Energy and Information Transfer Mechanisms in Turbulence

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April 15, 2025

Introduction

Self introduction

荒木 亮 (ARAKI, Ryo), born in Kyoto prefecture, Japan

Career

- Until 03/2020: Bachelor & Master @Osaka University🇯🇵
  - Supervisor: Prof. Susumu Goto
- Until 09/2023: PhD @École Centrale de Lyon🇫🇷 & Osaka Univ.🇯🇵
  - Supervisors: Dr. Wouter Bos and Prof. Susumu Goto
  - Thesis: “Temporal and Spatial Features of the Turbulent Kinetic Energy Cascade”
- Since 10/2023: **Fixed-term** Assistant Professor @Tokyo Univ. of Science
  - In Prof. Takahiro Tsukahara’s lab
  - **Looking for a next position!**

Today, I want to discuss ...

Part I

- Q. How do different **physical mechanisms** contribute to energy and information transfer in the inertial range of 3D turbulence?

Part II

- Q. What can we **universally** say about the nature of the information flow in turbulence in terms of nonequilibrium statistical mechanics?

Part III

- Ideas for future research & possible collaborations: Q. What is the universal bound on the finite-time predictability?

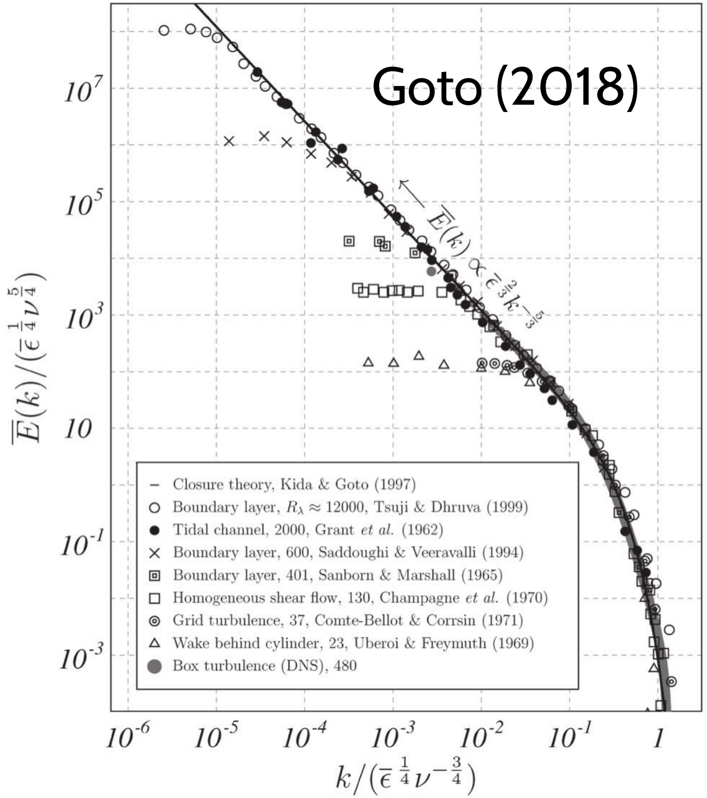
Collaborators

Adrián Lozano-Durán	Alberto Vela-Martín	Tomohiro Tanogami
@Caltech, USA	@UP3M, Spain	@Osaka U., Japan

Will attend StatPhys29 held in Firenze, July 13–18.

Physical Mechanisms of Energy/Information Transfer

Turbulence and its statistical universality



[...] big whirls have little whirls that feed on their velocity, and little whirls have lesser whirls and so on to viscosity [...]. Richardson (1922)

Causality and “forgetful” in energy cascade

Small scales “forget” about large scales

[The small scales] do not retain any information which relates to their great-great-great-grand parents. Davidson (2013)

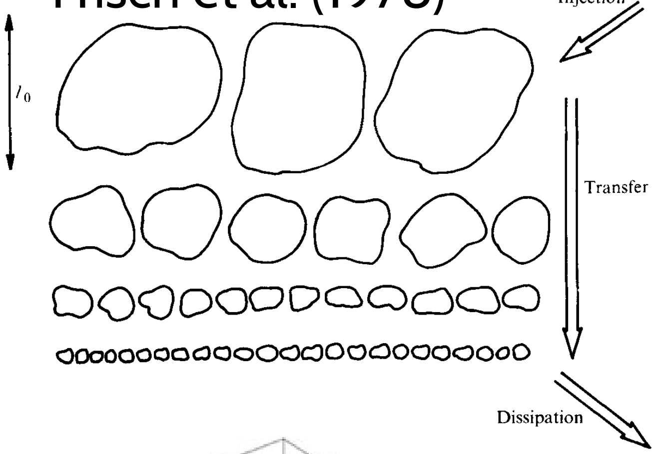
Small scales are determined by large scales

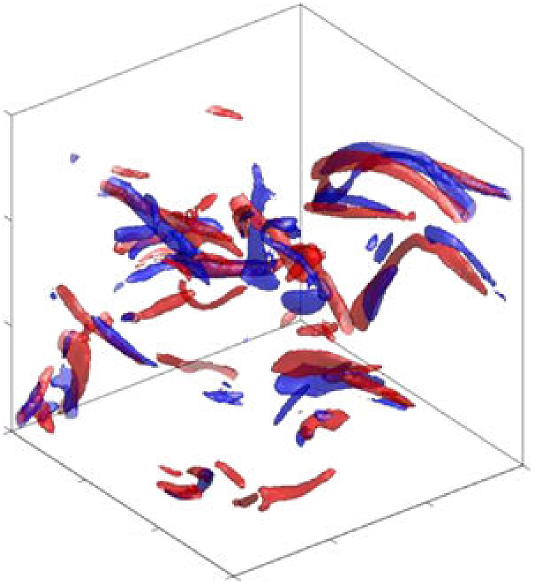
“Twin” turbulent simulations with the same large scales exhibit synchronised small scales.

Vela-Martín (2021)

Q. 1 Can we reconcile this paradox?

Frisch et al. (1978)





Brief introduction of information theory

Shannon entropy

$$H(X) := \int \mathrm{d}x\, p(x)[-\ln p(x)]$$

- Information of an event $x \in X$ is quantified by $-\ln p(x)$.
- $H(X)$ quantifies "average uncertainty" of a random variable X.

Quantities to quantify correlation and causality

- Mutual information $I[X : Y] := \int \mathrm{d}x\, p(x) \ln[p(x,y)/p(x)p(y)]$
- Transfer entropy $T_{X\to Y} := H\left(Y_{t+1} \mid \boldsymbol{Y}_t^{(k)}\right) - H(Y_{t+1}) \mid \boldsymbol{X}_t^{(l)}, \boldsymbol{Y}_t^{(k)})$

  where $\boldsymbol{X}_t^{(l)} := (X_t, X_{t-1}, \dots, X_{t-k+1})$
- Information flow $\dot{I}^X := \lim_{\mathrm{d}t \searrow 0} \frac{1}{\mathrm{d}t}(I[X_{t+\mathrm{d}t} : Y_t] - I[X_t : Y_t])$

Information flux

Discrete dynamical system

- Variable $\boldsymbol{Y} = [Y_1, Y_2, \dots, Y_N]$
- Time evolution $Y_j^{n+1} = f_j(\boldsymbol{Y}^n)$

Information flux

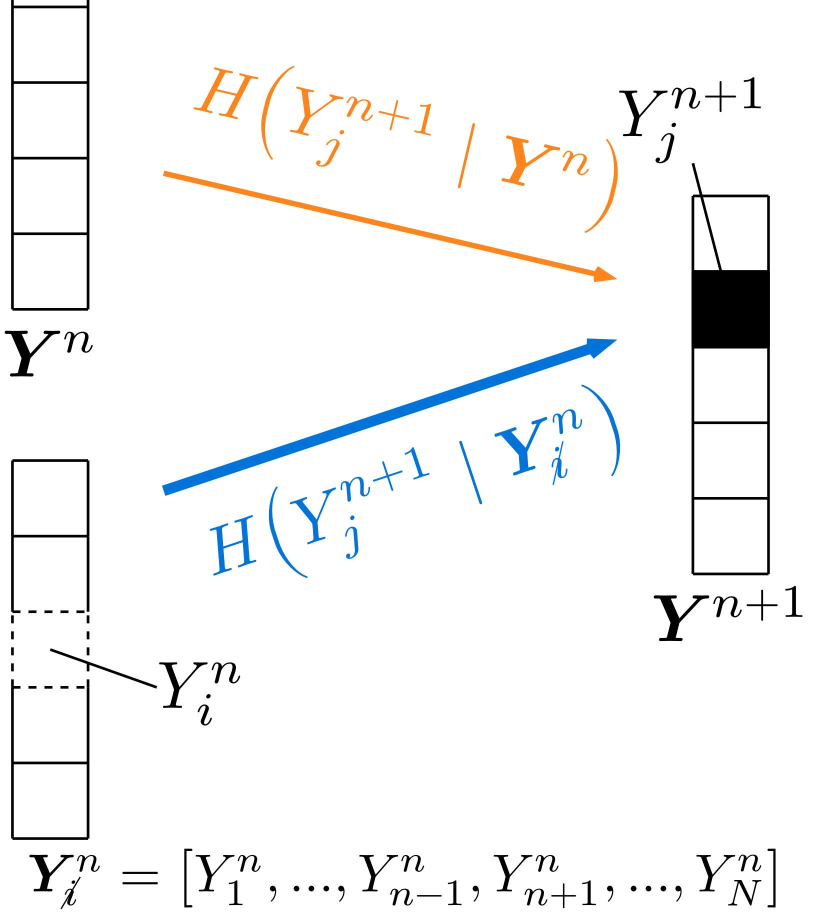
Uncertainty to observe Y_j^{n+1} is

- $H\left(Y_j^{n+1} \mid \boldsymbol{Y}^n\right)$ when $\boldsymbol{Y}^n$ is known
- $H\left(Y_j^{n+1} \mid \boldsymbol{Y}_{\not i}^n\right)$ when $\boldsymbol{Y}_{\not i}^n$ is known

$$\therefore T_{i \to j}^{Y} := H\left(Y_j^{n+1} \mid \boldsymbol{Y}_{\not i}^n\right) - H\left(Y_j^{n+1} \mid \boldsymbol{Y}^n\right)$$

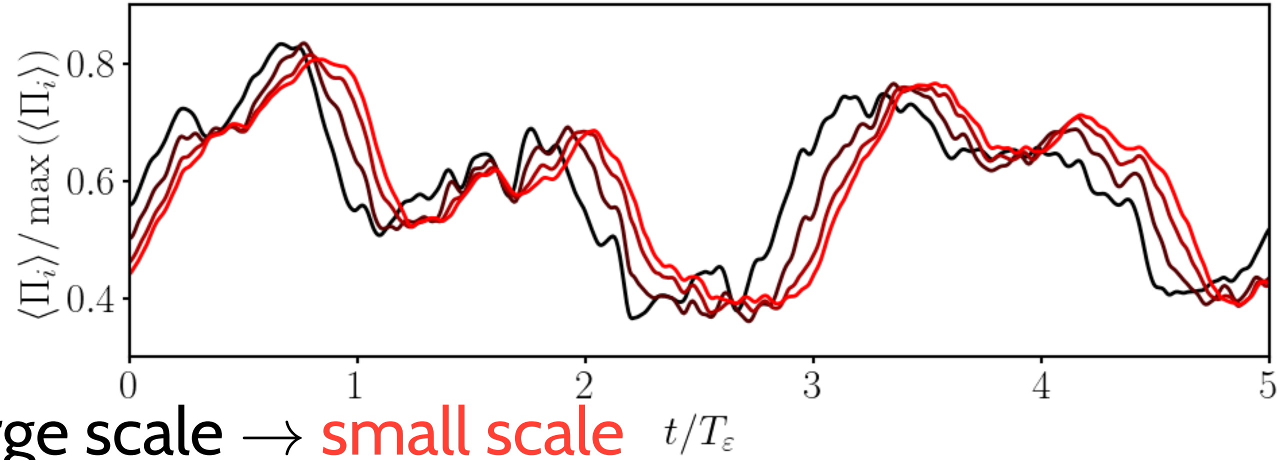
quantifies **decrease of uncertainty to observe Y_j^{n+1} by newly knowing Y_i^n.**

