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A random process for the construction of multiaffine fields

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We define a random process for the construction of multiaffine fields, given the scaling exponents for the structure functions. The difference with analogous processes for positive defined multifractal measures is stressed. In particular our methods can be used for the study of the scaling laws exhibited by the velocity field in three dimensional fully developed turbulence. We also discuss the probability distribution functions for the increments of the signal in the scaling range.

1. Introduction

Positive defined multifractal measures have an important role in many physical phenomena. In the context of chaotic dynamics, the build up of these measures on the invariant set of a system (e.g. a strange attractor) is well understood [1–3]. Moreover, simple probabilistic models (for instance the random beta model [1] or the two scale Cantor set) have been proposed to generate them recursively. A multifractal measure $d\mu(x)$ is characterized by the scaling of the coarse-grained weight

$$p_i(l) = \int_{\Lambda_i} d\mu \,, \tag{1}$$

where the set supporting the measure is partitioned in boxes Λ_i of size l. The signature of multifractality is the anomalous power law for small l:

$$\sum_{\Lambda_i} p_i(l)^q \sim l^{(q-1)d_q} \,, \tag{2}$$

with a non-constant function d_q.

The situation is much less clear for the scaling of field increments, such as the height of growing interfaces or the velocity in three dimensional fully developed turbulence. In order to avoid a misleading terminology, we shall call multiaffine the fields $\Phi(x)$ whose structure functions scale as

$$\langle |\Phi(x+r) - \Phi(x)|^q \rangle \sim r^{\zeta_q} \,, \tag{3}$$

where $\langle \quad \rangle$ is a spatial average, r varies in an appropriate scaling range and the exponent ζ_q is a non-linear function of q. The problem of field increments is the playground where the multifractal formalism was originally proposed by Parisi and Frisch [4], who considered the anomalous scaling of the velocity increments rather than the scaling of the energy dissipation (which is a positive defined measure [5]). The random beta model [1, 3] was introduced as a practical implementation of their ideas, but actually it only generates a multifractal measure without constructing the corresponding velocity field. In fact, the statistical properties of the velocity field $u(x)$ are obtained from the scaling exponents of the energy dissipation density $\epsilon(x)$ (i.e. $d\mu = \epsilon(x)\,dx$), via the dimensional relation

$$\frac{\delta u_x(l)^3}{l} \sim \epsilon_l(x) \equiv \frac{1}{l^3} \int_{\Lambda_l} \epsilon(y)\, d^3y\,. \qquad (4)$$

Here $\delta u_x(l) = |u(x+l) - u(x)|$, Λ_l is a box of size l centered in x and $\sim$ means that the two quantities have the same statistical properties. It follows from dimensional counting

$$\zeta_q = (\tfrac{1}{3}q - 1)d_{q/3} + 3 - \tfrac{2}{3}q\,. \qquad (5)$$

Multiplicative processes like the random beta model can be used to determine the generalized dimensions d_q and hence the ζ_q in terms of few free parameters (typically two or three). These models permit to obtain very good fits of the experimental or numerical data but give no information on the feature of the velocity fields.

In this paper we introduce a method to generate a multiaffine field in any dimension with a previously assigned set of exponents ζ_q for the structure functions. In the one dimensional case a different model has been recently introduced by Vicsek et al. [6] but the generalization to the multidimensional case seems to us rather difficult. We believe that it is important to have the possibility to generate a multiaffine signal with given scaling exponents in order to hope to provide a dynamical mechanisms for the explanation of intermittency in three dimensional turbulence. Moreover the existence of an algorithm for the construction of multiaffine fields is relevant to test new methods for the treatment of experimental data. Finally, multiaffine fields are interesting not only in turbulence but also in various growth phenomena, like ballistic deposition, growth of thin films by vapor deposition, two phase viscous flow in porous media, sedimentation of granular material [7].

The paper is organized as follows. In section 2 we discuss the algorithm used to construct the multiaffine function. In section 3 we present the analytical results concerning the scaling behaviour while section 4 contains some numerical results including the probability distribution functions of the increments of the signal in the scaling range. Section 5 contains the conclusions and a summary of the open problems.

