\zeta(p)$.[25,26] This simple model is also efficient in reproducing the gradient statistics of two numerical experiments available nowadays for the transversal velocity field.

III. pdf's IN RANDOM BETA MODELS

The random beta model is a prototype for the phenomenological description of the energy cascade in 3-D FDT. It reproduces the process of eddy fragmentation by fixing a hierarchy of scales $r_n=r_0 2^{-n}$, at which the main dynamical mechanisms are supposed to happen. In this approach the typical velocity difference at scale n is linked to that at scale $n-1$ through a random process independent of the scale, that is,

$$\Delta_{r_n}v=(r_n/r_{n-1})^{1/3}\beta_n^{-1/3}\Delta_{r_{n-1}}v$$

and then, iterating until the integral scale $r_0=1$:

$$\Delta_{r_n}v\equiv v_n=v_0 r_n^{1/3}\prod_{i=1}^{n}\beta_i^{-1/3}, \tag{10}$$

where β_i are independent, identically distributed, random variables.

The choice of their probability distribution $\mu(\beta)$ is dictated by *ad hoc* reasons. First, in order to recuperate a final scale invariant distribution one assumes independence from the step during the fragmentation process. Second, to obtain a weak deviation from the Kolmogorov scaling one adopts a bimodal form for $\mu(\beta)$. These two conditions lead to

$$\mu(\beta)=x_k\delta(\beta-1)+(1-x_k)\delta(\beta-\beta_0), \tag{11}$$

where x_k represents the probability of having a β value coincide to a *Kolmogorov-like* scaling, $h=1/3$, while $1-x_k$ describes the probability to have a stronger intermittent region with a local exponent $h_{\min}=(\log_2\beta_0+1)/3$. The prefactor $r_n^{1/3}$ in (10) together with (11) is chosen such

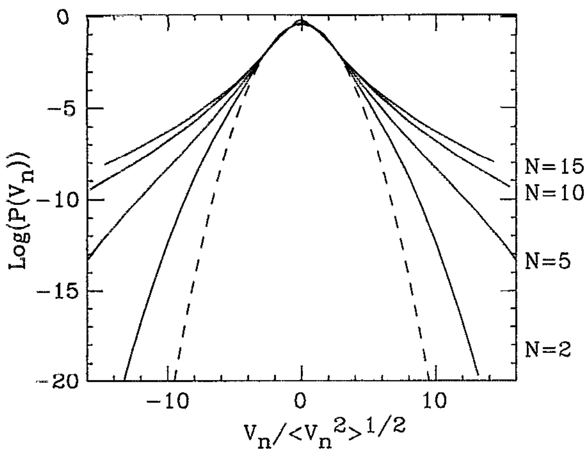

FIG. 2. Log-linear plot of the pdf for velocity increments in the inertial range $P(v_n)$. Here are plotted four cases correspondent to the choices $n=2$, 5, 10, and 15 with a Gaussian reference bell curve (dashed) in order to evidence the progressive departure toward more intermittent profiles as a function of the decreasing distance (increasing n) in the inertial range.

that the mean energy ratio ϵ_n transferred through contiguous scales, n and $n+1$, is constant along the cascade.[25]

Putting $x_k=0$ in Eq. (11) one recovers the usual beta model,[13] i.e., $\bar{h}=(\log_2\beta_0+1)/3$ in the whole fluid. In this situation the energy dissipation is concentrated in a fractal region of dimension $D_F=2+3\bar{h}$.

From (10) and (11) it is possible to carry out the structure function $\zeta(p)$ and the set of fractal dimensions $D(h)$.[25] A good agreement with experimental data[3,4] is obtained choosing

$$x_k=\tfrac{7}{8} \quad \text{and} \quad \beta_0=\tfrac{1}{2}\rightarrow h_{\min}=0. \tag{12}$$

Instead of proceeding in this way and inserting in (9) the resulting form for $D(h)$, we follow a more transparent path deriving the pdf, $P(v_n)$, for the velocity increment v_n in the inertial range[5] directly from (10).