Lozano-Durán & Arranz (2022)



Causality in energy flux time series

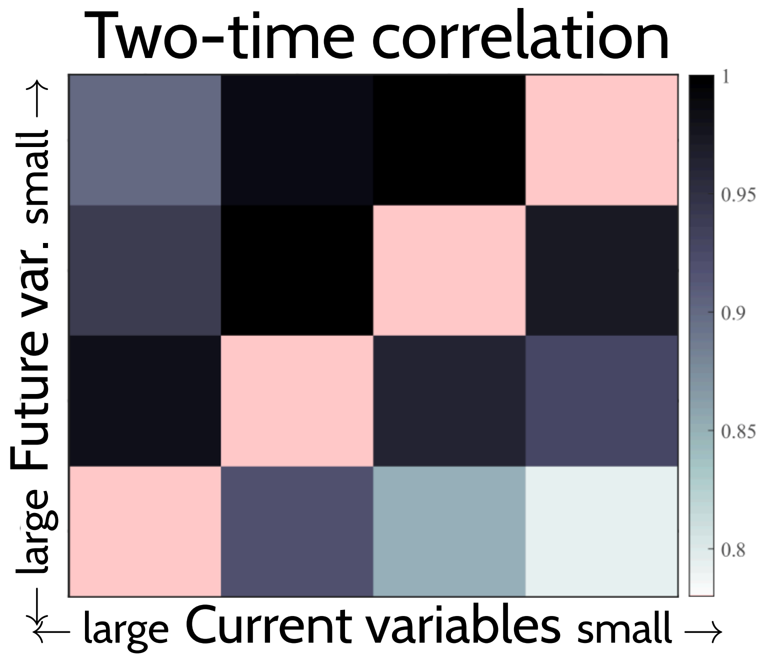
$$\Pi := -\mathring{\tau}_\ell(u_i, u_j)\bar{S}^\ell_{ij}$$

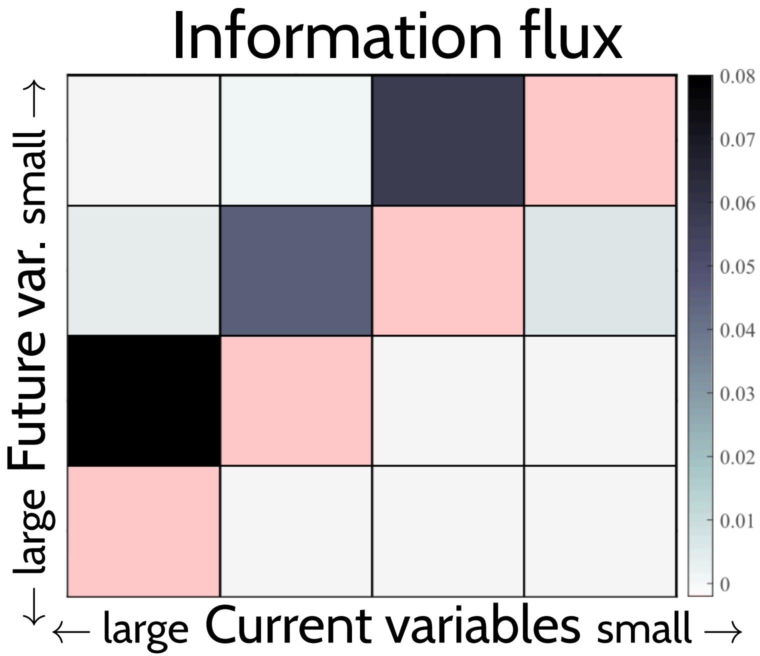


Large scale $\rightarrow$ small scale

- Coarse-grained velocity field $\bar{u}^\ell_i(\boldsymbol{x}) := \iiint_{-\infty}^{\infty} G_\ell(\boldsymbol{x}) u_i(\boldsymbol{x}+\boldsymbol{r})\, \mathrm{d}\boldsymbol{r}$
- Gaussian filter $G_\ell(\boldsymbol{x}) := \mathcal{N} \exp(-|\boldsymbol{r}|^2/2\ell^2)$
- Reynolds stress $\tau_\ell(u_i, u_j) := \overline{u_i u_j}^\ell - \bar{u}^\ell_i \bar{u}^\ell_j$
- Strain rate $\bar{S}^\ell_{ij} := \left[\partial_j \bar{u}^\ell_i + \partial_i \bar{u}^\ell_j\right]/2$ where non-diagonal part $\mathring{\tau}_{ij} := \tau_{ij} - \delta_{ij}\tau_{kk}/3$

Causality in energy flux time series

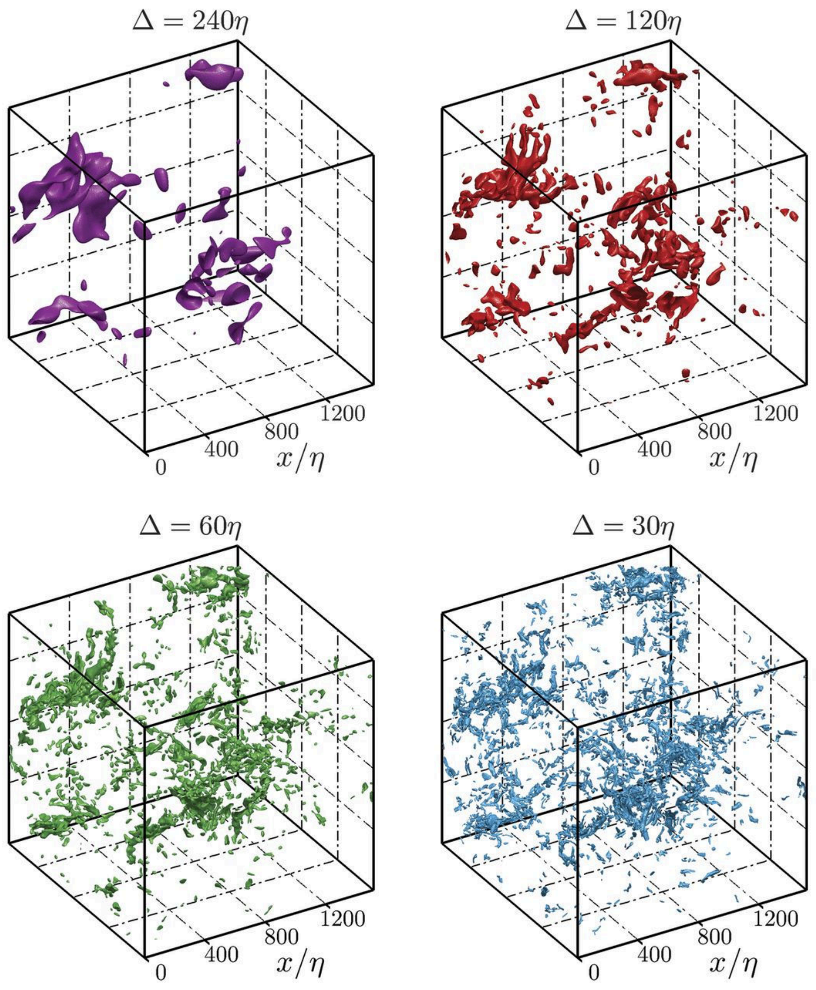




Summary 1 Information flux captures the forward scale-local causality, which two-time correlation function cannot.

Lozano-Durán & Arranz (2022)

Overview of my approach

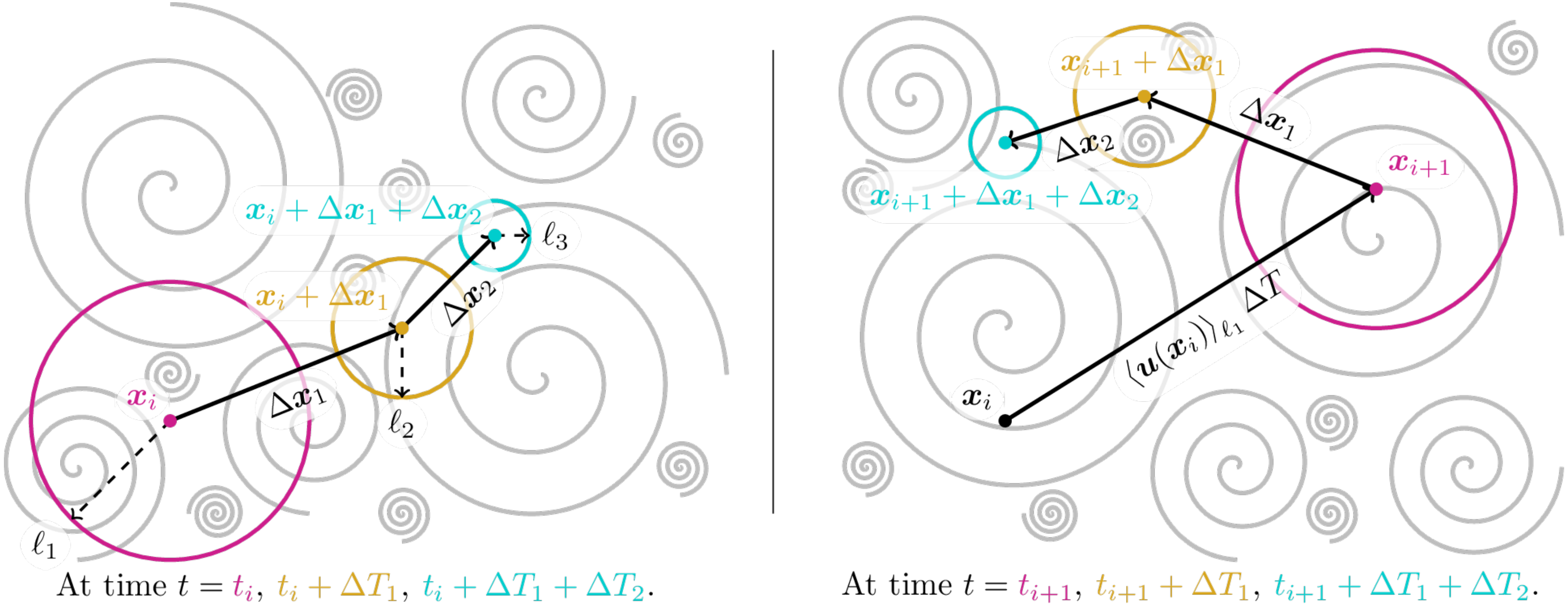


Turbulence dataset

- Time-resolved HIT @UPM
- Taylor-scale $\mathrm{Re}_\lambda = 315$
- Resolution $N^3 = 1024^3$
- Length $T_{\mathrm{sim}}/T_0 = 66$
- # of sample: $N_{\mathrm{sample}} = \mathcal{O}(10^3)$
- Linear forcing at low $\boldsymbol{k}$

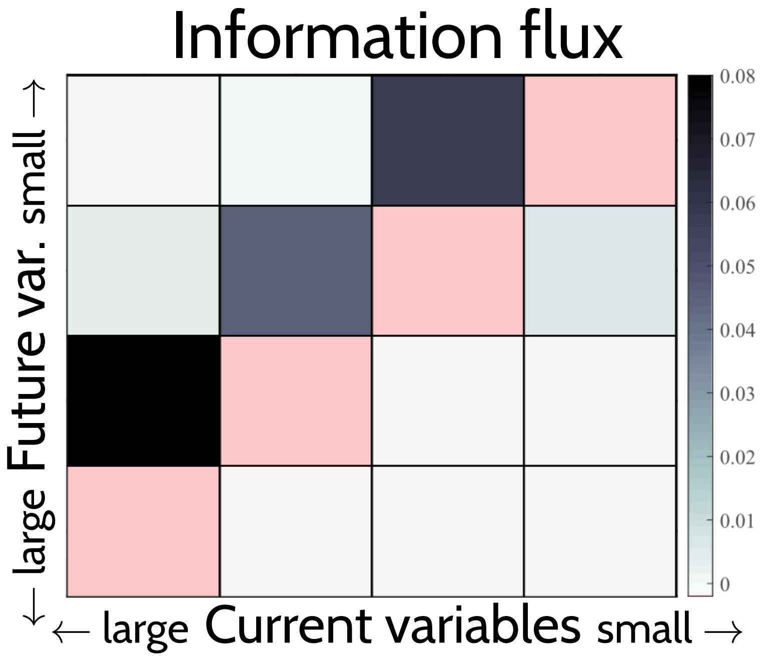
Cardesa et al. (2017)