2. The definition of the multiaffine field

In this section, we shall discuss how to define a one dimensional multiaffine process. The analysis of its scaling behaviour will be done in the next section. The construction proposed here could be extended straightforwardly in two or more dimensions.

Our algorithm for the construction of a multiaffine function is a generalization of the recursive method used for obtaining a self-affine function. Indeed, it can be proved [8] that the function

$$\Phi(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \gamma^{-nh}[1 - \exp(i2\pi\gamma^n x)] \times \exp(i\phi_n)\,, \quad \text{with } \gamma > 1\,, \qquad (6)$$

is a self-affine function, i.e. the increments $F(\gamma l) = |\Phi(x+\gamma l) - \Phi(x)|$ have the same statistical properties as $\gamma^h F(l)$, if the phases ϕ_n are independent identically distributed random variables in the interval $[0, 2\pi]$. It follows

$$\langle F(l)^q \rangle \sim l^{\zeta_q}\,, \quad \text{with } \zeta_q = hq\,. \qquad (7)$$

If there are an ultraviolet and an infrared cut-off, i.e. the index n in the sum runs from 0 to $N \gg 1$, the scaling properties (7) hold only for an appropriate range of scales $\gamma^{-N} \ll l \ll 1$.

Let us now consider our algorithm for the construction of multiaffine functions. For this purpose, we shall consider $\Phi(x)$ given by the following wavelet decomposition:

$$\Phi(x) = \sum_{j=-\infty}^{+\infty} \sum_{k=-\infty}^{+\infty} \alpha_{j,k}\psi_{j,k}(x) , \tag{8}$$

where

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) , \tag{9}$$

and $\psi(x)$ is the basis function with zero mean. In the discrete case, for $N = 2^n$ points x_i in the interval [0, 1], the sums in (8) are restricted from zero to $n-1$ for the index j and from zero to $2^j - 1$ for k.

If the basis function verifies certain properties, the functions $\phi_{j,k}$ will satisfy the following orthogonality conditions:

$$\int \psi_{j,k}(x)\, \psi_{j',k'}(x)\, \mathrm{d}x = \delta_{jj'}\delta_{kk'} , \tag{10}$$

and form a complete set of functions in L^2. A review and an introduction to the properties of orthogonal wavelet decomposition can be found in [9–11]. In the following we will not need the orthogonality property.

The set of coefficients $\alpha_{j,k}$ forms a dyadic structure as shown in fig. 1. We remark that in the discrete case the number of independent coefficients is $N-1$, as it should be, because the wavelet ψ has zero average. In the case of signals with non-zero average, a constant term must be included.

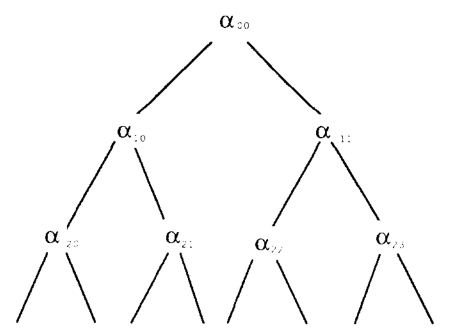

Fig. 1. The dyadic structure of the wavelet coefficients $\alpha_{j,k}$.

Let us now define our algorithm. Consider the random variable η with probability density distribution $P(\eta)$ and construct the following multiplicative process:

$$\begin{aligned} &\alpha_{1,0} = \epsilon_{1,0}\eta_{1,0}\alpha_{0,0} , \quad \alpha_{1,1} = \epsilon_{1,1}\eta_{1,1}\alpha_{0,0} , \\ &\alpha_{2,0} = \epsilon_{2,0}\eta_{2,0}\alpha_{1,0} , \quad \alpha_{2,1} = \epsilon_{2,1}\eta_{2,1}\alpha_{1,0} , \\ &\alpha_{2,2} = \epsilon_{2,2}\eta_{2,2}\alpha_{1,1} , \quad \alpha_{2,3} = \epsilon_{2,3}\eta_{2,3}\alpha_{1,1} , \end{aligned} \tag{11}$$

and so on. The $\eta_{j,k}$ are independent random variables having the same distribution $P(\eta)$, the coefficient $\alpha_{0,0}$ is arbitrary and $\epsilon_{j,k} = \pm 1$ with equal probability. The general term is $\alpha_{j,k} = \epsilon_{j,k}\eta_{j,k}\alpha_{j-1,k'}$ with $k' = [\frac{1}{2}k]$. It is easy to show that $|\alpha_{j,k}|$ are random variables with moments

$$\overline{|\alpha_{j,k}|^p} = 2^{j\log_2(\overline{\eta^p})}\, |\alpha_{0,0}|^p . \tag{12}$$