From (10) one realizes that $P(v_n)$ can be expressed starting from $\Pi(v_0)$ and $\mu(\beta)$:

$$P(v_n)=\int \Pi(v_0)dv_0 \int \delta\left(v_n-v_0 r_n^{1/3}\prod_{i=1}^{n}\beta_i^{-1/3}\right)$$
$$\times \prod_{i=1}^{n}\beta_i\mu(\beta_i)d\beta_i, \tag{13}$$

where the extra β_i factor multiplying the $\mu(\beta_i)$ density in the integral describes the ratio of the total volume occupied by the active fluid at step i. Inserting in the above expression the form (11) for $\mu(\beta)$ one obtains the sum

$$P(v_n)=\sum_{i=0}^{n}\binom{n}{i}x_k^{n-i}(1-x_k)^i\beta_0^{4i/3}r_n^{-1/3}$$
$$\times\exp(-C\beta_0^{2i/3}r_n^{-2/3}v_n^2), \tag{14}$$

where $C=(2\langle v_0^2\rangle)^{-1}$. This suggests that the final $P(v_n)$ is a superposition of n Gaussians, where n is linked to the r_n through the relation $n=-\log_2 r_n$.

In Fig. 2 we plot $P(v_n)$ for different choices of n. It is quite remarkable that, at decreasing r_n, they display the proper trend, developing more and more intermittent tails.

Let us stress, once more, that this distortion from the initial Gaussian profile of pdf for v_0 toward an exponential-like shape is due to the presence of more than one local exponent. Indeed, only the β fluctuations break the linear origin of relation (10) between v_n and v_0, producing a final non-Gaussian pdf.

In order to go from the pdf for v_n to that for the gradient S, one notes that the cascade stops at step N where the effective Reynolds number becomes order one, $(v_N r_N)/\nu\sim O(1)$. In this way it is possible to select, for each β's realization, the appropriate dissipative scale and to write for the gradient the following relation:

$$S=v_N/r_N\equiv\nu/r_N^2. \tag{15}$$

Inverting (15) one expresses N as a function of S, and evaluating (14) at the correspondent value one obtains

$$P(S)=\sum_{M=0}^{N(S)}\binom{N(S)}{i}x_k^{N(S)-M}(1-x_k)^M\left(\frac{\nu}{S}\right)^{(1+2m)/3}$$
$$\times\exp(-C\nu^{(2+m)/3}S^{(4-m)/3}), \tag{16}$$

where $m=M/N$ and $N(S)=\log(S/\nu)/(2\log 2)$.

Once again the K41 case is recovered by considering only one term of the sum, $M=0$, with $x_k=1$.

Even in the general case (16), since the S dependence of N is only logarithmic, there are no dominant terms in the sum, precluding any simple theoretical estimate of it.

Let us stress that in order to compare it with numerical data we have not chosen any free parameters using for $h_{\min}$ and x_k the same values obtained previously from a fit of $\zeta(p)$,[25,26] i.e., $h_{\min}=0$ and $x_k=7/8$.

In Fig. 1 we have plotted the numerical pdf obtained from a direct integration of NS equations[14] superimposing our fit given from Eq. (16). The agreement is quite satisfactory, while both the K41 theory and the beta model case cannot capture even the convexity of the curve.

The shape predicted for $P(S)$ is given by a superposition of stretched exponentials $\exp(-ct^{\alpha})$ each one characterized by $\alpha>1$. Despite this, the final curve displays the same convexity of a stretched exponential with $\alpha<1$, and this is the main visual difference with any single-exponent model, while the global trend differs from other theory[8,10] only in the limit of very large deviations.

Let us remark that the sum (16) is valid only for $S>\nu/r_0^2$ as one can realize from relation (15). This is a consequence of the fact that the random beta model is able to describe only active regions of the fluid, neglecting the laminar zones, $S\approx 0$; that is why a δ function, centered at $S=0$, has been inserted into Eq. (8).