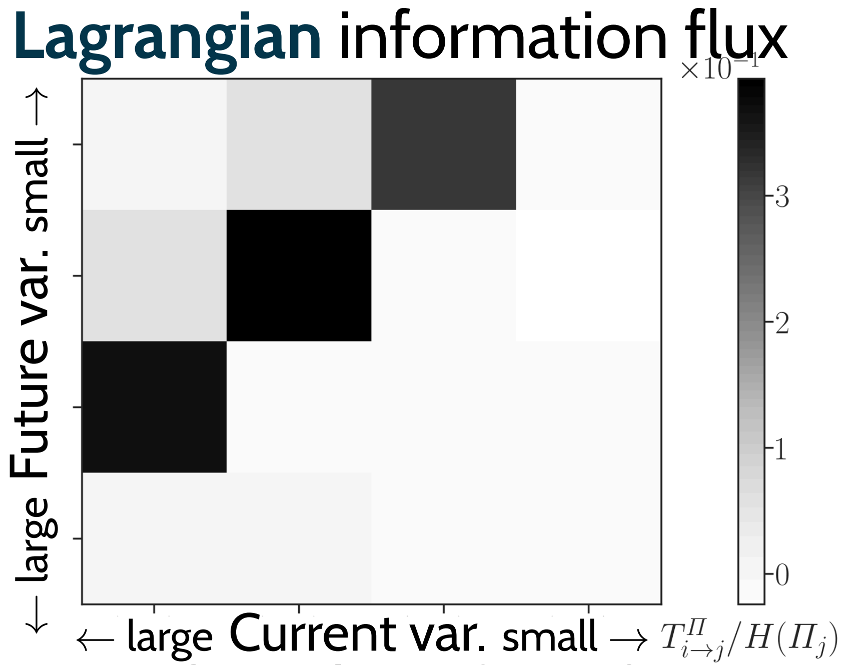
Lagrangian tracking of space-local energy flux



1. Compute space-local average of energy flux at $\boldsymbol{x}$.
2. Advect $\boldsymbol{x} \to \boldsymbol{x} + \Delta\boldsymbol{x}$ & compute the same quantity at smaller scale.
3. Repeat 1-2 to construct the Lagrangian dataset.

Scale-local information flux



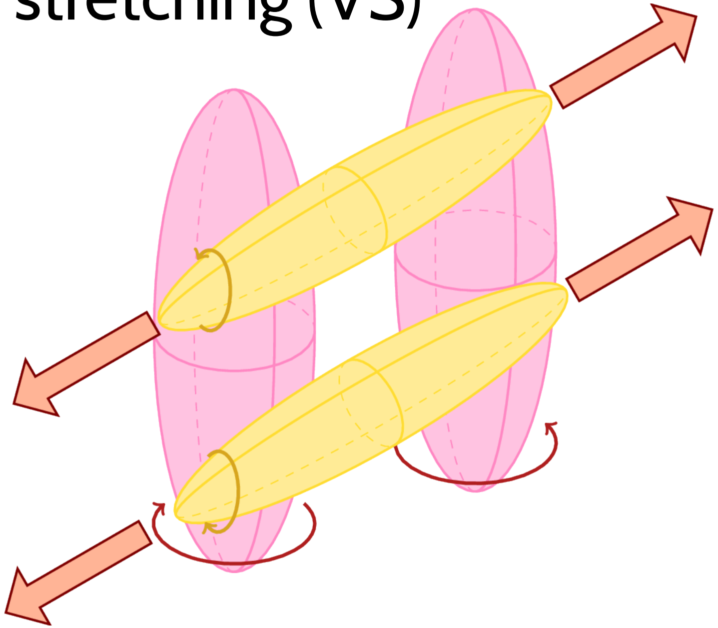


Summary 2 Lagrangian dataset captures the scale-local information flux without the self-induced ones.

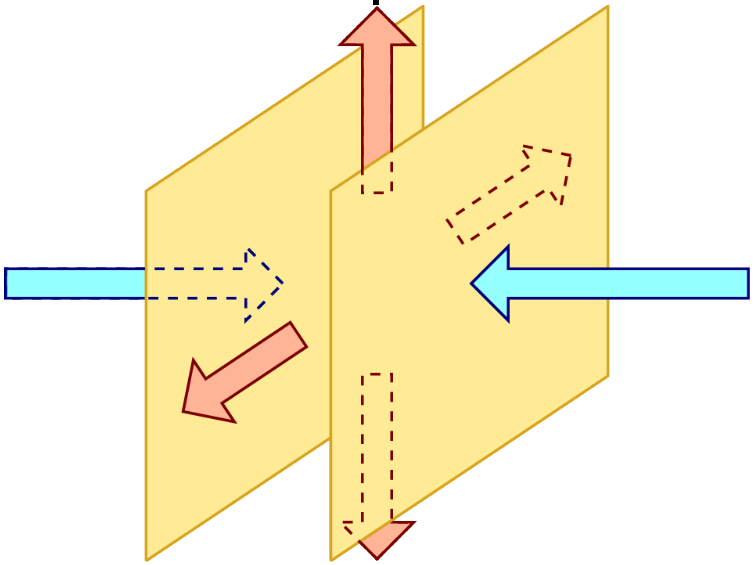
Physical mechanisms of energy cascade

Q. 2 What is the physical mechanism of the information flow?

Vortex stretching (VS)



Strain Self-Amplification (SSA)



Note: ***The*** physical mechanism of the cascade is still an open question.

Decomposed energy flux

$$\Pi := -\mathring{\tau}_\ell(u_i, u_j)\bar{S}^\ell_{ij}$$

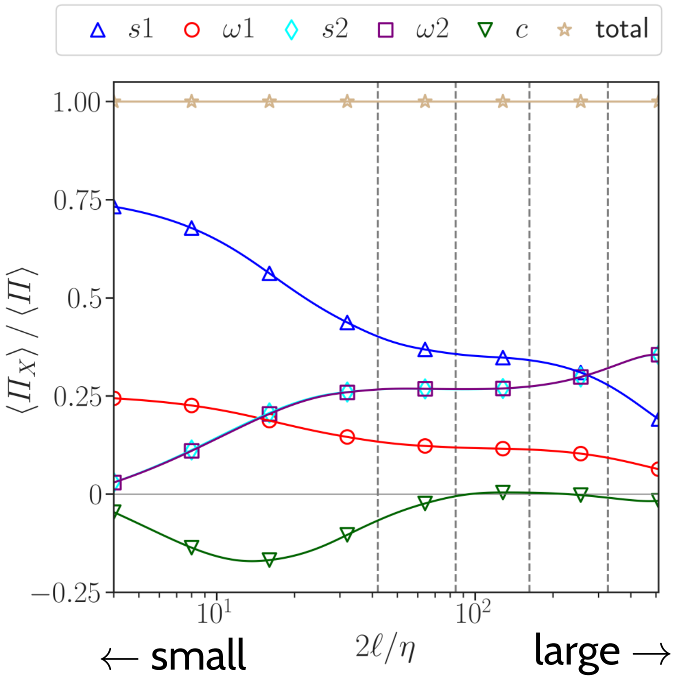
$$= \Pi^\ell_{s1} + \Pi^\ell_{\omega 1} + \Pi^\ell_{s2} + \Pi^\ell_{\omega 2} + \Pi^\ell_{c}$$

- Scale-local SSA

$$\Pi^\ell_{s1} := -\ell^2 \bar{S}^\ell_{ij}\bar{S}^\ell_{jk}\bar{S}^\ell_{ki}$$

- Scale-local VS

$$\Pi^\ell_{\omega 1} := \ell^2 \bar{\omega}^\ell_i \bar{S}^\ell_{ij}\bar{\omega}^\ell_j/4$$



Summary 3 $\Pi^\ell_{s1} > \Pi^\ell_{\omega 1}$: SSA transfers more energy to smaller scales than VS in the inertial range. Johnson (2021)

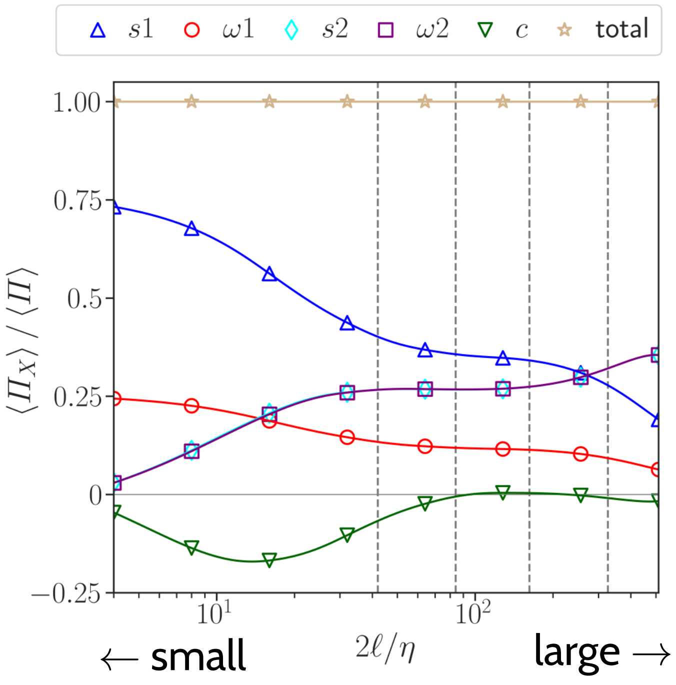
Decomposed energy flux

- Scale-nonlocal SSA

$$\Pi^{\ell}_{s2} := -\int_0^{\ell^2} \mathrm{d}\alpha\, \bar{S}^{\ell}_{ij}\tau_\beta\left(\bar{S}^{\sqrt{\alpha}}_{jk}, \bar{S}^{\sqrt{\alpha}}_{ki}\right)$$

- Scale-nonlocal VS

$$\Pi^{\ell}_{\omega 2} := -\frac{1}{4}\int_0^{\ell^2} \mathrm{d}\alpha\, \bar{S}^{\ell}_{ij}\tau_\beta\left(\bar{\omega}^{\sqrt{\alpha}}_{i}, \bar{\omega}^{\sqrt{\alpha}}_{j}\right)$$



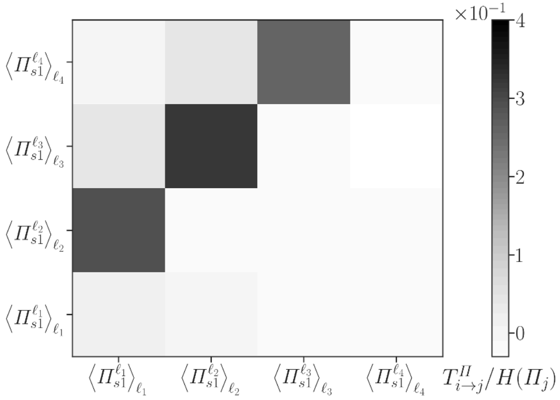
Summary 4 $\Pi^{\ell}_{s2} = \Pi^{\ell}_{\omega 2}$: Nontrivial relation in nonlocal terms.