The bar denotes the average over the ensemble of the realizations of the multiplicative process. Let us remark that the moment in (12) does not depend on k. In general, the scaling behaviour of the coefficients $\alpha_{j,k}$ does not imply that $\Phi(x)$ is multiaffine. However, from a heuristic point of view, one could guess that structure functions are power laws and calculate the exponents by supposing that the scaling properties at scale $r \sim 2^{-j}$ are dominated by the jth term in the sum of eq. (8). This leads to

$$\langle |\Phi(x+l) - \Phi(x)|^q \rangle \sim l^{\zeta_q} , \quad \text{with } \zeta_q = -\log_2(\overline{\eta^q}) - \tfrac{1}{2}q . \tag{13}$$

This argument will be confirmed in the next section.

The generalization of our algorithm to more than one dimension is straightforward. Following [9], the three dimensional field $\Phi(\boldsymbol{x})$ is decomposed as

$$\Phi(\boldsymbol{x}) = \sum_{j,k_1,k_2,k_3} \sum_{q=1}^{8} \alpha^{(q)}_{j,k_1,k_2,k_3} \psi^{(q)}_{j,k_1,k_2,k_3}(\boldsymbol{x}) , \tag{14}$$

where the index j refers to the dilation factor, the indices k's to translations in the three possible directions and the index q is needed to take into account the internal degrees of freedom. The coefficients $\alpha^{(q)}_{j,k_1,k_2,k_3}$ are obtained by a multiplicative process in the same spirit as in the one dimensional case. The condition of zero divergence can be imposed either by using divergenceless wavelets [12] or, as usual, in Fourier space.

3. Scaling behaviour of the signal

In order to show that $\Phi(x)$, defined in the previous section, is multiaffine we first consider the second order structure function.

$$S_2(r) \equiv \langle [\Phi(x+r) - \Phi(x)]^2 \rangle , \tag{15}$$

where $\langle \quad \rangle$ represents spatial average. Using the wavelet decomposition (8) and (9) we obtain

$$S_2(r) = \Big\langle \sum_{j,k} \{\alpha_{j,k} 2^{j/2} [\psi(2^j x + 2^j r - k) - \psi(2^j x - k)]\}^2 \Big\rangle . \tag{16}$$

Next we can observe that in our construction $\alpha_{j,k}$ are uncorrelated random variables with zero mean. Using the self-averaging property of the multiplicative process we have defined, i.e. the equivalence between spatial and ensemble average, we obtain

$$S_2(r) = \sum_{j,k} 2^j \overline{\alpha^2_{j,k}} \langle [\psi(2^j x + 2^j r - k) - \psi(2^j x - k)]^2 \rangle , \tag{17}$$

where the bar denotes ensemble average. The mean in the previous equation can be evaluated as

$$\langle [\psi(2^j x + 2^j r - k) - \psi(2^j x - k)]^2 \rangle = 2^{-j} G_2(2^j r) , \tag{18}$$

where $G_2(r) = \int [\psi(x+r) - \psi(x)]^2 \, dx$. Substituting (18) into (17) we obtain

$$S_2(r) = \sum_j \overline{\alpha^2_{j,k}} 2^j \, G_2(2^j r) , \tag{19}$$

where we have used the fact that the $\overline{\alpha^2_{j,k}}$ are independent of k and there are 2^j different values of k for a fixed j. By using (19) we can establish the scaling of $S_2(r)$:

$$\begin{aligned} S_2(2r) &= \sum_j \overline{\alpha^2_{j,k}} 2^j \, G_2(2^{j+1} r) \\ &= \sum_j 2^{j(\log_2 \overline{\eta^2} + 1)} \, G_2(2^{j+1} r) \\ &= 2^{-(\log_2 \overline{\eta^2} + 1)} \sum_j 2^{(j+1)(\log_2 \overline{\eta^2} + 1)} \, G_2(2^{j+1} r) \\ &= S(r) \, 2^{-(\log_2 \overline{\eta^2} + 1)} . \end{aligned} \tag{20}$$

The variable η has been defined in the previous section. It follows that

$$S_2(r) \propto r^{\zeta_2} \quad \text{with } \zeta_2 = -\log_2 \overline{\eta^2} - 1 . \tag{21}$$

The *naive* arguments at the end of the previous section correctly captures the scaling exponent, at least for the second moment.