In order to put our prediction in a more manageable form it is essential to write an explicit expression for $P(S)$ normalized to have $\langle S^2\rangle=1$. In the multifractal framework it is possible to show that, provided $\zeta(3)=1$, the following relation holds:

$$\nu\langle S^2\rangle\rightarrow A, \quad \text{when Re}\rightarrow\infty, \tag{17}$$

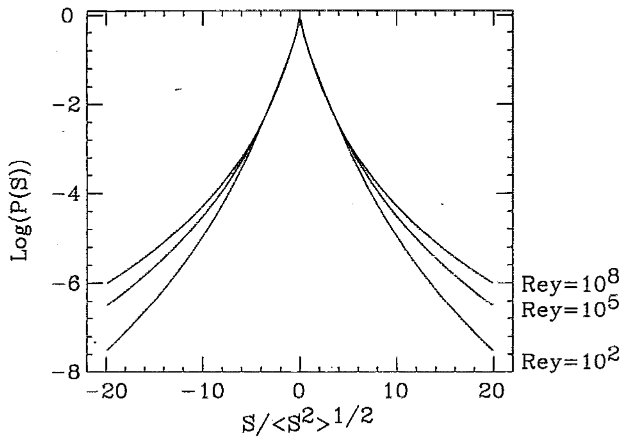

FIG. 3. Log-linear plot of the normalized pdf for transversal gradients, from relation (18), as a function of three different Reynolds numbers ($\mathrm{Re}=10^2$, 10^5, and 10^8). A is chosen $\simeq 1$, in order to eliminate nonuniversal effects. It is interesting to note that only by changing Re by some order of magnitude is it possible to detect a variation in the pdf tails.

where A is a nonzero nonuniversal constant. The above expression simply means that in the FDT regime the energy dissipation $\bar{\epsilon}\propto \nu\langle S^2\rangle$ reaches an asymptotic value.

Let us recall that the integral scale L_0 is set equal to one and therefore the large-scale Reynolds number is simply expressed by $\mathrm{Re}(L_0)=v_0/\nu$. Inserting it in (9) and taking into account (17) one obtains for the normalized pdf $P(S/\langle S^2\rangle^{1/2}=y)$ the following form:

$$P(y)\propto \int dh\,\rho(h)A^{h+D(h)/4-1}y^{h+D(h)/2-2}$$
$$\times \mathrm{Re}^{3[h+D(h)]/4-5/2}$$
$$\times \exp-\frac{y^{1+h}\,\mathrm{Re}^{(3h-1)/2}\,A^{(1+h)/2}}{2}. \tag{18}$$

At a first glance, the above expression does not seem clearer than (9), however, one can learn many interesting things from it.

First, it is straightforward to realize that in the K41 case [$h=1/3$ and $D(1/3)=3$] every explicit dependence of the pdf from the Reynolds number drops out; this is a different way to look at the Kolmogorov statement regarding the uniformity of energy dissipation in real space: in that condition the dissipative scale, η_k, would be fixed and everything becomes Reynolds independent.

In any case, unless the probability to find an exponent $h\neq 1/3$ is sensibly different from 0 [that is not the case for the parameters choice (12)], the pdf Reynolds dependence of (18) in the limit where $\mathrm{Re}\to\infty$ is very weak. In Fig. 3, we show a set of curves obtained for different values of Reynolds ($\mathrm{Re}=10^2,10^5,10^8$), and $D(h)$ following from the choices (12): it is evident that only by changing the Reynolds number of several order of magnitude it is possible to detect a notable variation in the pdf shapes.