$\beta := \sqrt{\ell^2 - \alpha}, \omega_i := \epsilon_{ijk}\partial_j u_k, \Omega_{ij} := [\partial_j u_i - \partial_i u_j]/2$

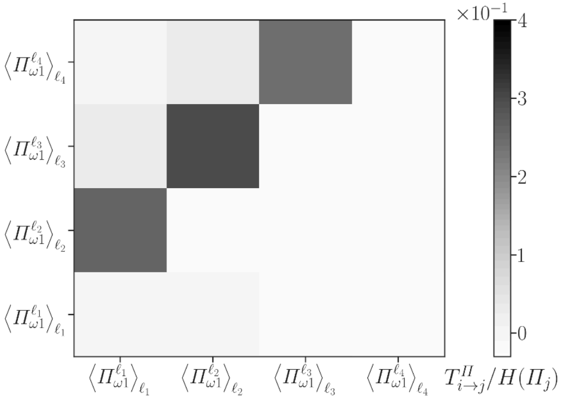
Johnson (2021)

Information flux associated with different mechanisms

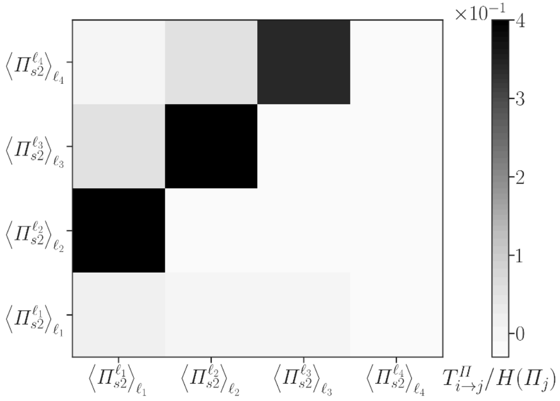
Local SSA Π^{ℓ}_{s1}



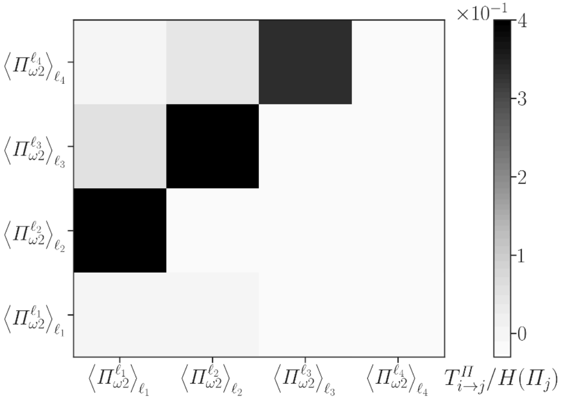
Local VS $\Pi^{\ell}_{\omega1}$

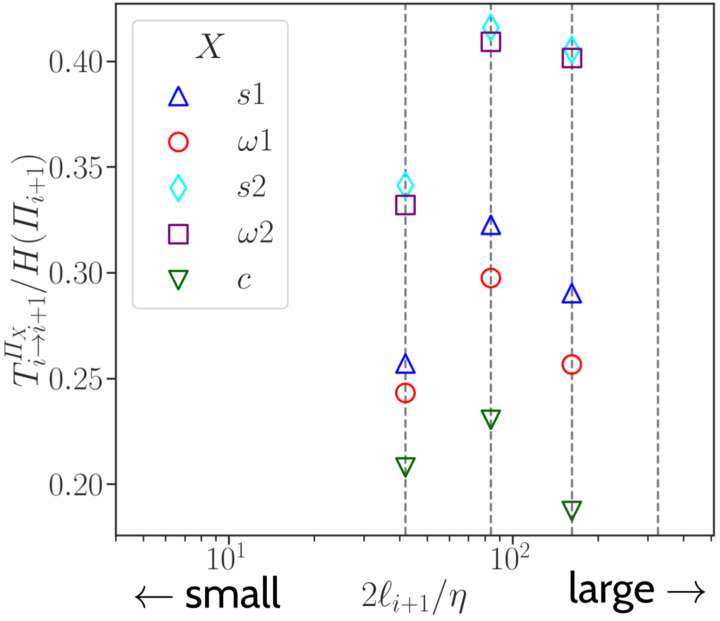


Nonlocal SSA Π^{ℓ}_{s2}



Nonlocal VS $\Pi^{\ell}_{\omega2}$

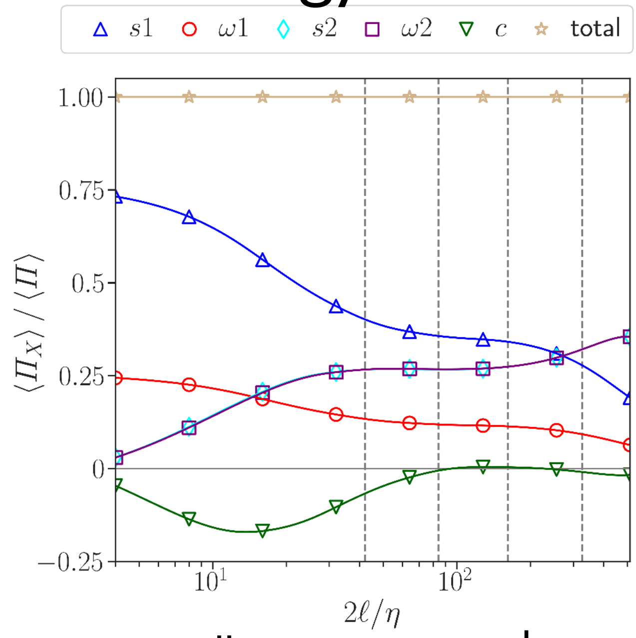




Summary 5
Nonlocal terms are more "causal"

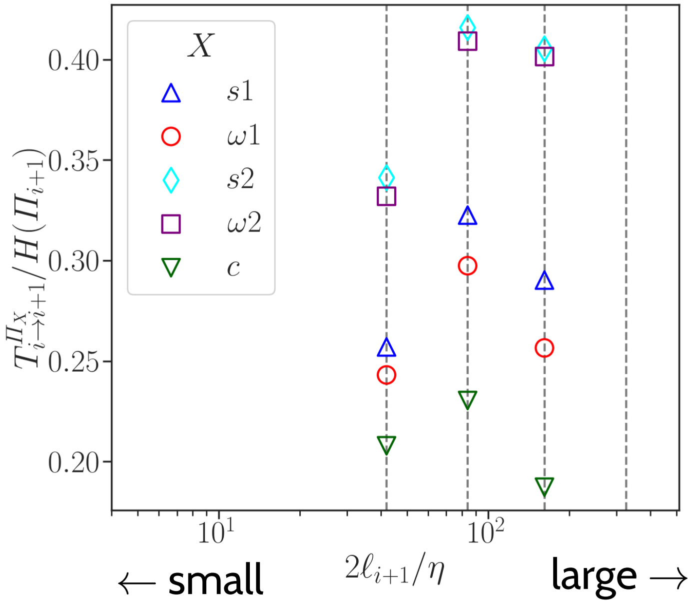
Discrepancy between energetic and causal mechanisms

Energy flux



← small large →

Information flux

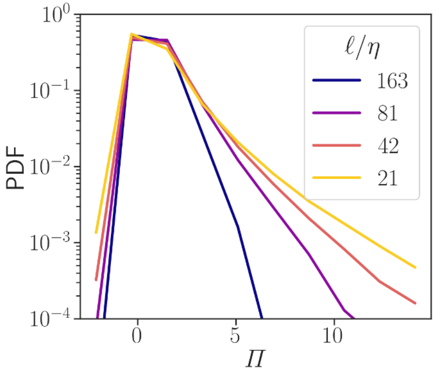


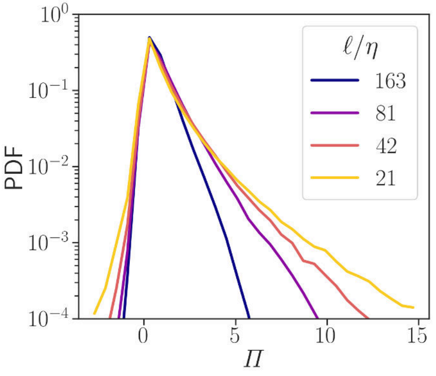
← small large →

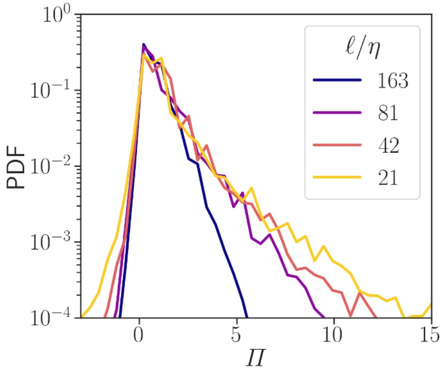
Summary 6 The most energetic mechanism $\neq$ the most causal mechanism (and vice versa)

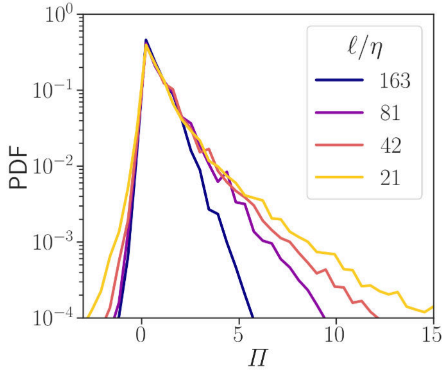
Estimation method dependency

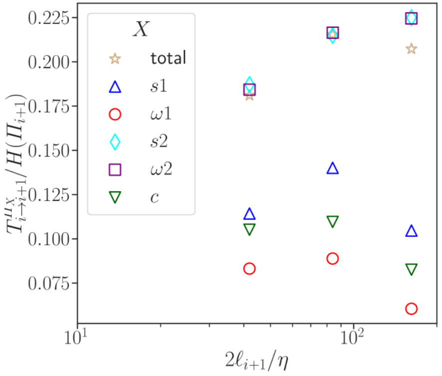
Direct binning (# of bin = 10)	Direct binning (# of bin = 30)	k-NN ($k = 15$)	k-NN ($k = 50$)

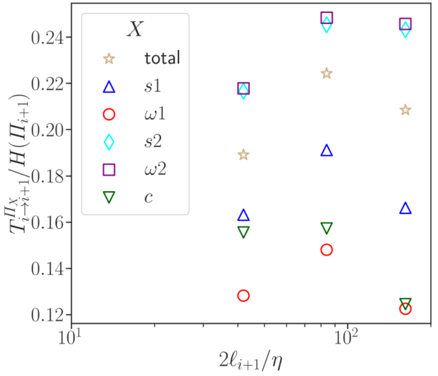


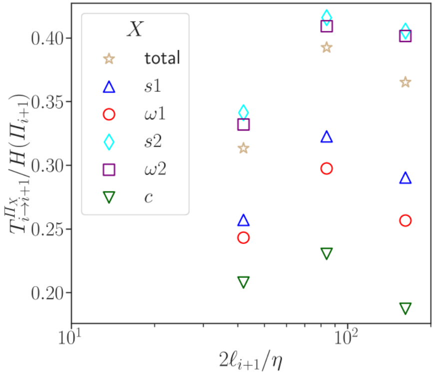


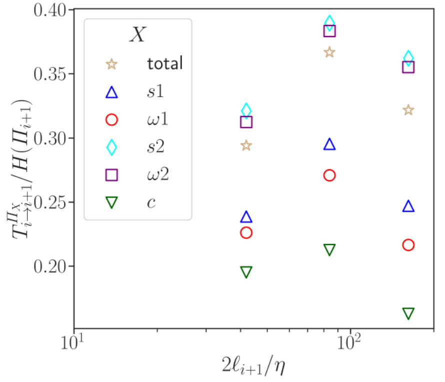










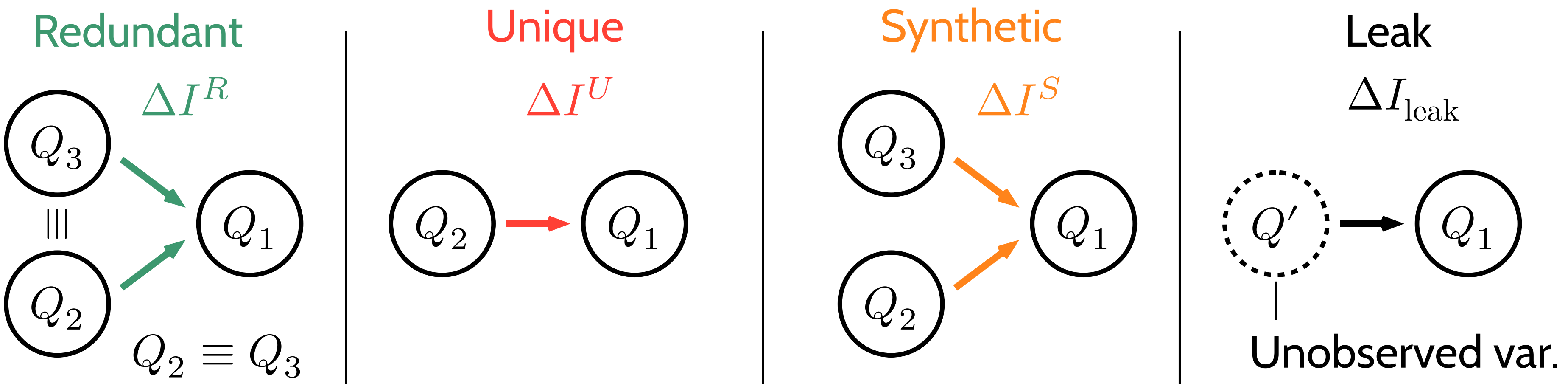


Synthetic-Unique-Redundant Decomposition (SURD)