In the same way we can compute the scaling of the fourth moment $S_4(r)$. After a long but straightforward computation we obtain

$$S_4(r) = \sum_j \overline{\alpha^4_{j,k}} 2^j \, G_4(2^j r) + 3 S_2^2(r) , \tag{22}$$

where $G_4(r) = \int [\psi(x+r) - \psi(x)]^4 \, dx$. By the same manipulations leading to (21) we finally have

$$S_4(r) = A_4 r^{\zeta_4} + 3 A_2 r^{2\zeta_2} , \tag{23}$$

with $\zeta_4 = -\log_2 \overline{\eta^4} - 2$. For $r \ll 1$ and using the convexity of the function ζ_q we have $r^{\zeta_4} \gg r^{2\zeta_2}$ so that

$$S_4(r) \propto r^{\zeta_4} . \tag{24}$$

As it is clear from (22) and (23), one can easily generalize the above computations for the cumulant structure functions $S^c_{2n}(r)$ defined by the relation

$$\langle \exp\{z[\Phi(x+r)-\Phi(x)]\}\rangle \equiv \exp\left(\sum_n \frac{z^n S^c_n(r)}{n!}\right). \tag{25}$$

The functions $S^c_{2n}(r)$ satisfy the scaling

$$S^c_{2n}(r) \propto r^{\zeta_{2n}}, \tag{26}$$

with $\zeta_{2n} = -\log_2 \overline{\eta^{2n}} - n$. For $r \ll 1$, due to the convexity of ζ_q, the leading contribution to S^{2n} is given by $S^c_{2n}(r)$. We have therefore shown that $\Phi(x)$ is multiaffine.

We conclude this section with the following remark. In section 1 we defined a signal as multiaffine if structure functions obtained performing a spatial average on a single realization have anomalous scaling. We stress that it is possible to have

$$\langle \overline{|\Phi(x+r)-\Phi(x)|^q}\rangle \sim r^{\tilde{\zeta}_q}, \tag{27}$$

with a nonlinear $\tilde{\zeta}_q$ even if the single realizations have no anomalous scaling. An example is provided by the following simple generalization of (6):

$$\Phi(x) = \lim_{N\to\infty} \sum_{n=-N}^{N} a_n[1-\exp(\mathrm{i}2\pi 2^n x)]\exp((\mathrm{i}\phi_n), \tag{28}$$

where the a_n are obtained by an uncorrelated multiplicative process $a_n = a_{n-1} b_n$ and b_n are independent identically distributed random variables. Numerical computations are in agreement with the rough heuristic argument presented at the end of section 2 and one has $\tilde{\zeta}_q = -\log_2 \overline{b^q}$ while for a single realization there is no anomalous scaling and the single exponent is $-\overline{\log_2 b}$.

Let us remark, finally, that the diadic structure (11) chosen for the multiplicative process is not the unique possibility. It is possible that also more complicated fragmentation processes induce a multiaffine scaling in the signal. While the absence of correlation, along the vertical structure of the tree, seems to be a necessary condition in order to obtain a scaling behaviour, there are not a priori restriction on the type of correlation along the horizontal direction (that spanned by the k index of the α_{ik} coefficients).

4. Numerical results

In the previous section we have shown that the function $\Phi(x)$ defined in section 2 satisfies the multiaffine scaling (3), with $\zeta_q = -\log_2 \overline{\eta^p} - \frac{1}{2}p$. Now we want to give a numerical example of $\Phi(x)$ and its scaling behaviour. For this purpose we consider $\psi(x)$ to be the mexican-hat function obtained by differentiation of a Gaussian:

$$\psi(x) = -\frac{\mathrm{d}^2}{\mathrm{d}x^2}\exp\left(-\frac{x^2}{2\sigma^2}\right), \tag{29}$$

where σ is the characteristic width of the Gaussian. Let us note that, although this choice does not produce an orthonormal set of functions, the scaling behaviour (26) is ensured. The width σ of the $\psi(x)$ is chosen to be slightly smaller than the initial interval where the signal is defined.