In order to quantify the Reynolds dependence of $P(y)$ given by (18) one can look at the flatness in major detail. It is straightforward to generalize the reasoning of Ref. 27 that leads to the estimate $\langle S^2\rangle\nu\to\mathrm{const}$ when $\mathrm{Re}\to\infty$, to the calculation of the fourth moment of S.[23] In fact, one has

$$S=\frac{v_{r_d}}{r_d}\propto r_d^{h-1}\propto \mathrm{Re}(1-h)/(1+h), \tag{19}$$

where the last relation comes from expression (5) connecting the Reynolds number and the dissipative scale. In the limit of $\mathrm{Re}\to\infty$, one can apply the usual saddle point estimate:

$$\langle S^4\rangle \propto \int dh\,\rho(h)\mathrm{Re}^{4(1-h)/(1+h)}\,P_{r_d}(h). \tag{20}$$

Finally the asymptotic scaling for the flatness, K, is given by

$$K(\mathrm{Re})=\langle S^4\rangle/\langle S^2\rangle^2=\mathrm{Re}^F,$$

with $F=0.135$. This result can be considered in good agreement with the experimental value $F_{\mathrm{exp}}=(0.35\pm 0.07)/2$ extrapolated from the data collected in Ref. 12. Let us note that in the K41 case we would have $F=0$.

IV. CLUSTERS OF ISOGRADIENTS

Recently, a new proposal concerning cluster statistics to test intermittency of gradient pdf has been suggested.[15] The appearance of clusters of high fluid activity with nonuniform probability can be interpreted as a different way of defining intermittency.

This new way of looking at data has been feasible only since large numerical simulations were performed. Experimental results are always a one-dimensional time sequence of data, which is possible to translate into spatial measurements solely via a suitable Taylor approximation; on the other hand, in the numerical simulations one can manage the whole spatiotemporal fluid history.

Cluster statistics can be applied to any physical observables, either vectors or scalars such as the energy dissipation, transversal and longitudinal gradients, or vorticity. We deal only with the statistics of the transversal gradient field, being the only one that can be considered independent of geometrical or dynamical constraints.

In the following we use the same clusters definition given in Ref. 15, in order to be able to compare their numerical measured statistics with those obtained using our prediction (16) as input.

In Ref. 15 a cluster of a scalar field $S(\mathbf{x})$ is defined as the connected region where the field $S(\mathbf{x})$ is greater than a threshold α.

One also introduces the relative volume $V(\alpha)/V_{\mathrm{tot}}$ as the ratio between the total number of points belonging to the clusters and the total number of points in the volume simulation.

The most significant quantity to be studied is the transversal gradient ratio $S(\alpha)/S_{\mathrm{tot}}$ versus the relative volume $V(\alpha)/V_{\mathrm{tot}}$ as a function of the threshold α. The former is

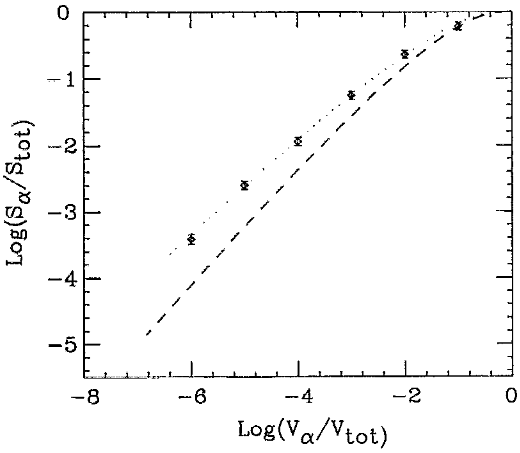

FIG. 4. Log–log plot of isogradient clusters concentration versus the ratio of volume occupied. Diamonds are the numerical points from Ref. 15. The dotted line corresponds to the multifractal fit obtained inserting (16) in (21) and (22); the dashed line is the K41 prevision.

defined as the total amount of transversal gradient belonging in clusters with $S>\alpha$ normalized to the value of S in the whole volume.

As a function of α, the above quantities have a trivial limit: when $\alpha\to 0$ both of them go to 1, i.e., the total volume is occupied by only one large cluster with the whole gradient concentrated on it.

For different degrees of intermittency one has different curves: a larger amount of gradients ratio at the same volume ratio indicates that the fluid is more intermittent because a larger concentration of it is contained in the same region.