Martínez-Sánchez et al. (2024)

Consider decomposing information flux from $Q_i(t)$ to $Q_j(t+\Delta t)$:

$$H\big(Q_j(t+\Delta t)\big) = \sum_{\boldsymbol{i}\in\mathcal{C}} \Delta I^{R}_{\boldsymbol{i}\to j} + \sum_{i=1}^{N} \Delta I^{U}_{i\to j} + \sum_{\boldsymbol{i}\in\mathcal{C}} \Delta I^{S}_{\boldsymbol{i}\to j} + \Delta I_{\mathrm{leak}\,\to j}$$

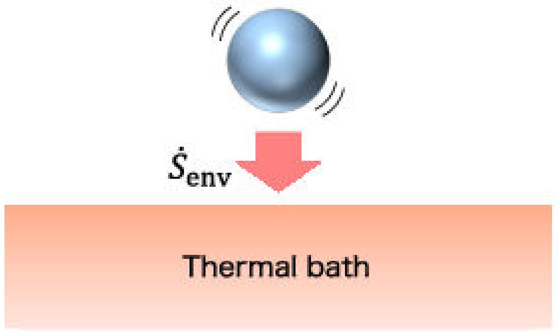


Results to be presented at EFDC2 @Dublin!

Information-Thermodynamic Nature of Information Flow in Turbulence

Stochastic thermodynamics (ST)

Thermodynamics for microscopic systems
with thermal fluctuations



Tanogami et al. (2023)

- (Underdamped) Langevin equation

$$\ddot{x} = -\frac{\gamma}{m}\dot{x} + F(x,t) + \boxed{\sqrt{2\gamma T}}\,\xi(t)$$

Fluctuation-dissipation relation

p: probability, $F(x,t)$: External force, $\xi(t)$: Noise

γ: Friction coefficient, m: Mass, T: Temperature

- Second law of ST

System's entropy change

$$\boxed{\frac{\mathrm{d}S}{\mathrm{d}t}} + \boxed{\dot{S}_{\mathrm{env}}} \geq 0$$

Environment's entropy change

References

- Peliti & Pigolotti (2021)
- Shiraishi (2023)
- Tasaki (2023)

Information thermodynamics (ITD)

Thermodynamics for microscopic subsystems with **information exchange**

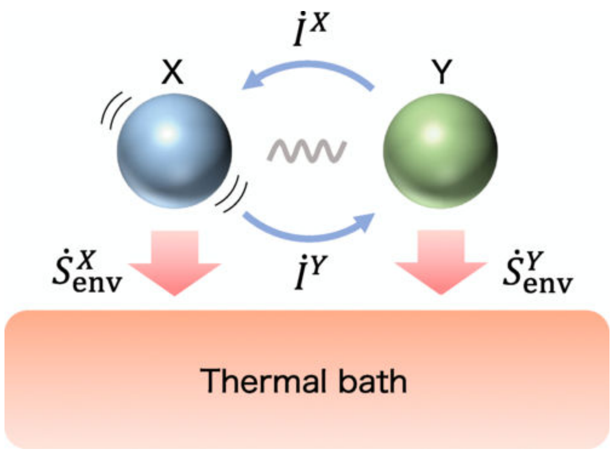
Second law of ITD (Shiraishi 2023)

Subsystem's entropy change

$$\underbrace{\frac{\mathrm{d}S^X}{\mathrm{d}t}}_{\text{Subsystem's entropy change}} + \underbrace{\dot{S}^X_{\mathrm{env}}}_{\text{Environment's entropy change}} \geq \underbrace{\dot{I}^X}_{\text{Information flow}}$$

Information flow

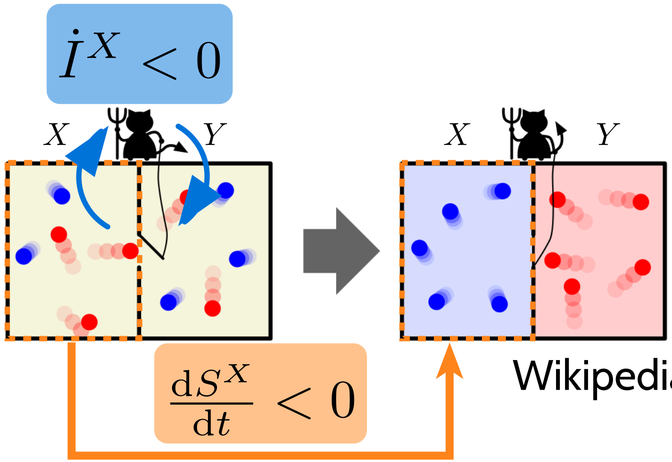
Environment's entropy change



Tanogami et al. (2023)

Example: Maxwell's demon

- Can reduce entropy $\rightarrow$ **Violate the second law of thermodynamics**
- Demon has to measure & feedback
  $\rightarrow$ **Satisfy the second law of ITD**



Wikipedia

Fluctuating Navier–Stokes equations (FNS eqs)

Explicitly taking **thermal fluctuations** into account

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \nu\nabla^2\boldsymbol{u} + \boldsymbol{f} + \boxed{\boldsymbol{\nabla}\cdot\ \boldsymbol{s}}$$

In Fourier space,

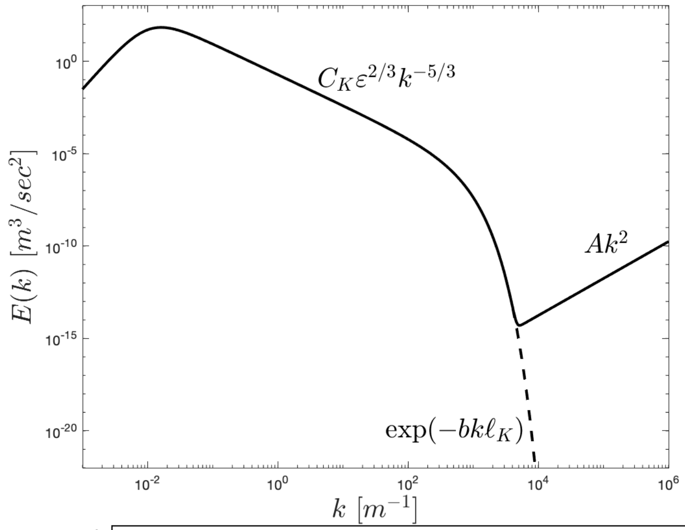
$$\frac{\partial \hat{\boldsymbol{u}}_{\boldsymbol{k}}}{\partial t} = \boldsymbol{B}_{\boldsymbol{k}}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*) - \nu k^2 \hat{\boldsymbol{u}}_{\boldsymbol{k}} + \hat{\boldsymbol{f}}_{\boldsymbol{k}} + \boxed{\sqrt{\frac{2\nu k^2 k_{\mathrm{B}} T}{\rho}}\hat{\boldsymbol{\xi}}_{\boldsymbol{k}}}.$$

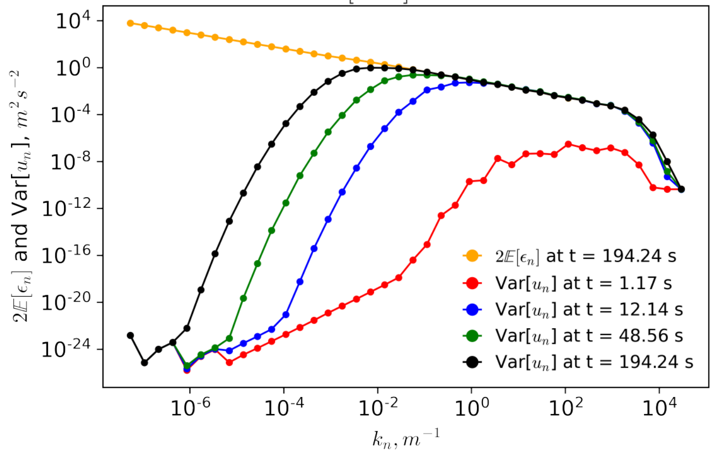
- $\hat{\boldsymbol{u}}_{\boldsymbol{k}}$: Fourier-space velocity at mode $\boldsymbol{k}$
- ν: Kinematic viscosity
- ρ: Mass density
- T: Temperature
- k_{B}: Boltzmann constant
- $B^a_{\boldsymbol{k}}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*) := -\mathrm{i}k^c\left(\delta^{ab} - \frac{k^a k^b}{k^2}\right)\sum_{\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{k}} \hat{u}^b_{\boldsymbol{p}}\hat{u}^c_{\boldsymbol{q}}$: Nonlinear term

Recent findings on the importance of thermal noise

Thermal fluctuations

- **dominate the far dissipation range** with energy equipartition (Bandak et al. 2022)
- are amplified to the largest scales **in finite time** (Bandak et al. 2024)
- **inhibit intermittency** in compressible turbulence (Srivastava et al. 2025)



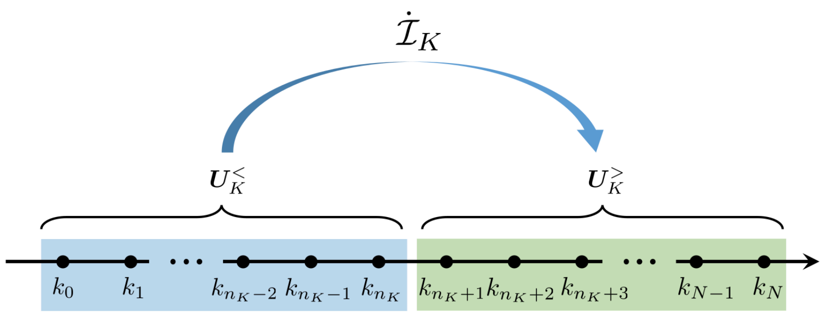


Summary 7 Thermal fluctuations impact dynamics and statistics of turbulence.

Information flow from large- to small-scale velocities

Divide full velocity field $\{\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*\}$ into

$$\{\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*\} = \boldsymbol{U}_K^< \cup \boldsymbol{U}_K^>$$



Tanogami (2024)

- Large scale modes $\boldsymbol{U}_K^<$ for $|\boldsymbol{k}| \leq K$
- Small scale modes $\boldsymbol{U}_K^>$ for $|\boldsymbol{k}| > K$

Mutual information $I[\boldsymbol{U}_K^<, \boldsymbol{U}_K^>] := \left\langle \ln \dfrac{p_t(\boldsymbol{U}_K^<, \boldsymbol{U}_K^>)}{p_t^<(\boldsymbol{U}_K^<) p_t^>(\boldsymbol{U}_K^>)} \right\rangle$

Information flow

$$\dot{I}_K^> := \lim_{\mathrm{d}t \searrow 0} \frac{I[\boldsymbol{U}_K^<(t) : \boldsymbol{U}_K^>(t+\mathrm{d}t)] - I[\boldsymbol{U}_K^<(t) : \boldsymbol{U}_K^>(t)]}{\mathrm{d}t}$$

When $\dot{I}_K^> > 0$, **small scales gain information** about large scales.