Next we construct $\Phi(x)$ by discretizing (8) in the interval $[0, 1]$ on a lattice of $N = 2^{16}$ points x_i:

$$\Phi(x_i) = \sum_{j=0}^{15} \sum_{k=0}^{2^j-1} \alpha_{j,k}\, \psi_{j,k}(x_i). \tag{30}$$

The $\alpha_{j,k}$ are defined according to the rule given in section 2, with probability $P(\eta) = y\,\delta(\eta - \eta_0) + (1-y)\,\delta(\eta-\eta_1)$. A typical realization of $\Phi(x_i)$ with $y = 0.125$, $\eta_0 = 2^{-1/2}$, $\eta_1 = 2^{-5/6}$ is given in fig. 2. To check the multi-affine scaling we have made 50 different realizations of $\Phi(x)$ and by averaging over them we have computed the structure functions $S_q(r)$ and the connected

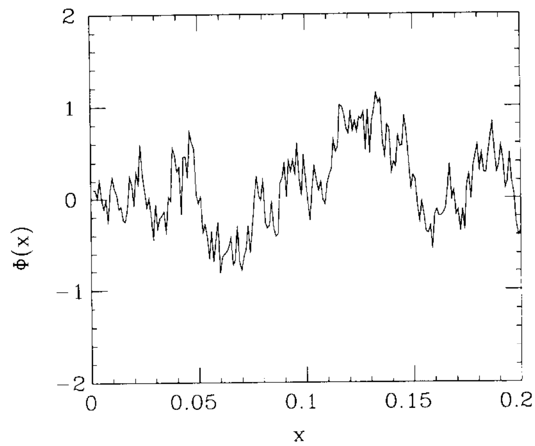

Fig. 2. A typical realization of the multiaffine signal. The multiplicators assume two values: $2^{-5/6}$ with probability 0.875 and $2^{-1/2}$ with probability 0.125.

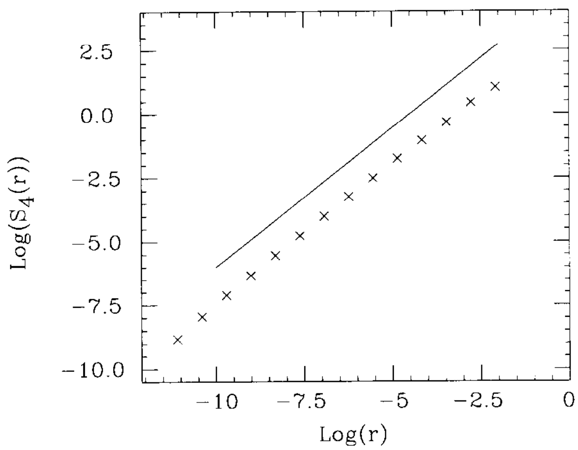

Fig. 4. The natural logarithm of the fourth order structure function $S_4(r)$ vs the natural logarithm of the scale. The slope of the line is given by (13).

structure functions $S_q^c(r)$ defined in the previous section. The theoretical and experimental scaling exponents are given in fig. 3, while in fig. 4 we show $S_4(r)$. The scaling exponents are consistent with the results discussed in section 3. Finally we mention that the energy spectrum has a scaling region with the correct exponent $-1-\zeta_2$.

We have calculated the probability distribution function (PDF) of the variables $\delta\Phi_l(x) = \Phi(x + l) - \Phi(x)$. Fig. 5 shows that it exhibits the typical shape of the PDF for the transverse velocity increments obtained in three dimensional turbulent flows at high Reynolds numbers [13]. For large l, the PDF is nearly Gaussian, while on small scales the PDF becomes more and more peaked around zero with relatively high tails (corresponding to the presence of strong intermittency in the velocity gradients and hence in the energy dissipation).