Starting from our multifractal result (16) one can give a prediction for the above-mentioned curve.

We have

$$\frac{S(\alpha)}{S_{\rm tot}}=\frac{\int_\alpha^\infty SP(S)dS}{\int_0^\infty SP(S)dS}, \tag{21}$$

where $P(S)$ is directly taken from (16) or from its equivalent integral expression (9), and for the volume ratio:

$$\frac{V(\alpha)}{V_{\rm tot}}=\frac{\int_\alpha^\infty P(S)dS}{\int_0^\infty P(S)dS}. \tag{22}$$

In Fig. 4 we show data taken from a direct numerical integration of Navier–Stokes equations[15] with the above prediction superimposed as a function of α, as well as the correspondent curve obtained in the K41 framework. The agreement with a multifractal choice for the $P(S)$ is very good, while the K41 case (dashed line), showing a less intermittent trend, clearly fails.

Here we do not claim that the cluster procedure of describing intermittency is an alternative to the more standard pdf way. What we would like to underline is that the prediction (9) is consistent with data coming from different simulations,[14,15] and consequently, it could be worthwhile to look for other possible numerical and experimental tests.

V. A SHELL MODEL

One of the first ideas proposed in order to grasp the qualitative future of the energy cascade in 3-D turbulent flows was using simple dynamical models more tractable than Navier–Stokes equations.[28] The aim is rather obvious: by constructing suitable Navier–Stokes approximations one might go beyond the phenomenological K41 theory. In the best case, one could use this well-understood model as a testing ground for theoretical methods and perhaps as a font of illuminating thoughts about dynamical aspects of turbulence.

Recently, a shell model attracted new interest and very promising results have been obtained both theoretically and numerically.[16–18,29]

We consider a shell model where the Fourier space is divided in N shells equispaced in a logarithmic scale.

Each shell k_n $(n=1,2,...,N)$ consists of the wave numbers k such that $K_0 2^n<k\leqslant K_0 2^{n+1}$. The velocity difference over a length scale $\approx k_n^{-1}$ is given by the scalar complex variable u_n. The energy is $E=\Sigma|u_n|^2/2$ and its power spectrum is $E(k_n)=\langle|u_n|^2\rangle/(2k_n)$. The Navier–Stokes equations are thus approximated by

$$\left(\frac{d}{dt}+\nu k_n^2\right)u_n=i(a_n u_{n+1}^* u_{n+2}^*+b_n u_{n-1}^* u_{n+1}^* + c_n u_{n-1}^* u_{n-2}^*)+f\delta_{n,4}, \tag{23}$$

where ν is the viscosity and f is a forcing (here on the fourth mode).

There are two main qualitative differences with Navier–Stokes equations:

- k is a scalar (no spatial structures);
- there are only nearest neighbor interactions among shells.

From demanding energy conservation when $\nu=f=0$, one has

$$a_n=k_n, \quad b_n=-\frac{k_{n-1}}{2}\, c_n=-\frac{k_{n-2}}{2}, \tag{24}$$

and $b_1=b_N=c_1=c_2=a_{N-1}=a_N=0$. The time evolution given by (23) exhibits a chaotic behavior on a strange attractor in the $2N$-dimensional phase space, with a maximum Lyapunov exponent proportional to $\nu^{-1/2}$.[17] Here we do not want to enter into much detail about the dynamical analysis and multifractal spectrum of this model[16,17]

Let us just summarize what has been observed in order to motivate our interest in it.[16,17,29]

(1) Equations (23) admit, when $f=0$ and $\nu=0$, an unstable fixed point given by the Kolmogorov scaling $u_n\propto k_n^{-1/3}$.

(2) The structure function $\zeta(p)$ has been calculated from a numerical integration showing intermittency and a remarkable agreement with that estimated in real 3-D fluids.

(3) There are a few positive Lyapunov exponents responsible for the chaotic behavior of the model, while a huge amount of them assume quasizero values.