Information-thermodynamic bound of information flow

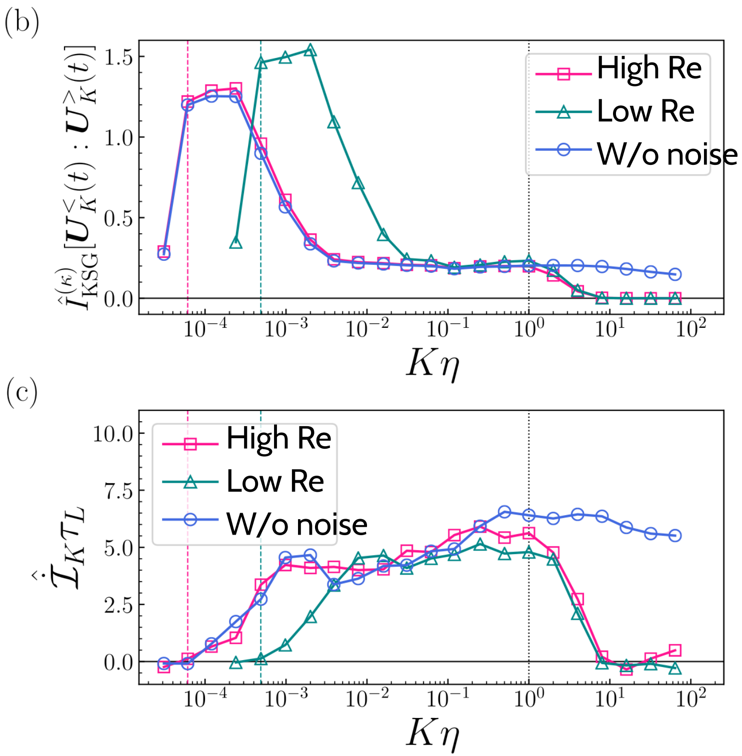
- In (nonequilibrium) steady state
- For wavenumber K in the inertial range

Information flow is bounded by

$$\boxed{\frac{\rho V \varepsilon}{k_{\mathrm{B}} T}} \geq \dot{I}_K^{>} \geq 0.$$

Environment's entropy change

Summary 8 Positive (macro $\to$ micro) information flow, bounded by the second law of ITD, exists in turbulence. (Tanogami & Araki 2024a)



V: Volume of fluid,
ε: energy dissipation rate

Sketch of the proof: Information flow

1. Consider the Fokker–Planck equation corresponding to FNS eqs.

$$\frac{\partial}{\partial t} p_t(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*) = \sum_{\boldsymbol{k} \in \mathcal{K}^+} \left[-\frac{\partial}{\partial \hat{\boldsymbol{u}}} \cdot \boldsymbol{J}_{\boldsymbol{k}}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*) - \mathrm{c.c.} \right]$$

$$\boldsymbol{J}_{\boldsymbol{k}}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*) = \left[\boldsymbol{B}_{\boldsymbol{k}}(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*) - \nu k^2 \hat{\boldsymbol{u}}_{\boldsymbol{k}} + \hat{\boldsymbol{f}}_{\boldsymbol{k}} \right] p_t(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*)$$
$$- \frac{1}{V} \frac{\nu k^2 k_{\mathrm{B}} T}{\rho} \left(\mathsf{I} - \frac{\boldsymbol{k}\boldsymbol{k}}{k^2} \right) \cdot \frac{\partial}{\partial \hat{\boldsymbol{u}}^*} p_t(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*)$$

$\mathcal{K}^+$: Set of independent Fourier modes, $\boldsymbol{B}_{\boldsymbol{k}}(\hat{u}, \hat{u}^*)$: Nonlinear term for mode $\boldsymbol{k}$

2. Define system's Shannon entropy

$$S[\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*] := -\int \mathrm{d}\hat{\boldsymbol{u}}\, \mathrm{d}\hat{\boldsymbol{u}}^*\, p_t(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*) \ln p_t(\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*)$$

Sketch of the proof: Information flow

3. Define environment's entropy change (due to viscous & thermal noise)

$$\dot{S}^{\mathrm{env}} := \sum_{\boldsymbol{k}} \frac{\rho V}{2k_{\mathrm{B}}T} \left\langle \hat{\boldsymbol{u}}^* \circ \left[\nu k^2 \hat{\boldsymbol{u}}^*_{\boldsymbol{k}} - \sqrt{\frac{2\nu k^2 k_{\mathrm{B}} T}{\rho}} \hat{\boldsymbol{\xi}}_{\boldsymbol{k}} \right] + \mathrm{c.c.} \right\rangle$$

4. Show the second law of stochastic thermodynamics

$$\dot{\sigma} := \frac{\mathrm{d}}{\mathrm{d}t} S[\hat{\boldsymbol{u}}, \hat{\boldsymbol{u}}^*] + \dot{S}^{\mathrm{env}} \geq 0$$

5. Derive the second law of information thermodynamics

$$\sum_{\boldsymbol{k} \in \mathcal{K}^+, k > K} \dot{\sigma}_{\boldsymbol{k}} = \frac{\mathrm{d}}{\mathrm{d}t} S[\boldsymbol{U}_K^>] + \dot{S}_{\mathrm{env}}^> \geq \dot{I}_K^>$$

where $\mathrm{d}_t S[\boldsymbol{U}_K^>] = 0$ (NESS) & $\dot{S}_{\mathrm{env}}^> \to \rho V \varepsilon / k_{\mathrm{B}} T$ as $K/k_\nu \to 0$.

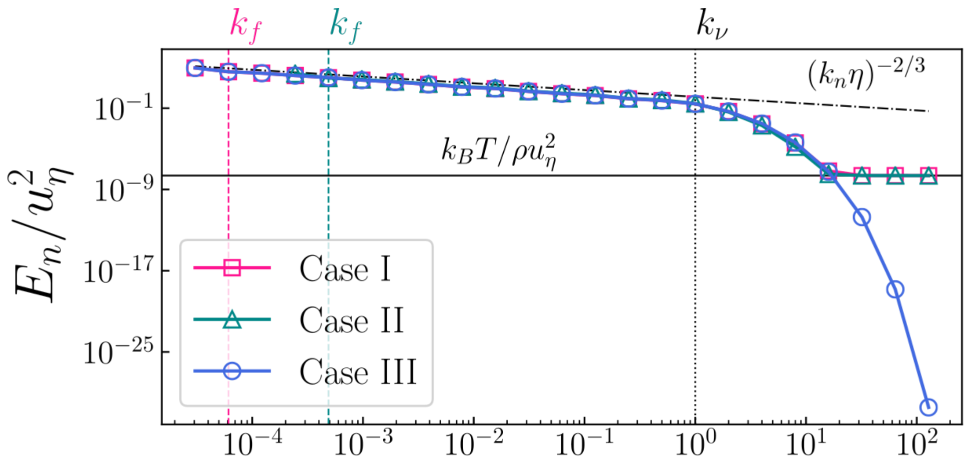
Fluctuating Sabra shell model (FSS model)

One-dimensional caricature of the fluctuating NS eqs.

$$\frac{\partial u_n}{\partial t} = B_n(u, u^*) - \nu k_n^2 u_n + f_n + \boxed{\sqrt{\frac{2\nu k^2 k_\mathrm{B} T}{\rho}}\xi_k},$$

$$B_n(u, u^*) = \mathrm{i}\left(k_{n+1} u_{n+2} u_{n+1}^* - \frac{1}{2} k_n u_{n+1} u_{n-1}^* + \frac{1}{2} k_{n-1} u_{n-1} u_{n-2}\right)$$

- u_n: 1D "velocity" at wavenumber k_n
- u_n^*: complex conjugate of u_n
- $k_n = k_0 2^n$: wavenumber



Tanogami & Araki (2024a)

Information flow and turbulence fluctuations

For FSS model,

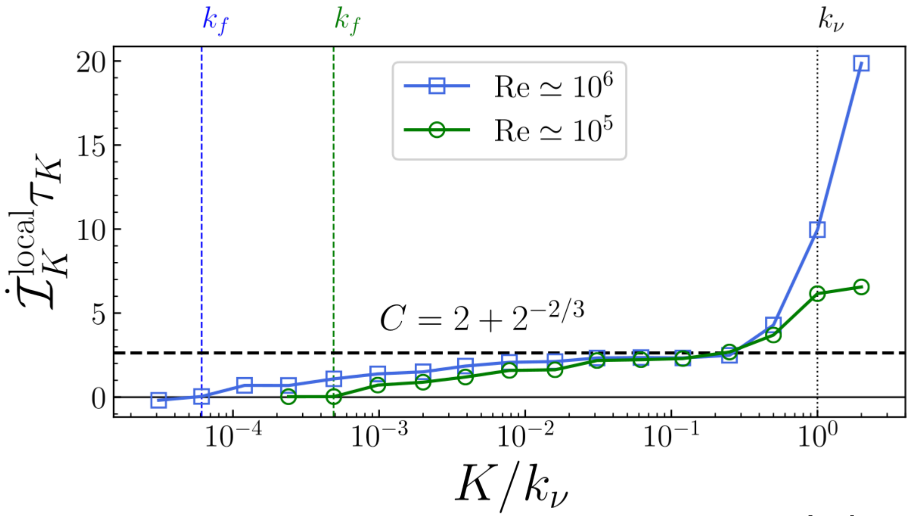
Assumption

PDF of the Kolmogorov multiplier

$$z_n := |u_n/u_{n-1}|\mathrm{e}^{\mathrm{i}\Delta_n}$$

is universal and independent of n.

$$\Delta_n := \arg u_n - \arg u_{n-1} - \arg u_{n-2}$$



Tanogami & Araki (2024b)

Statement

$$\dot{I}_K^{>} \leq C_p K \left\langle \left| u_{n_K} \right|^p \right\rangle^{\frac{1}{p}} \quad \text{for } p \geq 1$$

where C_p is a universal constant.

Summary 9 Information flow is bounded by turbulence fluctuations.

(Tanogami & Araki 2024b)

Sketch of the proof: Information flow & velocity fluctuations

1. Consider the Fokker–Planck equation corresponding to the large-scale modes $\boldsymbol{U}_K^< := \{u_n, u_n^* \mid 0 \le n \le n_K\}$ of the FSS model

$$\frac{\partial}{\partial t} p_t(\boldsymbol{U}_K^<) = \sum_{n=0}^{N} \left[-\frac{\partial}{\partial u_n} \overline{J}_n(\boldsymbol{U}_K^<) - \text{c.c.} \right],$$

$$\overline{J}_n(\boldsymbol{U}_K^<) := \left[\overline{B}_n(\boldsymbol{U}_K^<) + f_n \right] p_t(\boldsymbol{U}_K^<),$$

$$\overline{B}_n(\boldsymbol{U}_K^<) := \int \mathrm{d}\boldsymbol{U}_K^> \, B_n(u, u^*) p_t(\boldsymbol{U}_K^> | \boldsymbol{U}_K^<).$$

2. Impose assumption: PDF of the Kolmogorov multiplier

$$z_n := |u_n / u_{n-1}| \mathrm{e}^{\mathrm{i}\Delta_n}$$

is universal and independent of $n \to$ FP eq. is closed with $\boldsymbol{U}_K^<$.