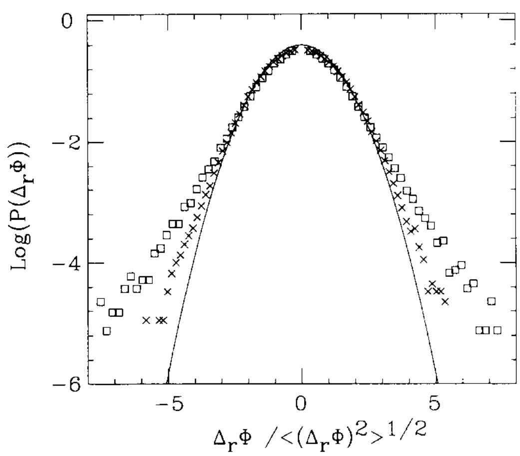

Fig. 5. Probability distribution functions of the increments of the multi-affine signal normalized in order to have unit variance. The signal is defined on 2^{16} points in the interval [0, 1]. The solid curve is a Gaussian reference curve, the crosses refer to a distance 2^{-8} and the squares to a distance 2^{-16}.

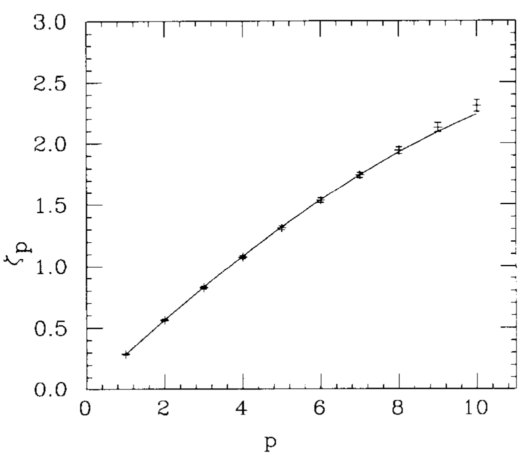

Fig. 3. The scaling exponents function ζ_p. The continuous curve is the prediction obtained by the arguments of section 3 and the crosses are the measured values.

5. Conclusions

In this paper we have shown how to define an algorithm to construct multiaffine fields based on wavelet decomposition. The idea underlying our approach is the definition of the coefficients of the wavelet in terms of a multiplicative process. Considering the one dimensional case, the wavelet coefficients $\alpha_{j,k}$ are random variables such that the moments of their absolute values $|\alpha_{j,k}|$ are multifractal. We have shown, both analitically and numerically, that structure functions exhibit power laws with anomalous scaling. We remark that our algorithm can be used to construct multiaffine fields with a priori definition of the scaling properties in any dimension.

The multiaffine functions considered in this paper do not satisfy any local scaling, that is to say for any point x the quantity $|\Phi(x+r) - \Phi(x)|$ has no scaling property in r. The original approach by Parisi and Frisch [4] should be considered only in a statistical sense. Recently there have been some attempts to extract exponents of local scaling from turbulent signal [14,15]. In the multiaffine function considered in this paper this approach is meaningless because, by construction, the scaling exponents can be identified only in a statistical sense and not locally.

Concerning orthogonal wavelet decomposition, we argue that a *statistical* analysis of wavelet coefficients of three dimensional turbulent signals coming from experiments or numerical simulations could give many interesting insights. For example, we expect that the moments of the coefficients have a *dominant* scaling behaviour consistent with the one observed for structure functions. However it is a matter of interest to see if the presence of different subdominant terms leads to a better scaling behaviour of the moments of wavelet coefficients with respect to structure functions. In our opinion, it is also important to study the probability distribution of the ratios $|\alpha_{j-1,k}|/|\alpha_{j,k}|$, looking in particular for a possible scale invariance of such probability. In real turbulence it is not difficult to imagine a certain degree of correlation among the wavelet coefficients at various scales. Such a correlation could be introduced also in the construction of the multiaffine signal, for instance, by considering a Markov process for the multiaffine variable η of section 2.

Finally, a dissipation range could be introduced in our algorithm by using the same ideas discussed in [13]. In this way a signal with the correct statistics for $\partial_x \Phi(x)$ can be generated and compared with outcomes from experiments and simulations.

Acknowledgements

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