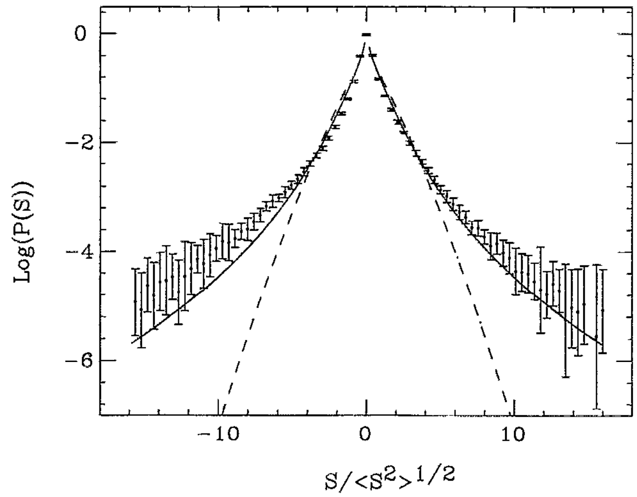

FIG. 5. Log-linear plot of the pdf for gradients in the Yamada–Ohkitani shell model.[16] The dots with error bars are the results from a numerical integration with $\mathrm{Re}=10^5$. The solid line is our multifractal theory, with the same choice of parameters used in Ref. 17 to fit the $\zeta(p)$ function; dashed curve is the K41 prediction.

(4) The chaoticity degree, estimated by the instantaneous maximum Lyapunov exponent, seems to be correlated with the burst in the energy dissipation and with instabilities on small scales.

(5) It seems possible to apply some closure techniques to carry out analytically the $\zeta(p)$ function.

We think that this nontrivial behavior is enough to motivate some consideration on its own. The fact that most of the above-listed features are shared with real turbulent flows give to this model an unexpected primary role between the possible dynamical approximations of energy cascade in 3-D turbulence. In the following we apply the same ideas presented for the 3-D velocity field to the set of the u_n variables.

In this case a real space representation is obtained via an inverse Fourier transform of the u_n field. That is,

$$u(x,t)=\sum_{n=0}^{N} u_n(t)\exp(ik_nx)+\text{c.c.} \tag{25}$$

and then for the gradient we obtain

$$S(x,t)=\sum_{n=0}^{N} iu_n(t)k_n\exp(ik_nx)+\text{c.c.} \tag{26}$$

We have performed a numerical integration of Eqs. (23) with $N=19$, $\nu=1.0\times10^{-6}$, and $f=1.0\times10^{-3}$, that correspond to Reynolds $\approx10^5$. In Fig. 5 we show the numerical result for the pdf with the K41 case superimposed as well as our prediction (16) obtained using the same $D(h)$, previously applied in fitting the $\zeta(p)$.[17] The measurements of both $\zeta(p)$ and pdf are completely consistent within each other in the random beta framework, i.e., the same choice for the RBM parameters reproduces both of them. The K41 theory, on the other hand, does not work.

VI. CONCLUSIONS

In this paper we have shown that it is possible to reproduce a vast number of statistical features regarding both the distribution function of velocity increments in the inertial range, and the pdf for transversal gradient in terms of the multifractal theory. We predict a Reynolds-dependent pdf for velocity gradients in agreement with the numerical simulations available up to now. We think that one of the most important open problems consists in generalizing the random beta model, including also fields with skewness not equal to 0 and with nontrivial spatial structure. This could be important in order to understand, respectively, what are the dynamical causes of small-scale fluctuations and how fundamental the coherent structures are in turbulence. How to include correlations in the fragmentation process is another open problem that could be important to obtain a dynamical knowledge of the mechanism at the origin of multifractality.

The good agreement between data produced with a scalar shell model and real turbulent flow suggests that the presence of organized structure is not necessary to explain either the scaling laws present in FDT or the intermittent tails in the pdf of the velocity gradients.

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