Sketch of the proof: Information flow & velocity fluctuations

3. Show the equivalence between scale-to-scale information flow $\dot{I}_K^{>}$ and phase-space contraction rate

$$\dot{I}_K^{>} = -\left\langle \sum_{n=0}^{n_K} \left(\frac{\partial}{\partial u_n} \overline{B}_n(\boldsymbol{U}_K^{<}) \right) + \mathrm{c.c.} \right\rangle \text{ for } k_f \ll K \ll k_\nu .$$

4. Show that $\dot{I}_K^{>}$ is upper bounded by the velocity structure function

$$\dot{I}_K^{>} \leq C_p K \left\langle \left| u_{n_K} \right|^p \right\rangle^{1/p} \text{ for } p \geq 1 \text{ by using}$$

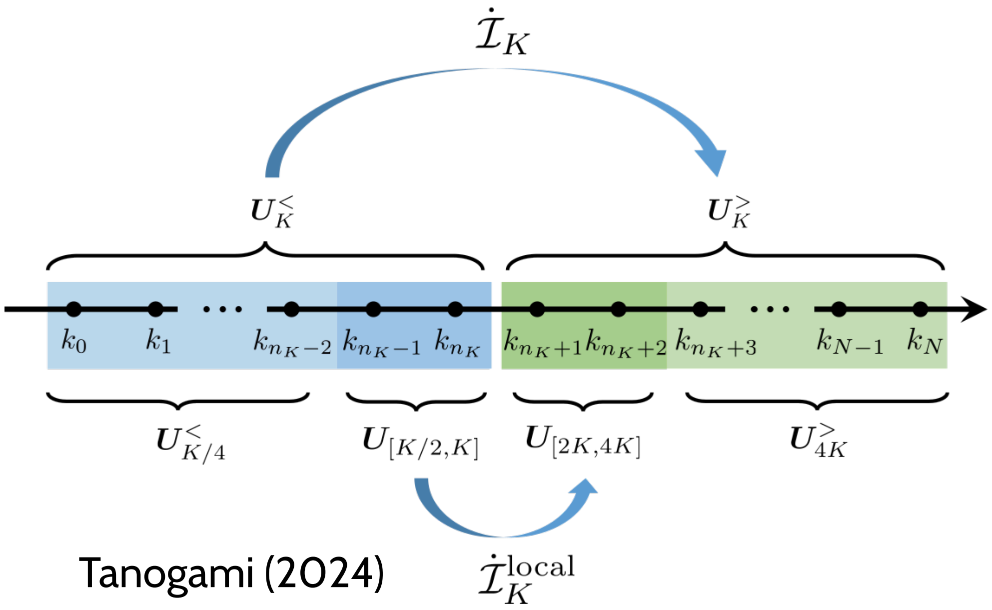
- Kolmogorov multiplier z_n
- Divergence of the "effective" nonlinear term $\overline{B}_n(\boldsymbol{U}_K^{<})$
- Hölder inequality $\langle fg \rangle \leq \langle f^p \rangle^{1/p} \langle g^q \rangle^{1/q}$ with $1/p + 1/q = 1$

Scale locality of information flow

Assumption: For FSS model,

Scale-local information flow

$$\dot{I}_K^{>,\ \mathrm{local}} := \lim_{\mathrm{d}t \searrow 0} \frac{I\left[\boldsymbol{U}^{<}_{[K/2,K]}(t) : \boldsymbol{U}^{>}_{[2K,4K]}(t+\mathrm{d}t)\right] - I\left[\boldsymbol{U}^{<}_{[K/2,K]}(t) : \boldsymbol{U}^{>}_{[2K,4K]}(t)\right]}{\mathrm{d}t}$$



Tanogami (2024)

Summary 10 Information flow is local in scale:

$$\dot{I}_K^{>} \approx \dot{I}_K^{>,\ \mathrm{local}}.$$

(Tanogami 2024)

Sketch of the proof: Scale locality

1. Decompose the information flow in scale local/nonlocal parts:

$$
\begin{aligned}
\dot{I}_K^{>} &= \dot{I}_K^{>,\ \text{local}} + \dot{I}_K^{>,\ \text{nonlocal}} \\
&= \dot{I}_{[2K,4K]}\left[\boldsymbol{U}_{[K/2,K]} : \boldsymbol{U}_{[2K,4K]}\right] \\
&\quad + \dot{I}_{[2K,4K]}\left[\boldsymbol{U}_{K/4}^{<} : \boldsymbol{U}_{[2K,4K]} \mid \boldsymbol{U}_{[K/2,K]}\right] - \dot{I}_K^{<}\left[\boldsymbol{U}_K^{<} : \boldsymbol{U}_{4K}^{>} \mid \boldsymbol{U}_{[2K,4K]}\right]
\end{aligned}
$$

2. Show **ultraviolet** locality (no direct influence from **high**-wavenumber modes)

$$
\dot{I}_K^{<}\left[\boldsymbol{U}_K^{<} : \boldsymbol{U}_{4K}^{>} \mid \boldsymbol{U}_{[2K,4K]}\right] = 0.
$$

3. Show **infrared** locality (no direct influence from **low**-wavenumber modes)

$$
\dot{I}_{[2K,4K]}\left[\boldsymbol{U}_{K/4}^{<} : \boldsymbol{U}_{[2K,4K]} \mid \boldsymbol{U}_{[K/2,K]}\right] = 0
$$

with the Kolmogorov multiplier's assumption.

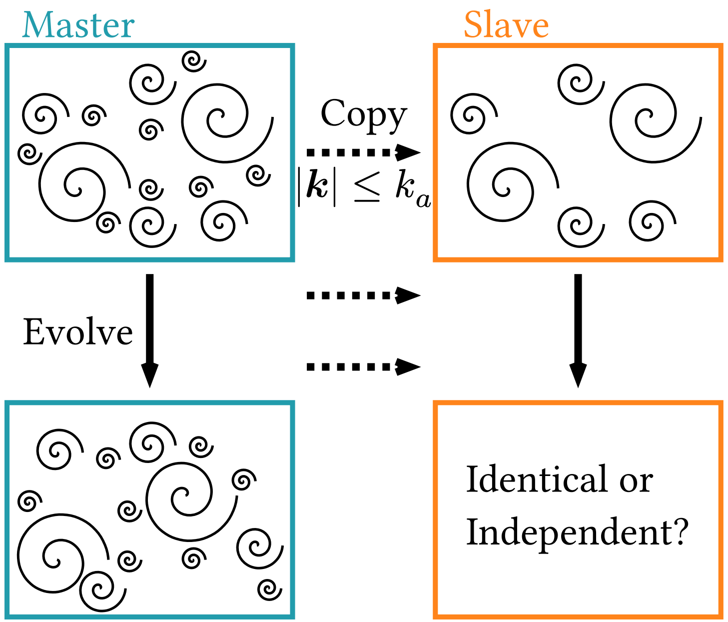
Ideas for Future Research

Data assimilation (DA) and chaos synchronisation

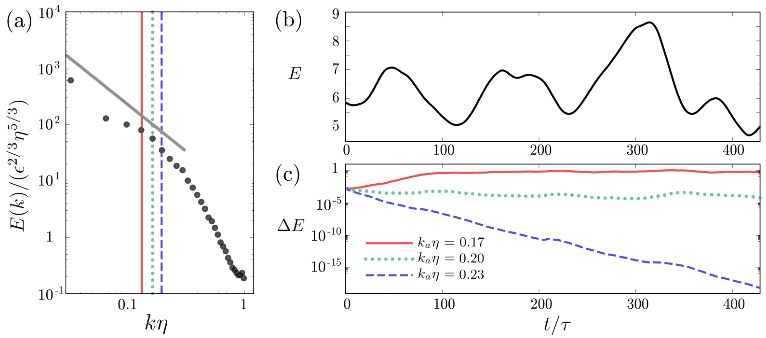
- Master: $\partial_t \boldsymbol{u}^{\mathrm{m}}(\boldsymbol{x}, t) =$ Navier–Stokes eq.
- Slave: $\partial_t \boldsymbol{u}^{\mathrm{s}}(\boldsymbol{x}, t) =$ Navier–Stokes eq.
  - $\hat{\boldsymbol{u}}^{\mathrm{s}} := \sum_{|\boldsymbol{k}| \le k_a} \hat{\boldsymbol{u}}^{\mathrm{m}}(\boldsymbol{k}, t) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$
- $\boldsymbol{u}^{\mathrm{s}}$ synchronises with $\boldsymbol{u}^{\mathrm{m}}$ for $k_a \eta > 0.2$

η: Kolmogorov scale

(Inubushi et al. 2023, Yoshida et al. 2005)



Q. 3 Can we understand the DA threshold $k\eta = 0.2$ by means of "information flow"?



Butterfly effect and spontaneous stochasticity

Butterfly effect (sensitivity on initial condition)

…slightly differing initial states can evolve into considerably different states.

— Lorenz EN. 1963. Deterministic nonperiodic flow. *Journal of the Atmospheric Sciences*. 20(2):130–41

Spontaneous stochasticity ("real" butterfly effect)

… two states of the system differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states of the system within a finite time interval, which **cannot be lengthened by reducing the amplitude of the initial error**.

— Lorenz EN. 1969. The predictability of a flow which possesses many scales of motion. *Tellus*. 21(3):289–307

Spontaneous stochasticity in turulence

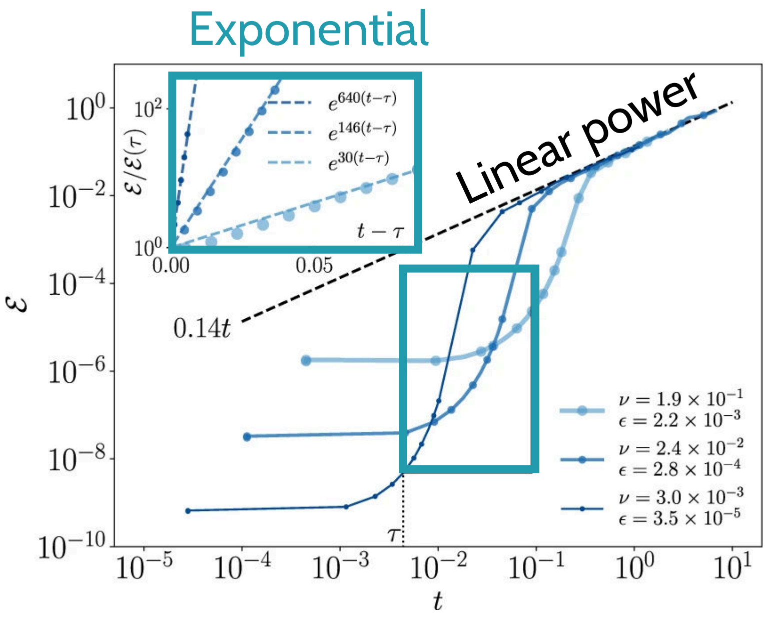
Numerical simulation of shear layer

$$\omega_1 := U\delta(y), \quad \omega_2 := [1 + \varepsilon\eta(x)]U\delta(y)$$

$\varepsilon \ll 1$: small parameter, $\eta(x)$: perturbation profile

Time series of error energy $\mathcal{E} := \|u_1 - u_2\|$

1. ε-dependent exponential divergence
   - Ruelle regime (Ruelle 1979)
2. Universal linear growth for $t > \mathcal{O}(10^0)$ independent of initial error
   - Lorenz–Kraichnan–Leith regime



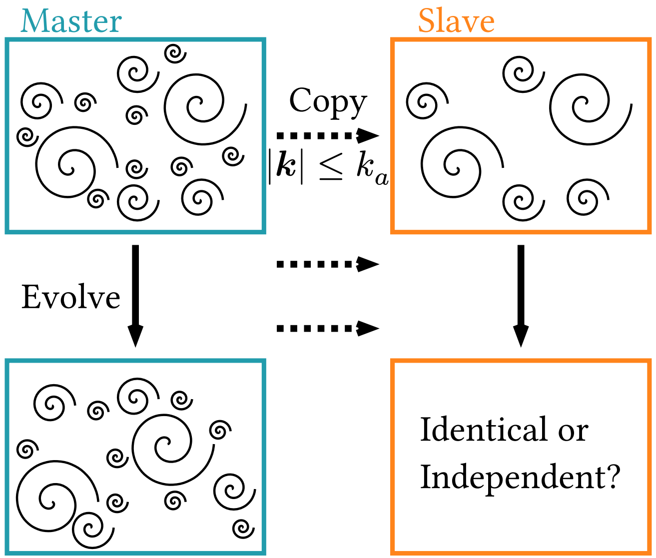
Thalabard S, Bec J, Mailybaev AA. 2020. From the butterfly effect to spontaneous stochasticity in singular shear flows. *Communications Physics*. 3(1):122

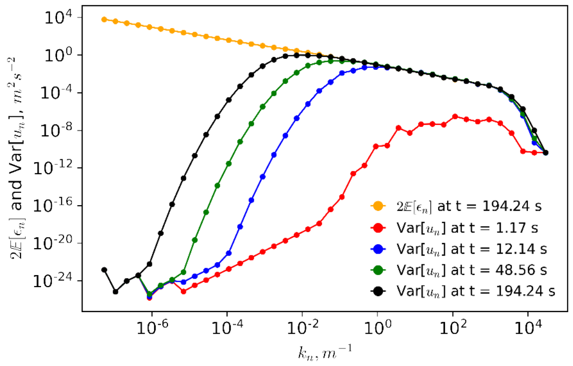
References: Bandak (2023), Palmer (2024)

Intrinsic limit of data assimilation in fluctuating hydrodynamics

- Successful DA threshold: $k_a \eta > 0.2$

  (Inubushi et al. 2023, Yoshida et al. 2005)
- Thermal fluctuations reach maximum scale in finite time (Bandak et al. 2024)
- How do **deterministic** and **fluctuating** NS differ for DA?
- How do spontaneous stochasticity and DA relate?





Q. 4 Can we seek **universal prediction boundary** of DA from fluctuating hydrodynamics?

Other research ideas on “Information hydrodynamics”

Q. 5 Information flow in 2D turbulence (Nakano et al. 2025)

Q. 6 Information-thermodynamic nature of thermal-noise-driven laminar-turbulent transition (Li et al. 2020)

Q. 7 Validation of the fluctuation theorem in fluctuating Navier–Stokes equations (Yao et al. 2023)

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Appendix

Turbulence model which preserves information

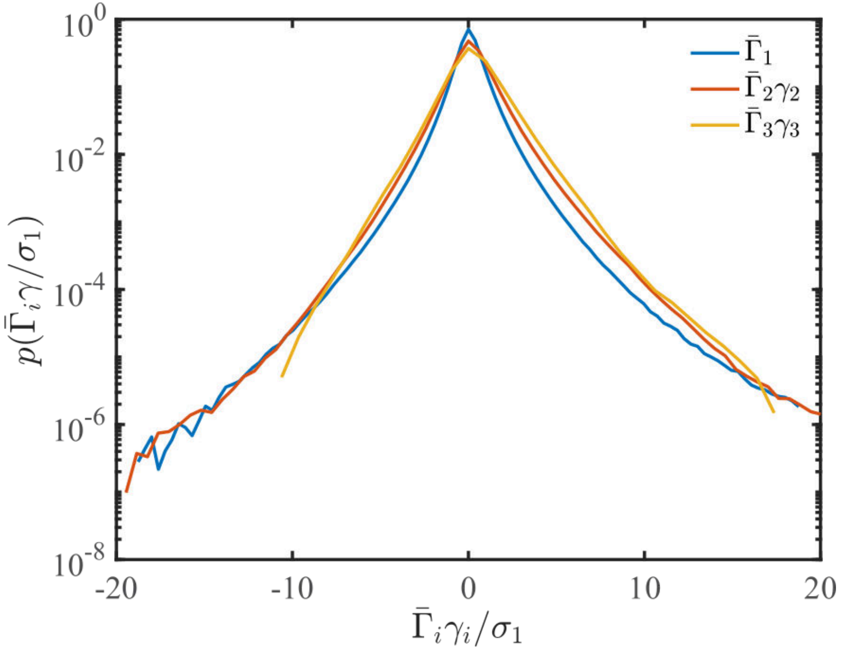
Reynolds stress model

$$\tau_{ij} = C_1(t) f_1(\boldsymbol{x}, t)\Delta^2 |\boldsymbol{S}| S_{ij} + C_4(t) f_4(\boldsymbol{x}, t)\Delta^2 \left(S_{ik}\Omega_{kj} - \Omega_{ik}S_{kj}\right)$$

where $\boldsymbol{S} := (\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^\top)/2, \boldsymbol{\Omega} := (\nabla \boldsymbol{u} - \nabla \boldsymbol{u}^\top)/2,$ Δ: grid resolution

Energy flux

$$\overline{\Gamma} := \left(\overline{u_i u_j} - \overline{u_i}\,\overline{u_j}\right)\overline{S}_{ij}$$
$$-2\nu \overline{S}_{ij}\overline{S}_{ij} + \tau_{ij}\overline{S}_{ij}$$



Lozano-Durán & Arranz (2022)

Modeling assumption

$$p(\overline{\Gamma}_1) \approx p(\overline{\Gamma}_2\gamma)/\gamma$$

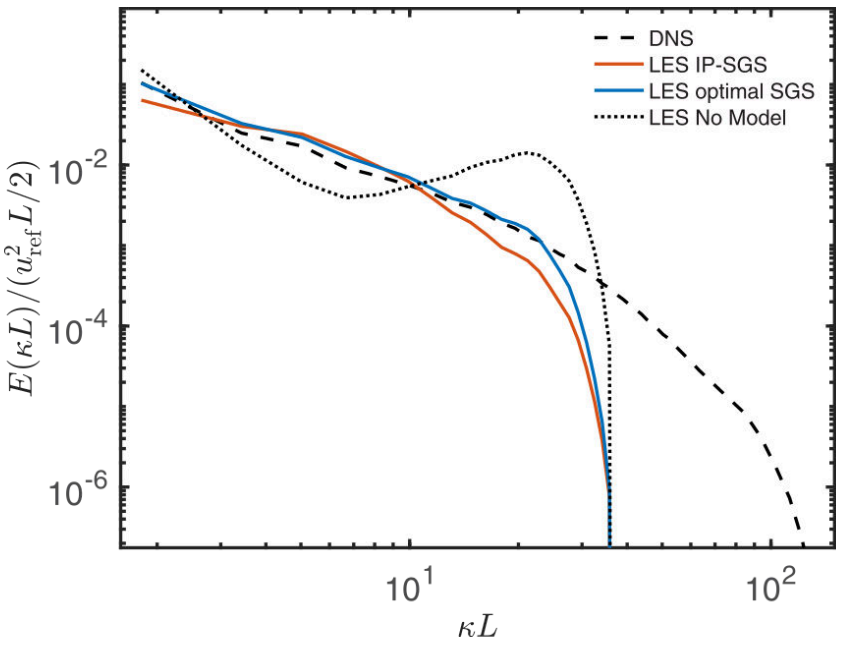
with scale factor $\gamma := (\Delta_1/\Delta_2)^{2/3}$

Information-Preserving SubGrid-Scale (IP-SGS) model

- "Similar PDF" $=$ minimum Kullback–Leibler divergence
- To estimate $C_i(t)$ & $f_i(\boldsymbol{x}, t)$, find

$$C_i, f_i = \arg\min_{C_i, f_i} \mathrm{KL}(\overline{\Gamma}_1 \parallel \overline{\Gamma}_2 \gamma)$$

Lozano-Durán & Arranz (2022)



Deficits of IP-SGS

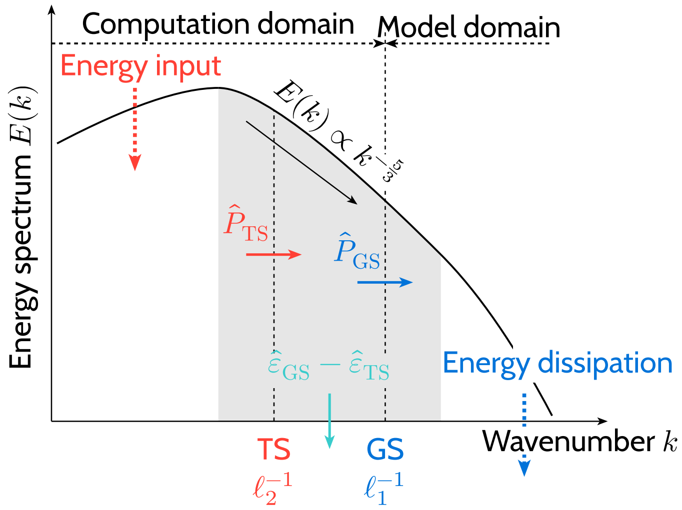
- A priori parameter α for $\gamma := (\Delta_2/\Delta_1)^\alpha$ exists
- No consideration of the near-wall behaviour

Kullback–Leibler divergence

$$\mathrm{KL}(P \| Q) := \int \mathrm{d}x\, p(x) \ln \frac{p(x)}{q(x)}$$

Local equilibrium hypothesis in terms of information theory

Instead of PDF similarity, we consider



Local equilibrium hypothesis (LEH)

Instantaneous local balance of energy fluxes at different scales:

$$\langle P_{\mathrm{TS}} + \varepsilon_{\mathrm{TS}} \rangle_v \approx \langle P_{\mathrm{GS}} + \varepsilon_{\mathrm{GS}} \rangle_v$$

GS: Grid Scale at ℓ_1, TS: Test scale at ℓ_2, v: small domain

Information-theoretic redifinition of LEH

"Similar PDF" $=$ maximum mutual information

$$\max I[P_{\mathrm{TS}} + \varepsilon_{\mathrm{TS}} : P_{\mathrm{GS}} + \varepsilon_{\mathrm{GS}}]$$

Mutual information

$$I[P:Q] := \iint \mathrm{d}x\,\mathrm{d}y\, p(x) \ln \frac{p(x,y)}{p(x)q(y)}$$

Detailed procedure of IP-CSM

Spatio-time dependent coefficient

$$f_1(\boldsymbol{x}, t) = |F_{\mathrm{CS}}|^{\frac{3}{2}}, f_4(\boldsymbol{x}, t) = |F_{\mathrm{CS}}|^2$$

$\rightarrow$ applicable to wall-bounded flow

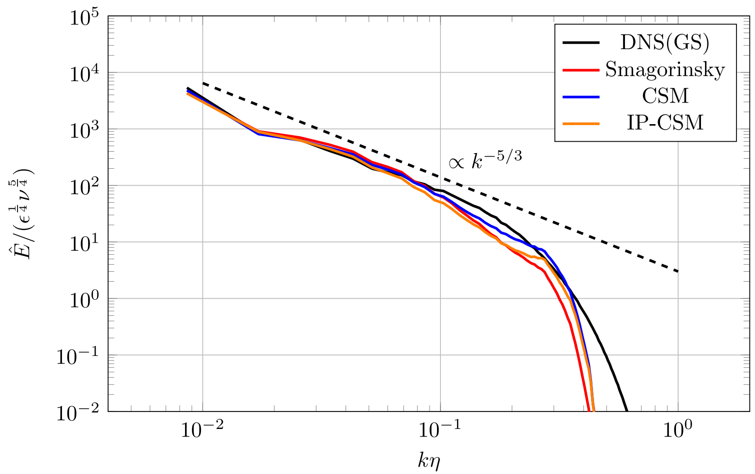
Coherent structure function (Kobayashi 2005)

$$F_{\mathrm{CS}} := \frac{(\Omega_{ij}\Omega_{ij} - S_{ij}S_{ij})/2}{(\Omega_{ij}\Omega_{ij} + S_{ij}S_{ij})/2}$$

Time-dependent parameter

$$C_1, C_4 = \arg\max_{C_1, C_4} I[\Gamma_1 : \Gamma_2(C_1, C_4)].$$

Summary 11 IP-CSM attains similar performance against existing SGS models.



Detailed procedure of IP-CSM

1. Compute Γ_1 from GS quantities and $\Gamma_2(C_1, C_4)$ using model quantities

$$\Gamma_1 = \int_v -\left(\overline{\overline{u_i}^{\ell_1}\overline{u_j}^{\ell_1}}^{\ell_2} - \overline{\overline{u_i}^{\ell_1}}^{\ell_2}\overline{\overline{u_j}^{\ell_1}}^{\ell_2}\right)\overline{S_{ij}}^{\ell_2} + \varepsilon_{\mathrm{TS}}\ \mathrm{d}v$$

$$\Gamma_2(C_1, C_4) = \int_v \overline{\boxed{\tau_{ij}^{\ell_1\ \mathrm{mod}}}}^{\ell_2}\overline{S_{ij}}^{\ell_2} - \overline{\boxed{\tau_{ij}^{\ell_1\ \mathrm{mod}}}S_{ij}}^{\ell_2} + \varepsilon_{\mathrm{GS}}\ \mathrm{d}v$$

2. Evaluate time-dependent $C_{i(t)}$ using gradient descent method

$$\boxed{C_1, C_4} = \arg\max_{C_1, C_4} I\boxed{(\Gamma_1 : \Gamma_2)}$$

3. Estimate the SGS stress $\rightarrow$ evolve LES

$$\boxed{\tau_{ij}^{\Delta\ \mathrm{mod}}} = \boxed{C_1}\,|F_{\mathrm{CS}}|^{\frac{3}{2}}\ \Delta^2|\boldsymbol{S}|S_{ij} + \boxed{C_4}\,|F_{\mathrm{CS}}|^2\ \Delta^2\left(S_{ik}\Omega_{kj} - \Omega_{ik}S_{kj}\right)$$